Games with Epistemic Moves

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1 Introduction

In this work, we propose a compact and natural representation of simple multi-step games where exchange of information is crucial. The main idea derives from *knowledge games* [4], where information is exchanged in order to meet a given objective. Quite often the objective concerns a security property, like letting know one's partner a secret without the intruder learning the secret on the way. Analysis of such games has been a part of research on *unconditional security* [5]. However, knowledge games lack "real life", non-epistemic context. In particular, they don't give account on how *relevant* is the information being exchanged. Even more importantly, it does not quantify how much information leaks on the way and how damaging the leak is. The intruder is guaranteed not to learn the secret *completely*, but there is no distinction between him almost knowing the secret afterwards or not knowing it at all. Moreover, the intruder can learn another information in the course of communication, which is as damaging as knowing the secret itself.

Here, we assume that knowledge games are played in the context of a "real", i.e. non-epistemic game. We model the game as a strategic game of incomplete information – that is, for every possible state of the world a matrix of payoffs is given, but the players may be uncertain about the current state of the world. As players reveal new information, their uncertainty is reduced, and they increase their ability to choose a good strategy in the real game. So, our models can be seen as an extension of strategic games of incomplete information: we give account of how the game is transformed when agents exchange pieces of their knowledge. One can also view them as a subclass of *extensive form* games with incomplete information, with non-terminal moves constructed akin to *public announcements* in the PAL logic [1].

So, from game theoretical perspective, we study a subclass of EF games with specific natural structure. From the perspective of reasoning about knowledge, we give context to exchange of information. By bringing it into the picture, we allow to assess how useful a communication is to the friendly party, and at the same time how damaging the information that has leaked can be.

2 Games with Epistemic Moves

We define a *game with epistemic moves* Γ as consisting of:

- An incomplete information game frame: a set of players (or agents) $\mathbb{A}\text{gt} = \{a_1, \ldots, a_k\}$ and sets of strategies Σ_a, one per agent $a \in \mathbb{A}\text{gt}$; a set of states St and indistinguishability relations $\sim_a \subseteq St \times St$ one per $a \in \mathbb{A}\text{gt}$; finally, a function that associates each state q with a real-valued strategic game pay_q over $\mathbb{A}\text{gt}$ and $\Sigma_1, \ldots, \Sigma_k$. We assume for simplicity that $\mathbb{A}\text{gt} = \{a, b, i\}$ and $|\Sigma_a| = 1$: Arnold communicates information to Bond who decides when and which strategy will be played, and Indiana is the intruder. We also assume that Arnold and Bond have the same objectives: $pay_q(a) = pay_q(b)$ for all q.
- A language of announcements L, typically obtained from atomic propositions $(p_1, p_2, \ldots)$, Boolean connectives $(\neg, \wedge)$, and epistemic operators (K_a, K_b, K_i) in the standard way.
- A valuation of formulae of L in states of Γ, i.e. $V : L \to 2^{St}$, constructed in the typical way.

A *pointed game* (Γ, q) is a game and a designated state in it.

In this paper, we assume that Γ is based on a finite nonempty set of states. Let $\approx_\Gamma \subseteq L \times L$ be the equivalence relation that deems formulae $\phi, \psi \in L$ equivalent iff $V(\phi) = V(\psi)$ in Γ. Let us define L_Γ as the set of abstraction classes in L wrt $\approx_\Gamma$. It is easy to see that, for a finite St_Γ, L_Γ is also finite (and bounded by $2^{|St|}$). We will call the elements of L_Γ the *epistemic actions* in Γ.

Epistemic transitions are defined by appropriate model updates [1, 4]: (1) Epistemic action α is *enabled* in state q of game Γ iff it is truthful, i.e., $q \in V(K_a\alpha)$; (2) The result of executing α in (Γ, q) is (Γ', q) where Γ' is obtained from Γ by removing all the states where $K_a\alpha$ does not hold. Thus, any game Γ implicitly defines a game graph where: (i) the vertices are pointed games reachable from (Γ, q) for some $q \in St_\Gamma$, (ii) information sets are abstraction classes of the indistinguishability relations, (iii) non-terminal edges are deterministic transitions induced by the epistemic actions of a, (iv) terminal edges are injective transitions labeled by pairs of "real" actions from b, i, and (v) payoffs in terminal vertices are defined accordingly. Note that the terminal strategic games can be split into equivalent k-step extensive games with imperfect information. Moreover, the graph does not have cycles, except for loops (α loops in (Γ, q) iff it does not change anybody's knowledge there). If we assume such actions to be irrelevant, we get the following as a consequence.

Theorem 1. *Every game with epistemic moves can be modeled as an extensive form game with incomplete information. If the former is finite, the latter is finite as well.*

We will call the EF game the *meta-game* of Γ. Moreover, strategies in the meta-game will be called *meta-strategies* of Γ. One can study such games from purely game-theoretical perspective, e.g. investigating various solution concepts like Nash equilibrium, correlated equilibrium, undominated strategies, Stackelberg equilibrium etc. We leave these ideas for future work. From the epistemic perspective, we can use the game structure to measure the usefulness of an action in a given non-epistemic context. An attempt is presented in the next section.

3 Relevance of Epistemic Actions

We will measure the *relevance* of epistemic action α in game Γ as the average marginal contribution of α in the outcome of meta-strategies that can use it. Let us construct a rating for subsets of epistemic actions as follows: $C : 2^{L_\Gamma} \to \mathbb{R}$ with $C(A)$ being the a's payoff guaranteed by the best joint meta-strategy of $\{a, b\}$ that uses only actions from A. We define two measures $f : L_\Gamma \to \mathbb{R}$:

Shapley relevance: $Sh(\alpha) = \sum_{A \subseteq L_\Gamma \setminus \{\alpha\}} \frac{|A|!(|L_\Gamma|-|A|-1)!}{|L_\Gamma|!} (C(A \cup \{\alpha\}) - C(A))$;
Banzhaf relevance: $Bn(\alpha) = \sum_{A \subseteq L_\Gamma \setminus \{\alpha\}} \frac{1}{2^{|A|}} (C(A \cup \{\alpha\}) - C(A))$.

Note that C is in fact a superadditive coalitional game with epistemic actions as players, and sets of these action as coalitions. Let us now recall some standard requirements on payoff division functions in coalitional games:

(Efficiency) $\sum_i f(i) = C(L_\Gamma)$.
(Symmetry) If i and j are interchangeable, then $f(i) = f(j)$.
(Dummies) For each dummy i: $f(i) = C(\{i\})$.
(Additivity) For any two games C_1, C_2: $f(C_1 \oplus C_2) = f(C_1) + f(C_2)$, where $C_1 \oplus C_2$ denotes the game defined by $(C_1 \oplus C_2)(A) = C_1(A) + C_2(A)$.

In consequence, the following is straightforward.

Theorem 2. *Sh is the only measure satisfying Efficiency, Symmetry, Dummies and Additivity.*

References

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