Some Complexity Results for Distance-Based Judgment Aggregation

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Abstract. Judgment aggregation is a social choice method for aggregating information on logically related issues. In distance-based judgment aggregation, the collective opinion is sought as a compromise between information sources that satisfies several structural properties. It would seem that the standard conditions on distance and aggregation functions are strong enough to guarantee existence of feasible procedures. In this paper, we show that it is not the case, though the problem becomes easier under some additional assumptions.

1 Introduction

It is often convenient to ascribe information-related stances (such as judgments, opinions, beliefs, etc.) to collectives of agents. The need for modeling collective opinions can be either external or internal. External agents may ascribe opinions to institutions and groups in order to simplify their model of the world and reason about it. Agents inside the group may need to reach consensus about issues of interest, and in particular to obtain collective decisions that will lead to consistent collective action. In this paper, we focus on one of the formal frameworks that try to explain how collective judgments are formed from individual stances, namely *judgment aggregation theory* [33].

Several formal theories within artificial intelligence have tried to explain how collective judgments arise from individual judgments. Epistemic logic [41] proposes to aggregate agents' views by aggregating the underlying models, i.e., indistinguishability relations over different valuations of atomic sentences. By different operations on epistemic relations we obtain different notions of group knowledge: mutual knowledge, common knowledge, distributed knowledge etc. [24]. On the other hand, Dempster-Shafer theory [9, 39] shows how probabilistic beliefs can be merged into a single collective belief. However, epistemic logic requires *complete* individual views, that is, everybody's opinions about *every* conceivable state of the world must be given as input. Dempster rule of combination admits incomplete models, but may yield logically inconsistent judgments, i.e., ones that violate logical interdependencies between issues (even if the input consists of consistent individual judgments). Thus, both theories make assumptions that turn out too strong for most cases of practical reasoning.

Another formal framework is provided by social choice theory. It develops and analyses (on a more abstract level) methods for reaching group decisions through aggregating individual stances. For instance, voting rules used in political elections are social choice methods. Deriving collective opinions from a partial representation of individual judgments on a set of mutually dependent issues may be obtained by a method of similar kind. More precisely, the problem of aggregating binary valuations assigned to each element of a set of logically related elements into a consistent set of valuations is studied by *judgment aggregation theory* [33].

Distance-based judgment aggregation [34, 38] comprises the largest class of judgment aggregation rules. Inspired by belief merging rules, the idea is to define the collective opinion as a "well-behaved compromise" among the individual opinions of the group members. That is, we assume that each member is willing to give up some of their judgments as long as the resulting aggregate judgment set does not stray too far from their individual ones. Distance-based aggregation rules are supposed to satisfy a number of structural constraints (see Section 3 for details) to make sure their output is indeed "well-behaved" in the mathematical sense. It seems – at least at the first glance – that the constraints should lead to computationally well-behaved procedures. In this paper, we show that it is not necessarily true.

Why is computational complexity important for aggregating judgments? Essentially, judgment aggregation provides an intuitive representation for decision problems in collective reasoning. In this context, its computational complexity is crucial. More specifically, judgment aggregation rules are *procedures* that determine the collective view based on individual inputs. The procedure is only useful if it returns the result in reasonable time. This is perhaps not that crucial in case of a jury consisting of 10 members and deliberating over 5 connected issues. Consider, however, a team of 100 robots reaching a collective decision based on the input from 400 sensors with different (but overlapping) range, or 500 stakeholders trying to agree on a company agenda. Scalability of the procedure becomes clearly of utmost importance.

The paper is structured as follows. In Section 3 we give the necessary definitions for a judgment aggregation problem and distance-based aggregation rules. In Section 4 we consider the problem of *verifying* whether a particular set of judgments can be selected as collective, for a collection of individual judgments, by a distance-based judgment aggregation rule. We also extend our results to aggregation of opinions expressed in multi-valued logics. In Section 5 we present our conclusions.

2 Related Work

Our paper fits in the area of *computational social choice* [6] which comprises interdisciplinary study of how computational analysis can be used to make social choice methods operational. Many contributions, e.g. [1, 7, 8, 23], have been made towards understanding the complexity-theoretic properties of voting rules. In comparison, complexity-theoretic properties of judgment aggregation are not so well explored.

Complexity analysis of distance-based judgment aggregation has, to the best of our knowledge, been focused on analysis of *particular* aggregation rules. The following papers have addressed the complexity of judgment aggregation procedures: [2–4, 20–22]. Out of these works, [2–4] focus on the complexity and parameterized complexity of decision problems related to control and bribery in *quota judgment aggregation rules* which generalize issue-by-issue majority judgment aggregation [12].. [20] studies the complexity of deciding whether the so called *premise-based judgment aggregation rule* (a special type of quota rule [14]) can be applied to a given judgment aggregation problem. [21] investigates the complexity of deciding if a given judgment is selected by two alternative rules: the quota rule and the most "typical" distance-based aggregation rule that uses the sum of Hamming distances to compute the "score" for each judgment set. The work [22] gathers and deepens the results of [21] and [20].

Summarizing, complexity-theoretic properties of judgment aggregation are only partially explored. This applies especially to distance-based judgment aggregation, where the only existing studies refer to particular "natural" judgment aggregators, mainly based on the sum of Hamming distances. In contrast, *we take the opposite approach and explore the bounds of the framework.* That is, we investigate what kind of complexity can be expected from *arbitrary* distance-based aggregation rules.

Besides papers that explicitly refer to the complexity of judgment aggregation procedures, we must also mention works on complexity of distance-based belief merging [28] and especially distance-based preference aggregation [1, 18, 19].

Relation to research on preference aggregation. The research on complexity in preference aggregation connects to the research on complexity of distance-based judgment aggregation through the result of [19] where it was shown that the Kemeny rule of voting coincides, for strict preference orders, with judgment aggregation based on the sum of Hamming distances. The complexity of the winner determination problem for the Kemeny preference aggregation rule, has been studied in [1] and [27], the latter proving it to be Θ_2^P complete.

It has been demonstrated that judgment aggregation is related to preference aggregation by showing when a preference aggregation problem can be translated to a judgment aggregation problem and vice versa [11, 26, 32]. Studies that formally establish the relationship between judgment aggregation rules and voting rules (or preference aggregation rules) on the general level are only now starting to be pursued [30], despite a number of discussions on the topic [13, 29, 35]. The general relationship between the complexity properties of preference aggregation rules and the complexity properties of the judgment aggregation rules that generalize them, is the next research step. We present some preliminary intuitions.

A judgment set can be used to characterize a strict preference order [11] by using a formula φ_b^a to represent that alternative a is preferred to alternative b. In complexity of preference aggregation, one is typically interested in the winner determination problem, that is, the problem of deciding whether an alternative is top ranked in at least one of preference orders produced by the preference aggregation rule. Considering only aggregation of strict preferences and following the analogy that a preference order is a judgment set, an alternative in preference aggregation corresponds to a judgment, and the winner determination problem can be interpreted as that of determining whether a particular judgment φ_b^a is a part of the collective judgment set produced by the judgment aggregation rule. The difficulty lies in the fact that a judgment aggregation rule can produce multiple collective judgment sets, some containing φ_b^a and some not. Therefore two different meaningful questions can be studied: (1) whether a judgment set as a whole can be selected as the collective opinion, corresponding to our definition of the winner set verification problem in Section 4, or (2) whether a given judgment is an element of all collective opinions, as in [22]. For preference aggregation, (1) corresponds to checking if a preference order is selected by the preference aggregation rule, while (2) is about determining whether a given alternative is highest ranked in all selected preference orders. Both decision problems are at least as hard as the problem of deciding whether an alternative is a winner of the election. Therefore we can expect decision problems in judgment aggregation to be no easier than their counterparts in preference aggregation.

Relations with belief merging Judgment aggregation has been related with belief merging [38]. Both theories are concerned with aggregating sets of formulas, however the demands on the aggregation results are different. In judgment aggregation, the agenda limits the scope of issues whose consistent aggregated truth-value is of interest. In belief merging, the agenda does not exist. The interest focus in merging is on determining, not sets of formulas like in judgment aggregation, but the (closed under deduction) set of formulas that are logically entailed by the sets of formulas being merged. The computational complexity analysis in belief merging is concerned with the decision problem of whether one particular formula (judgment) is entailed by a given collection of belief sets [28].

3 Preliminaries

We first give a brief exposition of judgment aggregation and distance-based judgment aggregation.

3.1 Judgment Aggregation

Let $\mathcal{L}$ be a propositional language over a countable set of atomic propositions $Prop$, and let T be a set of truth values such that $1 \in T$ (i.e., it includes the value for "absolutely true"). Any $v : Prop \to T$ is called a propositional valuation; we denote the set of valuations as PV. Each $v \in PV$ extends to a valuation $val_v : \mathcal{L} \to T$ for all formulae of $\mathcal{L}$. In most of the paper we will assume that $\mathcal{L}$ is the language of classical propositional logic, $T = \{0, 1\}$, and val_v is defined by the classical Boolean semantics of negation, conjunction, etc.

Judgment aggregation can be defined as follows.[3] Let N be a finite set of *agents*, $\mathcal{A} \subseteq \mathcal{L}$ a finite *agenda* of issues, and $\mathcal{C} \subseteq \mathcal{L}$ a finite set of *admissibility constraints*. A *judgment set* is a consistent and admissible combination of opinions on issues from $\mathcal{A}$, that is, some $js : \mathcal{A} \to T$ for which there exists a valuation

[3] Our definition of judgment aggregation combines features of logic-based aggregation [33] and algebraic aggregation [42]. It is easy to see that both formulations can be expressed in our notation.

$v \in PV$ such that: (i) $val_v(\varphi) = js(\varphi)$ for every $\varphi \in \mathcal{A}$, and (ii) $val_v(\psi) = 1$ for every $\psi \in \mathcal{C}$. The set of all judgment sets is denoted by JS. Now, a *judgment profile* is a collection of judgment sets, one per agent, i.e., $jp : N \to JS$. With a slight abuse of notation, we will denote the set of all such profiles by $JS^{|N|}$. Note that we can conveniently represent judgment profiles as $|Agt| \times |\mathcal{A}|$ matrices of elements from T. Finally, a *judgment aggregation rule* $\nabla : JS^{|N|} \to \mathcal{P}(JS) \setminus \{\emptyset\}$ aggregates opinions from all the agents into a collective judgment set (or sets). We allow for more than one "winning" set to account for nondeterministic or inconclusive aggregation rules.

Example 1. *Consider 3 robots guarding a building, that have just observed a person. Each robot must assess whether the person is authorized to be there (proposition p_1), if it has malicious intent (p_2), and whether to classify the event as dangerous intrusion (p_3). Additionally, it is assumed that a non-authorized person with malicious intent implies intrusion: $\neg p_1 \wedge p_2 \to p_3$ (note that the converse does not have to hold). A possible judgment profile is shown in Figure 1. The figure also shows that the most "obvious" aggregation rule (majority) results in an inadmissible judgment set.*

	p_1	p_2	p_3
robot 1	1	1	0
robot 2	0	0	0
robot 3	0	1	1
majority	0	1	0

Fig. 1. Guarding robots. $N = \{1, 2, 3\}$, $\mathcal{A} = \{p_1, p_2, p_3\}$, $\mathcal{C} = \{\neg p_1 \wedge p_2 \to p_3\}$

In case of binary (yes/no) judgments, this is equivalent to representing opinions as consistent and complete sets of propositional formulas. For example, the view of robot 1 in the Example 1 can be represented by the set $\{p_1, p_2, p_3\}$, the judgment set of robot 2 is $\{\neg p_1, \neg p_2, \neg p_3\}$, and for robot 3 it becomes $\{\neg p_1, p_2, p_3\}$. Issue-by-issue majority rule aggregates the sets into $\{\neg p_1, p_2, \neg p_3\}$ which is inconsistent with the constraint $\neg p_1 \wedge p_2 \to p_3$. Three-valued judgments can be modeled analogously by assuming that the third value is in place for p_i when neither p_i nor $\neg p_i$ occurs in the set (obviously, a set of judgments is then only required to be consistent but not necessarily complete). Representing judgments with more than 3 truth values by sets of formulas is not straightforward anymore.

There are two natural computational problems related to judgment aggregation: computing a "winning" judgment set and verifying that a judgment set is one of the winner sets. We look closer at the latter problem in Section 4.

3.2 Distance-Based Aggregation Rules

A distance-based aggregation rule [34, 38] looks for a collective opinion that does not stray too much from the individual judgments: Formally, such a rule is defined as $\nabla_{d,aggr}(jp) = \text{argmin}_{js \in JS} \left\{ aggr\big(d(js, jp[1]), \ldots, d(js, jp[|N|])\big) \right\}$, where d is a *distance function* [10, p.3-4 and 45], and *aggr* an *aggregation function* [25, p.3], cf. the definitions below.

Definition 1. *An* algebraic aggregation *is a function* $aggr : (\mathbb{R}^+)^n \to \mathbb{R}^+$ *such that:* ***(minimality)*** $aggr(0^n) = 0$, *and* ***(non-decreasing)*** *if* $x \leq y$, *then* $aggr(x_1, \ldots, x, \ldots, x_n) \leq aggr(x_1, \ldots, y, \ldots, x_n)$.

Definition 2. *A* distance *over set X is a function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ such that:* ***(minimality)*** *$d(x, y) = 0$ iff $x = y$,* ***(symmetry)*** *$d(x, y) = d(y, x)$, and* ***(triangle inequality)*** *$d(x, y) + d(y, z) \geq d(x, z)$.*

Well known aggregators are: min, max, sum, and product. Well known distances are the Hamming distance $d_H(x, y) = \sum_{i=1}^{m} \delta_H(x[i], y[i])$, and the drastic distance $d_D(x, y) = \max_{i=1}^{m} \delta_H(x[i], y[i])$, while $\delta_H(x, y) = 0$ if $x = y$ and 1 otherwise.

In belief-merging, it is not required that d satisfies triangle inequality, d is a pseudo-distance, but the only two concrete distances used in belief merging, the Hamming and drastic distance, satisfy it. How necessary this property is in judgment aggregation, is not well studied, but since we do not know of d's that are not distances, we decided to use distances within the scope of this paper.

Example 2. *Consider the robots from Example 1, and let us use d_H as the distance and $\sum$ as the aggregator. Then, the winner sets are $\{000, 011, 110\}$, all with score (i.e., aggregate distance) 3. In other words, the agents cannot do better than to accept one of their individual opinions.*

4 Verification of Collective Opinions in Distance-Based Judgment Aggregation

In computational social choice various complexity-theoretic aspects of voting theory are studied, such as how difficult it is to find a winner of elections or how difficult it is to manipulate an election. There are two natural computational problems related to judgment aggregation: the function problem of computing a "winning" judgment set, and the decision problem of verifying that a given judgment set is one of the winner sets. We look closer at the latter.

4.1 Winner Set Verification

In judgment aggregation the "winner" of an aggregation is the resulting collective opinion, i.e., a set of judgments. Consequently one can consider complexity issues from the stance of a judgment on a particular issue, but also from the stance of an entire set of judgments. If one is concerned with particular judgments, then the interesting complexity-theoretic one-judgment question to study is: how complex is it to determine if a judgment value $t \in T$ was assigned to issue $a \in \mathcal{A}$. This stance is taken in the complexity analysis of [21]. A similar stance, of whether a given belief is included in the merging result of belief bases, is taken when studying the complexity-theoretic properties of belief merging [28]. We adopt a different approach, and look at the verification problem for a given complex opinion, i.e., a judgment set.

We begin by defining formally the problem of *winner set verification*. Then, we investigate the "absolute" complexity that one may face in distance-based aggregation. It turns out that the problem is undecidable in general. On the other hand, the problem becomes more feasible under some reasonable restrictions on the distance and algebraic aggregation functions. Finally, we determine the complexity of winner set verification for some natural aggregators.

The winner set verification problem for agenda $\mathcal{A}$, set of constraints $\mathcal{C}$, logic L and a rule $\nabla^{d,aggr}$, is defined as follows.

Definition 3. WINVER$_\nabla$ *is the decision problem defined as follows:*
Input: *Agents N, agenda $\mathcal{A}$, constraints $\mathcal{C}$, judgment profile $jp \in JS^{|N|}(\mathcal{A},\mathcal{C})$, and judgment set $js \in JS(\mathcal{A},\mathcal{C})$.*
Output: *true if $js \in \nabla(jp)$, else false.*

What is the complexity of WINVER? One could expect that, under the assumptions in Definitions 1 and 2, distance-based aggregation should behave reasonably in computational terms. Unfortunately, it is not the case.

4.2 Negative Results

Theorem 1. *There is a distance which is not Turing computable.*

Proof. We construct the *Turing distance* d_{TR} as follows. First, we assume a standard encoding of Turing machines in binary strings; we use $TM(X)$ to refer to the machine represented by the string of bits $X \in \{0,1\}^m$. We also assume by convention that strings starting with 0 or ending with 1 represent only machines that always halt (e.g., they can represent various TM's with only accepting states).

Let $halts(X) = 0$ if the $TM(X)$ halts, and 1 otherwise. Now, for any $js, js' \in \{0,1\}^m$, we take

$$d_{TR}(js, js') = d_D(js, js') + halts(h(js, js')),$$

where d_D is the drastic distance (i.e., $d_D(js, js') = 0$ if $js = js'$ and 1 otherwise), and $h(js, js') = \big(\delta_H(js[1], js'[1]), \ldots, \delta_H(js[m], js'[m])\big)$ is the Hamming sequence for (js, js'). In other words, we XOR the binary strings corresponding to js and js', interpret the resulting string as a TM, and set the distance to 0 or 1 depending on whether the TM halts or not. On top of that, we add 1 whenever js, js' are not exactly the same.

We check that d_{TR} is a distance:
1. $d_{TR}(js, js) = d_D(js, js) + halts(0^m) = 0$;
2. $d_{TR}(js, js') = 0 \Rightarrow d_D(js, js') = 0 \Rightarrow js = js'$;
3. $d_{TR}(js, js') = d_{TR}(js', js)$: straightforward;
4. Triangle inequality: the nontrivial case is $js \neq js' \neq js''$, then $d_{TR}(js, js') + d_{TR}(js', js'') \geq 2 \geq d_{TR}(js, js'')$.

For incomputability, we observe that $TM(X)$ halts iff $d_{TR}(X, 0^{|X|}) \leq 1$. Consider the following cases: (1) $X = 0^n$: $TM(0^n)$ halts and $d_{TR}(0^n, 0^n) = 0$; (2) $X \neq 0^n$ and $TM(X)$ halts: then, $d_{TR}(X, 0^{|X|}) = 1 + halts(X) = 1$; (3) $X \neq 0^n$ and $TM(X)$ does not halt: then, $d_{TR}(X, 0^{|X|}) = 1 + halts(X) = 2$. □

Theorem 2. *There is a distance and an aggregation function for which* WINVER *is undecidable.*

Proof. We construct a Turing reduction from the halting problem. Given is a representation $X \in \{0,1\}^m$ of a Turing machine (same assumptions as in Theorem 1, i.e., every X starting with 0 or ending with 1 represents a TM that halts). We take d_{TR} as the distance, and $aggr = \sum$. Let the agenda $\mathcal{A} = \{p_1, \ldots, p_m\}$

consist of n unrelated atomic propositions, the set of constraints $\mathcal{C} = \emptyset$, and the judgment profile $jp = \{0^m, X\}$. Now, for $X = 1 \dots 0$ (the other cases of X trivially halt), we have that $TM(X)$ halts iff $js = 0^m, X$ are the only winner sets. To prove this, we first observe that: (i) there is no $Y \in \{0, 1\}^m$ with the aggregate distance less than 1 (since the aggregate distance for Y is a sum of nonnegative elements that includes $d_D(Y, X) + d_D(Y, 0^m)$ and $X \neq 0^m$ by assumption); (ii) for all candidate judgment sets $Y \notin \{0^m, X\}$ the aggregate distance is at least 2 (by the analogous argument); (iii) for $Y = 1^m$ the aggregate distance is always exactly 2, the score being $d_D(1^m, 0^m) + halts(1^m) + d_D(1^m, X) + halts(\overline{X}) = 1 + 0 + 1 + 0$. $TM(1^m)$ halts because 1^m ends with 1, and $TM(\overline{X})$ halts because $\overline{X}$ begins with 0. Now we prove the equivalence:

$\Rightarrow$: Assume that $TM(X)$ halts. Then, the aggregate distance for X is 1, and the same for 0^m (because $d_{TR}(X, 0^m) = 1$ and $d_{TR}(X, X) = d_{TR}(0^m, 0^m) = 0$). Thus, by (i), $0^m, X$ must be winners, and by (ii) no other judgment set can be a winner.

$\Leftarrow$: Assume that $TM(X)$ does not halt. Then, the aggregate distance for X is 2, and likewise for 0^m (because $d_{TR}(X, 0^m) = 2$). By (ii), $0^m, X$ must be winners, but they are *not the only winners* – by (iii), 1^m must be a winner too.

We have proved that $TM(X)$ halts iff $js = 0^m, X$ are the only winner sets. Suppose now that deciding WINVER terminates in finite time. Then, the halting of $TM(X)$ could be verified by 2^m WINVER checks, i.e., also in finite time – which is a contradiction. □

Thus, the standard requirements on distance metrics and aggregation function are not sufficient to guarantee even decidability of the winner set verification problem. Of course, the judgment aggregation rule used in the proof of Theorem 2 is artificial and unlikely to be ever used in any practical context. Still, it shows that the framework allows – at least theoretically – for such ill-behaved rules. Note that the effect should be the same if the distance is based on solving any other undecidable problem. For example, it can be based on a solution to a certain game, and if the game assumes imperfect information and perfect recall of players then solving it is in general undecidable [37, 15]. Or, the distance can be defined in terms of resources needed by a group of agents to achieve a given task (for undecidability, cf. e.g. [5]). We believe that these two examples of hypothetical distance-based rules are not so far-fetched anymore.

Distance-based aggregation rules that are actually used have much better computational properties, as we demonstrate in Section 4.3. Yet, Theorem 2 is important because it shows the *bounds* of the framework: in principle, the complexity of related decision problems can be very bad. This means that, when trying a *new* variant of distance-based aggregation, one should be cautious, and carefully examine its computational characteristic beforehand.

4.3 Positive Results

We now prove that, under reasonable conditions, winner set verification sits in the first level of the polynomial hierarchy. We recall that $\mathbf{P}^{\mathbf{NP}[k]}$ is the class of problems solvable by a polynomial-time deterministic Turing machine asking at most k adaptive queries to an **NP** oracle. Clearly, $\mathbf{NP} \subseteq \mathbf{P}^{\mathbf{NP}[k]} \subseteq \mathbf{\Delta_2^P} = \mathbf{P}^{\mathbf{NP}}$.

Theorem 3. *If $aggr$ and d are computable in polynomial time then* WINVER *for* $\nabla_{d,aggr}$ *is in* $\mathbf{P}^{\mathbf{NP}[2]}$.

Proof. We prove the inclusion by showing Algorithm 1 for WINVER, which uses two oracles, given in Algorithms 2 and 3. Note that the js in the input of Algorithms 3 is always consistent.

Algorithm 1: Winver()

Input: js,jp,N,$\mathcal{A}$,$\mathcal{C}$,d,$aggr$
Output: **true** if js is a winner for jp under $aggr$, **false** otherwise

if *Consistent(js,$\mathcal{A}$,$\mathcal{C}$)* **and not** *ExistsBetter(js,jp,N,$\mathcal{A}$,$\mathcal{C}$,d,$aggr$)* **then**
  return true else return false

Algorithm 2: Oracle Consistent()

Input: js,$\mathcal{A}$,$\mathcal{C}$
Output: **true** if js is consistent for $\mathcal{A}$ and $\mathcal{C}$, **false** otherwise

guess a valuation $v \in PV$ for the atomic propositions in $\mathcal{A}$
if $val_v(\varphi) = js(\varphi)$ *for every* $\varphi \in \mathcal{A}$ **and** $val_v(\psi) = 1$ *for every* $\psi \in \mathcal{C}$ **then**
  return true else return false

Algorithm 3: Oracle ExistsBetter()

Input: js,jp,N,$\mathcal{A}$,$\mathcal{C}$,d,$aggr$
Output: **true** if there is a judgment set 'closer' to jp than js, **false** otherwise

guess $js' \in JS$
guess a valuation $v' \in PV$ for the atomic propositions in $\mathcal{A}$
if $val_{v'}(\varphi) = js'(\varphi)$ *for every* $\varphi \in \mathcal{A}$ **and** $val_v(\psi) = 1$ *for every* $\psi \in \mathcal{C}$
and$aggr\big(d(js', jp[1]), \ldots, d(js', jp[|N|])\big) < aggr\big(d(js, jp[1]), \ldots, d(js, jp[|N|])\big)$
then
  return true else return false

For combinations of most typical distances and aggregators, the following is a straightforward consequence.The problem of checking if a judgment is in at least one collective judgment set is already known to be **NP**-complete for $d = d_H, aggr = \sum$ [21].

Corollary 1. *If* $aggr \in \{\min, \max, \sum, \prod\}$ *and* $d \in \{d_H, d_D\}$ *then* WINVER *for* $\nabla_{d,aggr}$ *is in* $\mathbf{P}^{\mathbf{NP}[2]}$.

4.4 Aggregation of Non-Binary Judgments

In this section, we briefly report that all the results from Sections 4.2 and 4.3 carry over to the case of judgments interpreted in a given k-valued logic.[4] In particular, we note that the algorithm in Section 4.3 depends neither on the set of truth values, nor on the way valuations of complex formulas derive from valuations of atomic propositions. Also, the Turing distance used in Section 4.2 is built on pointwise comparison of judgment sets that always results in a binary string. Thus, we can state the following.

[4] We do not discuss motivation for using such judgments, and instead refer the interested reader e.g. to [36, 16, 31, 40, 17].

Theorem 4. *For every $k \in \mathbb{N}$, there is a distance over $\{0, \ldots, k-1\}^m$ which is not Turing computable.*

Proof. Analogous to the proof of Theorem 1.

Theorem 5. *Let $k \in \mathbb{N}$, and $\mathcal{L}$ a k-valued logic constructed like in Section 3.1. Then, there is a distance and an aggregation function for judgment sets in $\mathcal{L}$ such that* WINVER *is undecidable.*

Proof. Analogous to the proof of Theorem 2.

Theorem 6. *If aggr is an aggregation function over over $\{0, \ldots, k-1\}^n$, d is a distance metric over $\{0, \ldots, k-1\}^m$, and both aggr and d are computable in polynomial time, then* WINVER *for $\nabla_{d,aggr}$ is in $\mathbf{P}^{\mathbf{NP}[2]}$.*

Proof. The claim is demonstrated by the same algorithm as in the proof of Theorem 3.

5 Conclusions

Complexity-theoretic properties of voting procedures are a frequent topic of study in computational social choice. In contrast, the complexity of judgment aggregation has drawn attention only recently. In this paper, we explore the complexity bounds of an important family of judgment aggregation rules, namely those based on minimization of aggregate distance. More precisely, we study the decision problem of verifying if a given judgment set can be selected as the collective opinion. It turns out that feasibility of distance-based aggregation in general cannot be guaranteed, and should not be taken for granted. However, by assuming some requirements on the possible outcomes of the distance and aggregation functions, we can tame the complexity reasonably. We also show that the pattern of complexity does not change when the framework is extended to multi-valued judgments.

To our best knowledge, this paper is the first to analyze the complexity of verifying distance-based aggregate judgments on an abstract level. There are not many concrete judgment aggregation rules proposed in the literature; this aspect of the judgment aggregation theory has only now begun to be developed. Our results suggest that, when devising a new judgment aggregation rule, we should expect complexity traps, and carefully look for rules that are relatively efficient.

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