Strategic Games and Truly Playable Effectivity Functions

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Abstract. A well known (and often used) result by Marc Pauly states that playable effectivity functions correspond to strategic games in a precise sense. That is, for every playable effectivity function E there exists a strategic game that assigns to coalitions exactly the same power as E, and vice versa. While the latter direction of the correspondence is correct, we show that the former does not always hold in the case of infinite game models. We point out where the proof of correspondence goes wrong, and we present examples of playable effectivity functions in infinite models for which no equivalent strategic game exists. Then, we characterize the "truly playable" class, that does correspond to strategic games. Moreover, we discuss a construction that transforms any playable effectivity function into a truly playable one while preserving the power of most (but not all!) coalitions.

1 Introduction

Several logics for reasoning about coalitional power have been proposed and studied in the last two decades. Eminent examples are: Alternating Time Temporal Logic (ATL) [1], Coalition Logic (CL) [11], and Seeing To It That (STIT) [2], used in computer science and philosophy to reason about properties of multi-agent systems. A crucial feature of these logics is the correspondence between their models and the game structures they are meant to reason about.

In particular, the connection between the semantics of Coalition Logic and the games it is supposed to describe relies on *Pauly's representation theorem* [11] which states that playable effectivity functions correspond exactly to strategic games. Moreover, the correspondence has been used when obtaining further results for CL: if the semantics can be defined equivalently in terms of strategic games and playable effectivity functions, they can be used interchangeably when proving properties of the logic. A similar remark applies to ATL and STIT, connected to Coalition Logic by a number of simulation results [3,6,7].

The correspondence between strategic games and effectivity functions is important even without the logical context. Effectivity functions generalize basic models of cooperative game theory, whereas strategic games are models of non-cooperative game theory. Pauly's result is relevant as, in its raw form, it puts forward a characterization of *coalitional games that can be implemented by strategic games*, therefore establishing a strong connection between the two families of game models.

In this paper, we show that the representation theorem is not correct as it stands. More precisely, we show that there are some playable effectivity functions with no corresponding strategic games. We point out where Pauly's proof of correspondence goes wrong, and we present examples of playable effectivity functions in infinite sets of outcomes, for which no equivalent strategic games exist. Then, we define a more restricted class of effectivity functions (that we call *truly playable*) and we show them to correspond precisely to strategic games. We discuss several alternative characterizations of truly playable functions. Moreover, we present a construction that recovers the correspondence in the sense that it transforms any playable function into a truly playable one while preserving the power of most (but not all) coalitions.

2 Preliminaries

We start by introducing the game-theoretic notions of strategic game and effectivity function, and discussing their relevant features.

2.1 Strategic Games

Strategic games (also: normal form games) are basic models of non-cooperative game theory [10]. After [11], we focus on abstract game forms, where the effect of strategic interaction between players is represented by abstract outcomes from a given set and players' preferences are not specified. For simplicity we refer to them as strategic games.

Definition 1 (Strategic game). *A* strategic game *G is a tuple $(N, \{\Sigma_i | i \in N\}, o, S)$ that consists of a nonempty finite set of players N, a nonempty set of strategies Σ_i for each player $i \in N$, a nonempty set of outcomes S, and an outcome function $o : \prod_{i \in N} \Sigma_i \to S$ which associates an outcome with every strategy profile.*

The outcome function is often assumed to be a bijection and it is consequently dispensed with [10]. Some works, mostly aiming at formalizing the condition of *non-imposedness* in social choice theory [9],[4] assume surjectivity. As different scenarios may require different assumptions, we use the definition of strategic game in its most general form, as done in [11].

Additionally, we follow [11] and define coalitional strategies σ_C in G as tuples of individual strategies σ_i for $i \in C$, i.e., $\Sigma_C = \prod_{i \in C} \Sigma_i$. Note that (regardless

[4] The condition of non-imposedness is also referred to as *citizen sovereignty* and it allows players to freely choose among all possible alternatives in a decision process.

of possible conceptual interpretations of the empty coalition $\emptyset$, cf. [4] for a discussion) this definition allows for only one strategy $\sigma_\emptyset$ of $C = \emptyset$, namely the empty function.

2.2 Effectivity Functions

Effectivity functions have been introduced in cooperative game theory [9] to provide an abstract representation of the powers of coalitions to influence the outcome of the game.

Definition 2 (Effectivity function). *An* effectivity function *is a function* $E : 2^N \to 2^{2^S}$*, that associates a family of "outcome sets" of states from* S *with each set of players.*

Intuitively, elements of $E(C)$ are *choices* available to coalition C: if $X \in E(C)$ then by choosing X the coalition C can force the outcome of the game to be in X. Effectivity functions are usually required to satisfy additional properties, consistent with this interpretation. As far as we are concerned we will examine the class of *playable* effectivity functions, as defined in [11].

Definition 3 (Playability [11]). *An effectivity function* E *is* playable *if and only if the following conditions hold:*

Outcome Monotonicity $X \in E(C)$ *and* $X \subseteq Y$ *implies* $Y \in E(C)$*;*
N-maximality $\overline{X} \notin E(\emptyset)$ *implies* $X \in E(N)$*;*
Liveness $\emptyset \notin E(C)$*;*
Safety $S \in E(C)$*;*
Superadditivity *if* $C \cap D = \emptyset$*,* $X \in E(C)$ *and* $Y \in E(D)$*, then* $X \cap Y \in E(C \cup D)$*.*

Each strategic game $\mathcal{G}$ can be canonically associated with an effectivity function, called the α-effectivity function of $\mathcal{G}$ and denoted with $E^\alpha_{\mathcal{G}}$.

Definition 4 (α-Effectivity in Strategic Games). *For a strategic game* G*, its (coalitional)* α-effectivity function $E^\alpha_G : 2^N \to 2^{2^S}$ is defined as follows: $X \in E^\alpha_G(C)$ if and only if there exists σ_C such that for all $\sigma_{\overline{C}}$ we have $o(\sigma_C, \sigma_{\overline{C}}) \in X$.

It is claimed in [11] that playable effectivity functions correspond to strategic games. That is, for every effectivity function E there is a strategic game G such that $E = E^\alpha_G$, and vice versa [11, Theorem 2.27]. We will show further that this claim is not correct.

2.3 Nonmonotonic Core of an Effectivity Function

The key notion on which our considerations rely is that of "nonmonotonic core", introduced in [11]. Looking at playable effectivity functions, we can observe that their representation contains much redundancy. In particular, the fact that $E(C)$ is outcome monotonic suggests that we can succinctly represent the effectivity function in terms of minimal sets, i.e., the elements of $E(C)$ that form an antichain under set inclusion. The nonmonotonic core is aimed to provide such a representation.

Definition 5 (Nonmonotonic core). *Let E be an effectivity function. The* nonmonotonic core $E^{nc}(C)$ *for $C \subseteq N$ is the set of minimal sets in $E(C)$:*

$$\{X \in E(C) \mid \neg\exists Y (Y \in E(C) \text{ and } Y \subsetneq X)\}.$$

We will show in Section 3.1 that not all effectivity functions have a nonempty nonmonotonic core. Moreover, even when it is nonempty, not all sets in an effectivity function need to contain a subset in the nonmonotonic core (cf. Section 4.3). Thus, E^{nc} does not always behave well as a representation of the effectivity function, unless it is "complete" in the following sense.

Definition 6 (Complete nonmonotonic core). *The nonmonotonic core $E^{nc}(C)$ is* complete *if for every $X \in E(C)$ there exists $Y \in E^{nc}(C)$ such that $Y \subseteq X$.*

Note that if $E(C)$ has a complete nonmonotonic core then $E^{nc}(C)$ *can* be used as a succinct representation of $E(C)$. Complete nonmonotonic cores turn out to be fundamental when establishing the proper correspondence between strategic games and effectivity functions.

The nonmonotonic core of the empty coalition is of particular interest to us. For it, the following holds.

Proposition 1. *For every playable effectivity function E:*

1. *$E(\emptyset)$ is a filter.* [5]
2. *$E^{nc}(\emptyset)$ is either empty or a singleton.*

Proof. (1) $E(\emptyset)$ is non empty by safety; it is closed under supersets by outcome monotonicity, and under intersections by superadditivity (with respect to the empty coalition).

(2) Suppose $E^{nc}(\emptyset)$ is non-empty, and let $X, Y \in E^{nc}(\emptyset)$. Then, coalition $\emptyset$ is effective for each of X and Y, hence, by superadditivity, it is effective for $X \cap Y$. By the definition of $E^{nc}(\emptyset)$, it follows that $X = X \cap Y = Y$.

Proposition 2. *For every α-effectivity function $E^{\alpha}_{G} : 2^N \to 2^{2^S}$, the following hold:*

1. *The nonmonotonic core of $E^{\alpha}_{G}(\emptyset)$ is the singleton set $\{Z\}$ where $Z = \{x \in S \mid x = o(\sigma_N)$ for some $\sigma_N\}$.*
2. *$E^{\alpha}_{G}(\emptyset)$ is the principal*[6] *filter generated by Z.*

Proof. For both claims it suffices to observe that $Z \in E^{\alpha}_{G}(\emptyset)$ and that $Z \subseteq U$ for every $U \in E^{\alpha}_{G}(\emptyset)$. Therefore, $E^{nc}(\emptyset) = \{Z\}$ for $E = E^{\alpha}_{G}$ and $E^{\alpha}_{G}(\emptyset)$ is the principal filter generated by Z.

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[5] A family F of subsets of Ω is a *filter* if and only if (1) $\Omega \in F$, (2) $\emptyset \notin F$ (3) F is closed under finite intersection, and (4) F is closed under supersets. These structures are sometimes referred to as proper filters, to distinguish them from improper filters, that do not satisfy condition (2) and consequently coincide with 2^{Ω} (cf. e.g. [5]).

[6] Filter F is *principal* if and only if there exists $X \subseteq \Omega$ such that F is the set of all supersets of X. Then, F is said to be *generated* by X. Filters that are not principal are referred to as *non-principal*.

3 Problem with Correspondence

In this section we show that the playability conditions are not sufficient in infinite models to make effectivity functions correspond to strategic games.

3.1 A Counterexample to Representation Theorem

We begin by quoting the claim that we are going to dispute.

Theorem 1 (Pauly's Representation Theorem [11]). *A coalitional effectivity function E α-corresponds to a nonempty strategic game if and only if E is playable.*

Thus, the theorem states that every playable effectivity function is equal to the α-effectivity function of some game (Pauly calls this equivalence relation *α-correspondence*), and that each game has an α-effectivity function that is playable. While the latter claim is easily true, the former one turns out incorrect.

Proposition 3. *There is a playable effectivity function E for which $E \neq E^{\alpha}_{G}$ for all strategic games G.*

Proof. Consider a coalitional frame with a single player a that has the set of natural numbers $\mathbb{N}$ as the domain (i.e., $N = \{a\}, S = \mathbb{N}$), and the effectivity defined as follows:

- $E(\{a\}) = \{X \subseteq \mathbb{N} \mid X \text{ is infinite}\}$;
- $E(\emptyset) = \{X \subseteq \mathbb{N} \mid \overline{X} \text{ is finite}\}$.

In other words, the grand coalition $\{a\}$ is effective for all infinite subsets of the natural numbers, while the empty coalition can enforce all the cofinite subsets.

We claim that E is playable and that it does not correspond to any strategic game. Let us first verify the playability conditions. Outcome monotonicity, N-maximality, liveness and safety are straightforward to check. For superadditivity, notice that we have only two cases to verify:

1. $C = \{a\}, D = \emptyset$. Superadditivity holds here because intersection of an infinite and a cofinite set is infinite.
2. $C = \emptyset, D = \emptyset$. Superadditivity in this case holds because intersection of two cofinite sets is cofinite.

On the other hand, $E^{nc}(\emptyset) = \emptyset$ because there are no minimal cofinite sets. This implies, by Proposition 2, that $E \neq E^{\alpha}_{G}$ for all strategic games G.

3.2 Tracing the Problem

When showing that playable effectivity functions exactly correspond to strategic games, the difficult direction is from effectivity functions to games. Below, we summarize the relevant part of the proof of Theorem 2.27 from [11], and show where it goes wrong. We outline the construction of a strategic game $\mathcal{G}$ given an effectivity function E (Steps 1–4); then, the argument supposed to show that E α-corresponds to $\mathcal{G}$ (Steps 5–6).

Step 1: the players and the domain remain the same. The game $\mathcal{G} = (N, S, \Sigma_i, o)$ inherits the set of outcomes and the set of players as occurring in the effectivity function E.

Step 2: coalitions choose a set from their effectivity function. Now, a family of functions is defined:

$$F_i = \{f_i : \mathcal{C}_i \to 2^S \mid \text{ for all } C \text{ we have that } \; f_i(C) \in E(C)\}$$

where $\mathcal{C}_i = \{C \subseteq N \mid i \in C\}$. Each function f_i assigns choices to all coalitions of which i is a member. F_i simply collects all such assignments.

Step 3: coalitions are partitioned according to their choices. Let $f = (f_i)_{i\in N}, f_i \in F_i$, be a tuple of such assignments, one per player. The next step is to define the set $P_\infty(f)$ which results from iterative partitioning of the set of players in the coarsest possible way such that players in the same partition are assigned same coalitional choices:

$$\begin{aligned}
P_0(f) &= \langle N \rangle \\
P_1(f) &= P(f, N) = \langle C_1^1, \ldots, C_{k_1}^1 \rangle \\
P_2(f) &= \langle P(f, C_1^1), \ldots, P(f, C_{k_1}^1) \rangle = \langle C_2^2, \ldots, C_{k_2}^2 \rangle \\
&\;\;\ldots \\
P_\infty(f) &= P_r(f) \text{ such that } P_i(f) = P_{i+1}(f) \text{ for all } i \geq r,
\end{aligned}$$

where each $P(f, C)$ returns the coarsest partitioning $\langle C_1, \ldots, C_m \rangle$ of coalition C such that for all $l \leq m$ and for all $i, j \in C_l$ it holds that $f_i(C) = f_j(C)$.

That is, a subset of C is part of the partition $P(f, C)$ iff its members agree modulo f.

Step 4: an outcome is chosen in the intersection of coalitional choices. Now, strategy sets for each player and the outcome function are defined as follows. Each player in N is given a set of strategies of the form (f_i, t_i, h_i) where $f_i \in F_i$ is an assignment of coalitional choices for player i (see point (ii)), t_i is a player (possibly different from i), and $h_i : 2^S \setminus \emptyset \to S$ is a selector function that selects an arbitrary element from each nonempty subset of S.

The outcome of strategy σ_N is now defined as:

$$o(\sigma_N) = h_{i_0}\big(\bigcap_{l=1}^{k} f(C_l)\big),$$

where i_0 is a uniquely chosen player, h_{i_0} is the outcome selector from i_0's strategy, and C_l are partitions from $P_\infty(f)$.

This concludes the construction of a game $\mathcal{G}$ which should α-correspond to the effectivity function E. Steps 5–6 are supposed to prove that $E = E^\alpha_{\mathcal{G}}$.

Step 5: choices are not removed by the construction. First, an attempt to prove $E(C) \subseteq E^{\alpha}_{\mathcal{G}}(C)$ for arbitrary coalition C is presented:

> For the inclusion from left to right, assume that $X \in E(C)$. Choose any C-strategy $\sigma_C = (f_i, t_i, h_i)_{i\in C}$ such that for all $i \in C$ and for all $C' \supseteq C$ we have $f_i(C') = X$.(*) By coalition monotonicity, such f_i exists.(**) Take now any $\overline{C}$-strategy, $\sigma_{\overline{C}} = (f_i, t_i, h_i)_{i\in\overline{C}}$. We need to show that $o(\sigma_C, \sigma_{\overline{C}}) \in X$. To see this, note that C must be a subset of one of the partitions C_l in $P_\infty(f)$. Hence, $o(\sigma_N) = h_{i_0}(G(f)) = h_{i_0} \bigcap_{l=1}^{k} f(C_l) \in X$. [11, p.29]

The deduction of the last sentence is where the proof goes wrong. The problem is that, for $C = \emptyset$ the only available strategy is the empty strategy $\sigma_\emptyset$ which vacuously satisfies condition (*). And, for any agent i, a choice assignment f_i satisfying the condition must exist. However, *there is no guarantee that any i will indeed choose f_i in its strategy* since the coalition C for which we can fix its strategy does not include any players. In consequence, one cannot deduce that $h_{i_0}\big(\bigcap_{l=1}^{k} f(C_l)\big) \in X$; this could be only concluded if the intersection contains at least one player whose choice $f_i(C_l)$ is X (or a subset of X).

To see this more clearly, let us consider the effectivity function in the counterexample from Section 3.1. Note that $\sigma_{\overline{C}} = \sigma_{\{a\}} = (f_a, a, h_a)$ such that $f_a(\{a\}) \in E(\{a\})$. Let us now take $X = \mathbb{N} \setminus \{1\}$, $f_a(\{a\}) = \mathbb{N}$, and $h_a(\mathbb{N}) = 1$. Now, $o(\sigma_N) = o(\sigma_{\{a\}}) = 1 \notin X$, which invalidates the argument from [11] quoted above.

Another case where the reasoning fails is $C = N$. Consider a state space S with $\{x\} \notin S$, and an effectivity function E such that $\{x\} \notin E(N)$. Now, let strategy profile σ_N consist of $\sigma_i = (f_i, t_i, h_i)$ where everybody assumes choosing the whole state space in all circumstances (i.e., $f_i(C) = S$ for all i and C) and applies the same selector h_i such that $h_i(S) = x$. Now we get that $o(\sigma_N) = x$, so $\{x\} \in E^{\alpha}_{\mathcal{G}}(N)$, and hence $E(N) \neq E^{\alpha}_{\mathcal{G}}(N)$.

Step 6: choices are not added by the construction. The proof of the other direction ($E^{\alpha}_{\mathcal{G}}(C) \subseteq E(C)$) fails too, because in order to establish the inclusion for $C = N$, it is reduced to inclusion in step 5 for $C = \emptyset$, and we have just shown that it does not necessarily hold.

This concludes our analysis of the proof of Pauly's representation theorem in [11]. The construction of the strategic game corresponding to a given effectivity function fails because the game might endow the empty coalition and the grand coalition of players with inappropriate powers. We consider this analysis important for two reasons. First, we have identified precisely what was wrong with the construction of the proof. Second, we will reuse the sound parts of the original construction when proving a revised version of the correspondence in Section 4.2 and to obtain some additional results in Section 4.4.

3.3 A Look at Consequences

We have observed that playability conditions are not sufficient to characterize strategic games. This raises some relevant issues for studying game models and logics for reasoning about games:

1. What are the "truly playable" effectivity functions that really correspond to strategic games? How can we characterize these functions in an abstract way? This issue is discussed in Section 4.
2. Conversely, how can we generalize the counterexample from Section 3.1 in order to characterize the class of playable but not truly playable effectivity functions? Section 4.3 deals with this question.
3. Is it possible to "reconstruct" playable effectivity functions into truly playable ones, without modifying the coalitional abilities much? We propose such a procedure in Section 4.4, and show that it preserves the powers of most coalitions.
4. Finally, what is the impact on strategic logics, Coalition Logic in particular? Does changing from playable to truly playable models yield a different notion of validity or semantic consequence? Are axiomatizations from [11,7] sound and complete for truly playable models? What logical constructs are needed to distinguish between playable and truly playable structures? These questions will be treated in a subsequent work.

4 Truly Playable Effectivity Functions

In this section we introduce an additional constraint on playable effectivity functions, that will enable us to prove the correspondence with strategic games.

4.1 Characterizing True Playability

The subset of playable effectivity functions that α-correspond to strategic games can be characterized in terms of the nonmonotonic core of the empty coalition. Alternatively, it can be characterized in terms of effectivity of the grand coalition of all the agents.

Definition 7. *An effectivity function E is* truly playable *iff it is playable and $E(\emptyset)$ has a complete nonmonotonic core.*

We will formally prove the correspondence between strategic games and truly playable functions in Section 4.2.

Several equivalent characterizations of truly playable effectivity functions are given in Proposition 4. For one of them, we will need the additional notion of a *crown*. Intuitively, an effectivity function is a crown if every choice of the agents in the grand coalition includes at least one state that the grand coalition can enforce precisely. Formally, this means that N can only force some singleton sets and all their supersets. By forming an anti-chain of singletons and drawing the cones we obtain a "crown" as in Figure 1, hence the term.

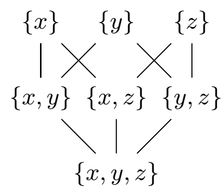

Fig. 1. A crown

Definition 8. *An effectivity function* $E : 2^N \to 2^{2^S}$ *is a* crown *if and only if* $X \in E(N)$ *implies* $\{x\} \in E(N)$ *for some* $x \in X$.

Proposition 4. *The following are equivalent for every playable effectivity function* $E : 2^N \to 2^{2^S}$.

1. *E is truly playable.*
2. $E(\emptyset)$ *has a non-empty nonmonotonic core.*
3. $E^{nc}(\emptyset)$ *is a singleton and* $E(\emptyset)$ *is a principal filter, generated by* $E^{nc}(\emptyset)$.
4. *E is a crown.*

Proof.
(1) $\Rightarrow$ (2): immediate, by safety.
(2) $\Rightarrow$ (3): Let $Z \in E^{nc}(\emptyset)$ and let $X \in E(\emptyset)$. Then, by superadditivity, $Z \cap X \in E(\emptyset)$, and $Z \cap X \subseteq Z$, hence $Z \cap X = Z$ by definition of $E^{nc}(\emptyset)$. Thus, $Z \subseteq X$. So, $E(\emptyset)$ is the principal filter generated by Z, hence $E^{nc}(\emptyset) = \{Z\}$.
(3) $\Rightarrow$ (1): immediate from the definitions.
(3) $\Rightarrow$ (4): Let $E^{nc}(\emptyset) = \{Z\}$ and suppose $\{x\} \notin E(N)$ for all $x \in X$ for some $X \subseteq S$. Then, by N-maximality, $S \setminus \{x\} \in E(\emptyset)$, i.e. $Z \subseteq S \setminus \{x\}$ for every $x \in X$. Then $Z \subseteq S \setminus X$, hence $S \setminus X \in E(\emptyset)$. Therefore, $X \notin E(N)$ by superaditivity and liveness. By contraposition, E is a crown.
(4) $\Rightarrow$ (3): Let $Z = \{z \mid \{z\} \in E(N)\}$ and let $X \in E(\emptyset)$. Take any $z \in Z$, which is nonempty by liveness and the fact that E is a crown. By superadditivity we obtain that $\{z\} \cap X \in E(\emptyset)$, hence $z \in X$ by liveness. Thus, $Z \subseteq X$. Moreover, $Z \in E(\emptyset)$, for else $S \setminus Z \in E(N)$ by N-maximality, hence $\{x\} \in E(N)$ for some $x \in S \setminus Z$, which contradicts the definition of Z. Therefore, $E(\emptyset)$ is the principal filter generated by Z, hence $E^{nc}(\emptyset) = \{Z\}$.

We also observe that on finite domains playability and true playability coincide.

Corollary 1. *Every playable effectivity function* $E : 2^N \to 2^{2^S}$ *on a finite domain* S *is truly playable.*

Proof. Straightforward, by Proposition 4.3 and the fact that every filter on a finite set is principal.

Finally, note that in a truly playable function the nonmonotonic core for coalitions different from $\emptyset, N$ does not have to be complete, and neither does it have to be nonempty, as Example 1 demonstrates.

Example 1. Consider the following effectivity function for $N = \{a, b\}, S = \mathbb{N}$:

- $E(\emptyset) = \{\mathbb{N}\}$,
- $E(\{a\}) = E(\{b\}) =$ all cofinite subsets of $\mathbb{N}$,
- $E(\{a, b\}) = 2^{\mathbb{N}} \setminus \emptyset$.

It is easy to see that E is truly playable, but the nonmonotonic core of $E(\{a\})$ is empty, and hence also not complete.

4.2 Truly Playable Functions Correspond to Strategic Games

The proof of Theorem 2.27 from [11] fails when we consider the effectivity function of the empty coalition (resp. of the grand coalition). However the proof *is* correct for the other cases. We will now show that the additional condition of true playability yields correctness of the original construction from [11].

Theorem 2. *A coalitional effectivity function E α-corresponds to a strategic game if and only if E is truly playable.*

Proof. By Propositions 2 and 4, for any strategic game $\mathcal{G}$ its α-effectivity function $E^{\alpha}_{\mathcal{G}}$ is truly playable.

For the other direction, given a truly playable effectivity function E, we slightly change Pauly's procedure outlined in Section 3.2 (Steps 1–4). We impose an additional constraint on players' strategies $\sigma_i = (f_i, t_i, h_i)$, namely, we require that $h_i(X) = x$ for some $\{x\} \in E(N)$. In other words, the selector functions only select the "jewels" in the crown. Note that for $C \notin \{\emptyset, N\}$ the new procedure yields game $\mathcal{G}'$ with exactly the same $E^{\alpha}(C)$ as the original construction $\mathcal{G}$ from [11] because:

- We do not add any new choice sets to $E^{\alpha}_{\mathcal{G}}(C)$. Indeed, that could only happen because the selectors chosen by agents outside C are restricted to $\{x \mid \{x\} \in E(N)\}$, and hence we can have that $X \cap \{x \mid \{x\} \in E(N)\} \in E^{\alpha}_{\mathcal{G}'}(C)$ in the new construction for some $X \in E^{\alpha}_{\mathcal{G}}(C)$ from the previous construction. However, by true playability of E and Proposition 4 we have that $\{x \mid \{x\} \in E(N)\} \in E(\emptyset)$, and thus by superadditivity all the states $y \notin \{x \mid \{x\} \in E(N)\}$ can be removed from C's strategies that yielded X in $\mathcal{G}$. But then these states will also be removed from the intersection $\bigcap_{l=1}^{k} f(C_l)$, and so $X \cap \{x \mid \{x\} \in E(N)\} \in E^{\alpha}_{\mathcal{G}}(C)$ already in the previous construction.
- We do not remove any choice sets from $E^{\alpha}_{\mathcal{G}}(C)$. Indeed, that could only happen because of removing an $X \in E^{\alpha}_{\mathcal{G}}(C)$ which contains "superfluous" elements and replacing it with $X \cap \{x \mid \{x\} \in E(N)\}$. But then, X must also be in $E^{\alpha}_{\mathcal{G}'}(C)$ because $E^{\alpha}_{\mathcal{G}'}(C)$ is closed under supersets.

It remains now to show that the procedure constructs a strategic game $\mathcal{G}$ such that $E(C) = E^{\alpha}_{\mathcal{G}}(C)$ for all $C \subseteq N$, that is, to show that steps 5 and 6 work well in case of truly playable structures.

Ad. Step 5. We show that $E(C) \subseteq E^{\alpha}_{\mathcal{G}}(C)$ for $C = \emptyset$ and $C = N$, the only cases in which the original proof failed for playable structures.

Assume that $X \in E(\emptyset)$. We need to prove that $X \in E^{\alpha}_{\mathcal{G}}(\emptyset)$. By true playability we know that there exists $Y \in E^{nc}(\emptyset)$ such that $Y \subseteq X$. By Proposition 4, $E^{nc}(\emptyset) = \{Y\}$ and $E(\emptyset) = \{Z \mid Y \subseteq Z\}$. We will show now that $Y = \{x \mid \{x\} \in E(N)\}$ (*). First, suppose that $x \in Y$ and $\{x\} \notin E(N)$, then by N-maximality $S \setminus \{x\} \in E(\emptyset)$, a contradiction. Second, let $\{x\} \in E(N)$ and $x \notin Y$, then by superadditivity $\emptyset \in E(N)$ which contradicts liveness.

Now, consider any strategy profile σ_N. We have $o(\sigma_N) = h_{i_0}\big(\bigcap_{l=1}^{k} f(C_l)\big) \in Y$ because every h_i returns only elements in Y by construction.

For the case $C = N$, assume that $X \in E(N)$. We need to prove that $X \in E^{\alpha}_{\mathcal{G}}(N)$. By true playability we have that there exists $x \in X$ such that $\{x\} \in E(N)$. Now, let σ_N consist of strategies $\sigma_i = (f_i, t_i, h_i)$ such that $f_i(N) = x$ for every i. It is easy to see that $o(\sigma_N) = x$, and hence $\{x\} \in E^{\alpha}_{\mathcal{G}}(N)$. Thus, $X \in E^{\alpha}_{\mathcal{G}}(N)$ because $E^{\alpha}_{\mathcal{G}}(N)$ is closed under supersets.

Ad. Step 6. Dually to Step 5, we show that $E^{\alpha}_{\mathcal{G}}(C) \subseteq E(C)$. That is, assuming $X \notin E(C)$ we show that $X \notin E^{\alpha}_{\mathcal{G}}(N)$. We do it by a slight modification of the original proof from [11].

Suppose first that $C = N$. Then, $\overline{X} \in E(\emptyset)$ by N-maximality, and by Step 5 we have $\overline{X} \in E^{\alpha}_{\mathcal{G}}(\emptyset)$. Since $E^{\alpha}_{\mathcal{G}}$ is truly playable, we have also that $X \notin E^{\alpha}_{\mathcal{G}}(N)$.

Assume now that $C \neq N$, and let $j_0 \in \overline{C}$. Let σ_C be any strategy for coalition C. We must show that there is a strategy $\sigma_{\overline{C}}$ such that $o(\sigma_C, \sigma_{\overline{C}}) \notin X$. To show this, we take $\sigma_{\overline{C}} = (f_i, t_i, h_i)_{i \in \overline{C}}$ such that for all $C' \supseteq \overline{C}$ and for all $i \in \overline{C}$ we have $f_i(C') = S$. We also choose t_{j_0} such that $((t_1 + \ldots + t_n) \bmod n) + 1 = j_0$. Note that $\overline{C}$ must be an element of one of the partitions C_l in $P_\infty(f)$, say C_{l_0}. Moreover, there must be a partitioning $\langle C_1, \ldots, C_k \rangle$ of $N \setminus C_{l_0}$ such that $G(f) = f(C_{l_0}) \cap \bigcap_{l=1}^{k} f(C_l) = \bigcap_{l=1}^{k} f(C_l)$. Since $f(C_l) \in E(C_l)$ we get that $G(f) \in N \setminus C_{l_0}$ by superadditivity. By coalition-monotonicity and the fact that $N \setminus C_{l_0} \subseteq C$, we also have $G(f) \in E(C)$. Finally, by (*) and superadditivity we obtain $G(f) \cap \{x \mid \{x\} \in E(N)\} \in E(C)$.

Since $X \notin E(C)$ and $E(C)$ is closed under supersets, it must hold that $G(f) \cap \{x \mid \{x\} \in E(N)\} \not\subseteq X$. Thus, there is some $s_0 \in S$ such that: $s_0 \in G(f)$, $\{s_0\} \in E(N)$, and $s_0 \notin X$. Now we fix h_{j_0} so that $h_{j_0}(G(f)) = s_0$. Then, $o(\sigma_C, \sigma_{\overline{C}}) = h_{j_0}(G(f)) = s_0 \notin X$ which concludes the proof.

4.3 Non-Truly Playable Structures

In this section we focus on the class of playable but not *truly playable* effectivity functions, hereafter called "non-truly playable". From Proposition 4 we know that a playable effectivity function E is truly playable if and only if the filter $E(\emptyset)$ is principal and generated by $E^{nc}(\emptyset)$. Hence, playability and true playability coincide on finite domains. There exist, however, non-truly playable effectivity functions on infinite domains, and we have already discussed an example of such a function in Section 3.1.

Non-truly playable effectivity functions have a simple abstract characterization, following from Proposition 4:

Proposition 5. *Effectivity function $E : 2^N \to 2^{2^S}$ is non-truly playable if and only if it is playable and $E(\emptyset)$ is a non-principal filter.*

To see a more generic class of examples, consider an infinite domain S, and let $\mathcal{F}$ be any non-principal filter on S. Then we define an effectivity function $\mathcal{E}_\mathcal{F}$ on S as follows.

- $E_\mathcal{F}(\emptyset) = \mathcal{F}$.
- $E_\mathcal{F}(N) = \{X \mid \overline{X} \notin \mathcal{F}\}$
- For each C with $\emptyset \subsetneq C \subsetneq N$ take $E_\mathcal{F}(C)$ to be any set of sets such that $E_\mathcal{F}(\emptyset) \subseteq E_\mathcal{F}(C) \subseteq E_\mathcal{F}(N)$ that is closed under outcome monotonicity and that are pairwise closed under regularity and superadditivity.

Proposition 6. *$E_\mathcal{F}$ is playable but not truly playable.*

Proof. That $E_\mathcal{F}$ is not truly playable follows by the fact that $E_\mathcal{F}(\emptyset)$ is a non-principal filter. In order to check that $E_\mathcal{F}$ is playable we only need to check superaddivity for $\emptyset$ and N, as the other conditions follow by construction.

Assume $X \in E_\mathcal{F}(\emptyset)$ and $Y \in E_\mathcal{F}(N)$. We have to prove that $X \cap Y \in E_\mathcal{F}(N)$. Suppose that $X \cap Y \notin E_\mathcal{F}(N)$. But then, by definition of $E_\mathcal{F}(N)$ we have that $\overline{X \cap Y} \in E_\mathcal{F}(\emptyset)$. By de Morgan's law we have that $\overline{X} \cup \overline{Y} \in E_\mathcal{F}(\emptyset)$. But as $Y \in E_\mathcal{F}(N)$ we know that $\overline{Y} \notin E_\mathcal{F}(\emptyset)$. However $E_\mathcal{F}(\emptyset)$ is a filter so $X \cap (\overline{X} \cup \overline{Y}) \in E_\mathcal{F}(\emptyset)$. From this follows that $\overline{Y} \in E_\mathcal{F}(\emptyset)$. Contradiction.

Here are some examples of non-principal filters on $\mathbb{N}$:

- For any $k \in \mathbb{N}$ let $E_k(\emptyset) = \{X \subseteq \mathbb{N} \mid X \text{ is cofinite in } \mathbb{N} \text{ and } k \in X\}$.
- More generally, for any $K \subseteq \mathbb{N}$ which is not cofinite in $\mathbb{N}$ let $E_K(\emptyset) = \{X \subseteq \mathbb{N} \mid X \text{ is cofinite in } \mathbb{N} \text{ and } K \subseteq X\}$.

In the case of single player, $N = \{a\}$, the construction above immediately extends these filters to non-truly playable effectivity functions on $\mathbb{N}$:

- $E_k(\{a\}) = \{X \subseteq \mathbb{N} \mid X \text{ is infinite or } k \in X\}$,
- $E_K(\{a\}) = \{X \subseteq \mathbb{N} \mid X \text{ is infinite or } K \cap X \neq \emptyset\}$,

We know from the proof of proposition 4 that for all playable E, if the nonmonotonic core of $E(\emptyset)$ is nonempty, then it must be complete. The above examples show that this does not have to be the case for other coalitions. For instance, observe that the nonmonotonic core of $E_1(\{a\})$ is $E_1^{nc}(\{a\}) = \{\{1\}\}$; still, $\{1\} \nsubseteq Even \in E_1(\{a\})$. Similarly, we can show that even infinite nonmonotonic core does not guarantee its completeness. Indeed, it is easy to see that $E_{Even}^{nc}(\{a\}) = \{\{k\} \mid k \in Even\}$ while we also have $Odd \in E_{Even}(\{a\})$. Thus, a playable effectivity function can have its nonmonotonic core for the grand coalition nonempty and consisting entirely of singletons, and yet not be a crown – hence remaining non-truly playable.

4.4 From Playable to Truly Playable Effectivity Functions

In this section we show that one can reconstruct a non-truly playable effectivity function into a truly playable one with "minimal" modifications. To do so, we interpret choices of the grand coalition containing multiple outcome states as ones that involve inherent nondeterminism. That is, we interpret $\{x_1, x_2, \dots\} \in E(N)$ as a choice where no agent has control over which state out of $x_1, x_2, \dots$ will become the outcome; as a consequence any of these states can possibly be encountered in the next moment. Under such assumption, it is possible to recover true playability by a simple extension of Pauly's procedure. The extension consists in adding an extra player $\mathbf{d}$ (the "decider") who settles the nondeterminism and decides which of $x_1, x_2, \dots$ is going to become the next state.

Proposition 7. *Let $E : 2^N \to 2^{2^S}$ be a playable effectivity function. There exists a truly playable effectivity function $E' : 2^{N\cup\{\mathbf{d}\}} \to 2^{2^S}$ with additional player $\mathbf{d} \notin N$, such that:*

- *$E'(C) = E(C)$ for every $C \subseteq N, C \neq \emptyset$,*
- *$E'(\emptyset) = \{S\}$, and*
- *$E'(N \cup \{\mathbf{d}\}) = 2^S \setminus \{\emptyset\}$.*

Proof. Given a playable E, we construct a strategic game whose α-effectivity function satisfies the properties above. Then, existence of a truly playable effectivity function follows immediately. The idea is to take the construction from the proof of Theorem 2.27 in [11] and reassign selection of the outcome state to the additional player $\mathbf{d}$.

Let $h : 2^S\setminus\{\emptyset\} \to S$ be any selector function that selects an arbitrary element from the argument set. In our case, h will designate the "default" outcome for each subset of S. Now, the game $\mathcal{G}$ is constructed as follows:

- $N' = N \cup \{\mathbf{d}\}$;
- Strategies of player $i \neq \mathbf{d}$ are simply the player's assignments of coalitional choice, i.e., $\Sigma_i = F_i$;
- Strategies of the decider are state selections: $\Sigma_{\mathbf{d}} = S$;
- The transition function is based on the same partitioning of N as before, that yields $\langle C_1, \dots, C_k\rangle$. Then, the game proceeds to the state selected by the decider if his choice is consistent with the choices of the others, otherwise it proceeds to the appropriate "default" outcome:
$$o(\sigma_N, s) = \begin{cases} s & \text{if } s \in \bigcap_{i=1}^k f(C_l) \\ h(\bigcap_{i=1}^k f(C_l)) & \text{else.} \end{cases}$$

Now, it is easy to see that for every $\emptyset \subsetneq C \subsetneq N$ indeed $E^\alpha_{\mathcal{G}}(C) = E(C)$ because that was the case in the original construction, and the only difference now is that $\mathbf{d}$ "took over" the selection of a state in $\bigcap_{i=1}^k f(C_l)$ from a collective choice of N. For $C = N$, we also have $E^\alpha_{\mathcal{G}}(N) = E(N)$ since for every σ_N we get by superadditivity that $\bigcap_{i=1}^k f(C_l) \in E(N)$, and every state from the intersection can be potentially selected by $\mathbf{d}$. Moreover, $\{s\} \in E^\alpha_{\mathcal{G}}(N \cup \{\mathbf{d}\})$ for every $s \in S$ because $\{s\}$ is enforced by $\sigma_{N \cup \{\mathbf{d}\}} = \langle f_1, \ldots, f_{|N|}, s \rangle$ such that $f_i = S$ for all $i \in N$. Thus, by outcome monotonicity, $E^{\alpha}_{\mathcal{G}}(N \cup \{\mathbf{d}\}) = 2^S \setminus \{\emptyset\}$. Finally, by true playability of $E^{\alpha}_{\mathcal{G}}$, we have $E^{\alpha}_{\mathcal{G}}(\emptyset) = \{\{s \mid \{s\} \in E^{\alpha}_{\mathcal{G}}(N \cup \{\mathbf{d}\})\}\} = \{S\}$. We observe additionally that $E^{\alpha}_{\mathcal{G}}(\mathbf{d}) = \{\{s\} \cup \{h(X) \mid X \in E^{\alpha}_{G}(\mathbf{d}) \text{ and } s \notin X\} \mid s \in S\}$.

5 Conclusions

In this paper, we revisit the correspondence between two classes of abstract game forms: strategic games from noncooperative game theory on one hand, and effectivity functions from cooperative game theory on the other. Our contribution is twofold. First, we correct a well-known result from [11] relating strategic games and playable effectivity functions. We show that strategic games do not correspond to all playable functions, but to a strict subset of the class, which we call *truly playable effectivity functions*. Second, we provide several abstract characterizations of truly playable functions, most notably in terms of principal filters. We also show that the remaining playable effectivity functions (that we call non-truly playable) are induced by non-principal filters, and hence only scenarios with infinitely many possible outcomes can fall in that class.

The importance of our work is mainly theoretical. Essentially, it implies that all the claims that have been proved using Pauly's correspondence between playable effectivity functions and games should be revisited and possibly reinterpreted in the light of the results presented here. Example of such issues include: axiomatization for Coalition Logic in the class of multi-player game models, axiomatization of ATL in coalitional effectivity models, and the respective finite model properties.[7] In practical terms, this also means that, whenever a decision procedure is built on those theoretical results, the designer should be aware of the correct correspondence between the two classes of game models, which is especially relevant for satisfiability-checking algorithms. Tableaus for extensions of Coalition Logic, like the one for a combination of CL and description logic $\mathcal{ALC}$ from [8], are good examples of such procedures.

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[7] We note that these particular properties are not affected by the difference between playable and truly playable models, which has been established in a follow-up to this paper (under submission).

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