Game-Theoretic Network Centrality

Tomasz P. Michalak
Department of Computer Science, University of Oxford
Institute of Informatics, University of Warsaw

UNIVERSITY OF OXFORD

DEPARTMENT OF COMPUTER SCIENCE

Plan of the Talk

1. Introduction to the Shapley value & its computation

2. The Shapley value as a game-theoretic network centrality measure

3. Applications and computations

1

Shapley value
& its computational aspects

Characteristic Function Games

Given 3 agents, the **set of agents** is:

N = $\{a_1, a_2, a_3\}$

The possible **coalitions** are:

a_1	a_2	a_3	$a_1 a_2$	$a_1 a_3$	$a_2 a_3$	$a_1 a_2 a_3$
5	5	5	12	12	12	24

<? ? ?>

A **solution** of a coalitional game:

STABILITY → THE CORE

STABILITY

Characteristic Function Games

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A **solution** of a coalitional game:

STABILITY → THE CORE

Such a division of payoff which no sub-coalition wants to deviate from

$a_1 a_2 a_3$

24

<10 7 7>

a_2 : **Wait!**
But it is not fair!

a_3 : My contribution to every coalition in the game is the same as a_1

a_1 : **Great!**
I like this core division!`

Characteristic Function Games

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N = $\{a_1, a_2, a_3\}$

The possible **coalitions** are:

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5	5	5	12	12	12

A **solution** of a coalitional game:

FAIRNESS → SHAPLEY VALUE

A unique division of payoff
That meets fairness criteria
(axioms)

$a_1 a_2 a_3$

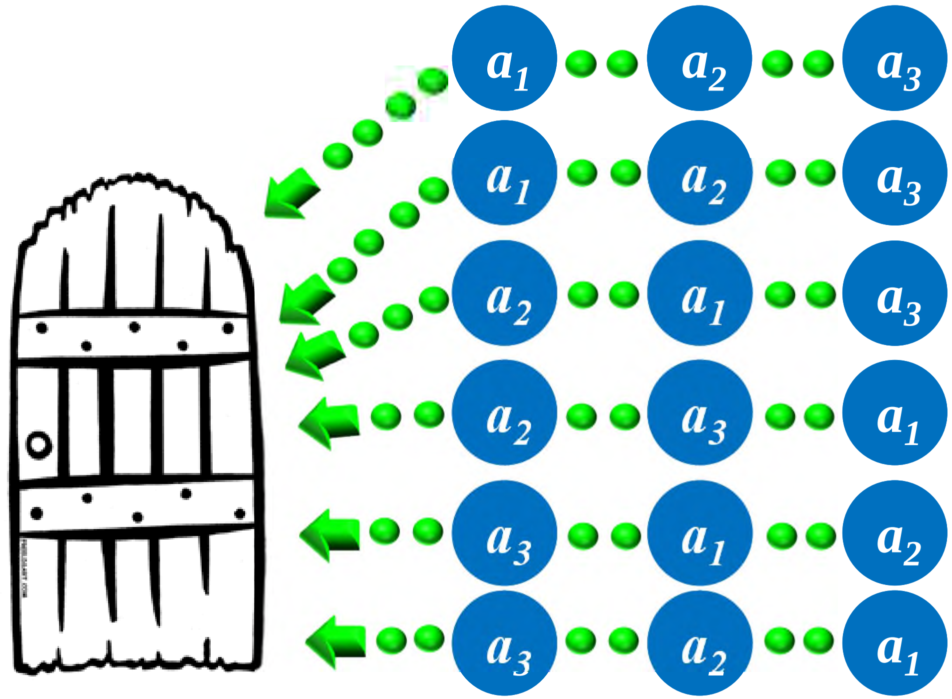
24

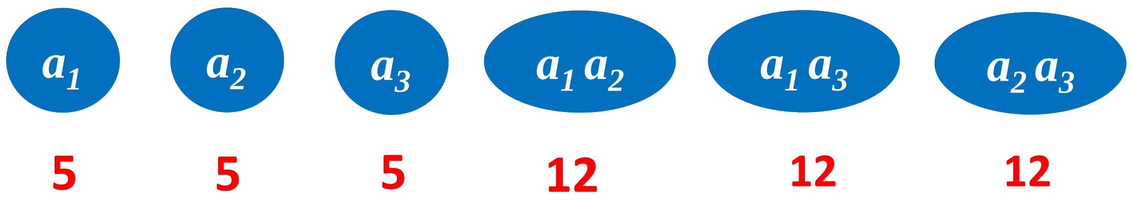
<8 8 8>

Fairness criteria:

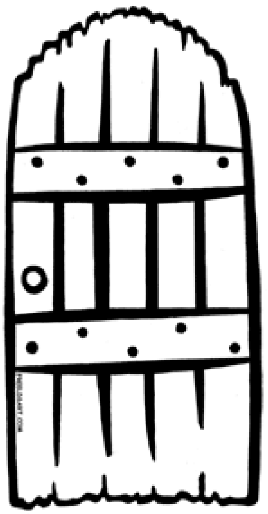
- Symmetry
- Null-player
- Additivity
- Efficiency

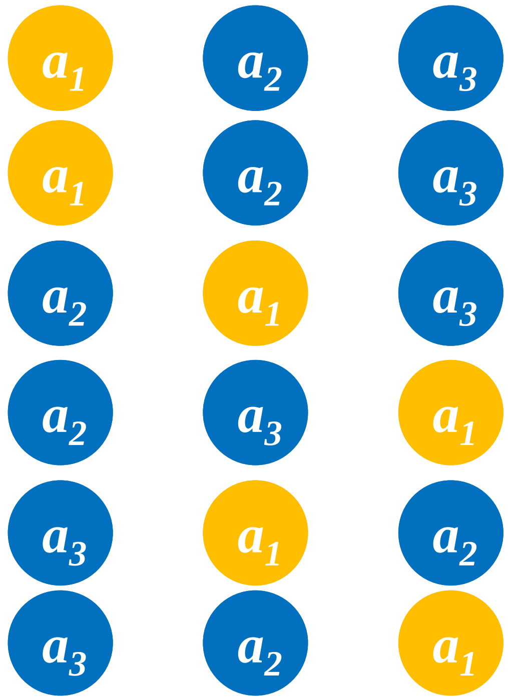
Shapley Value – Definition

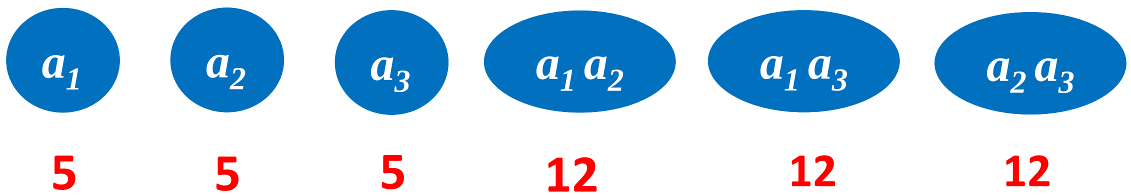




Shapley Value – Definition







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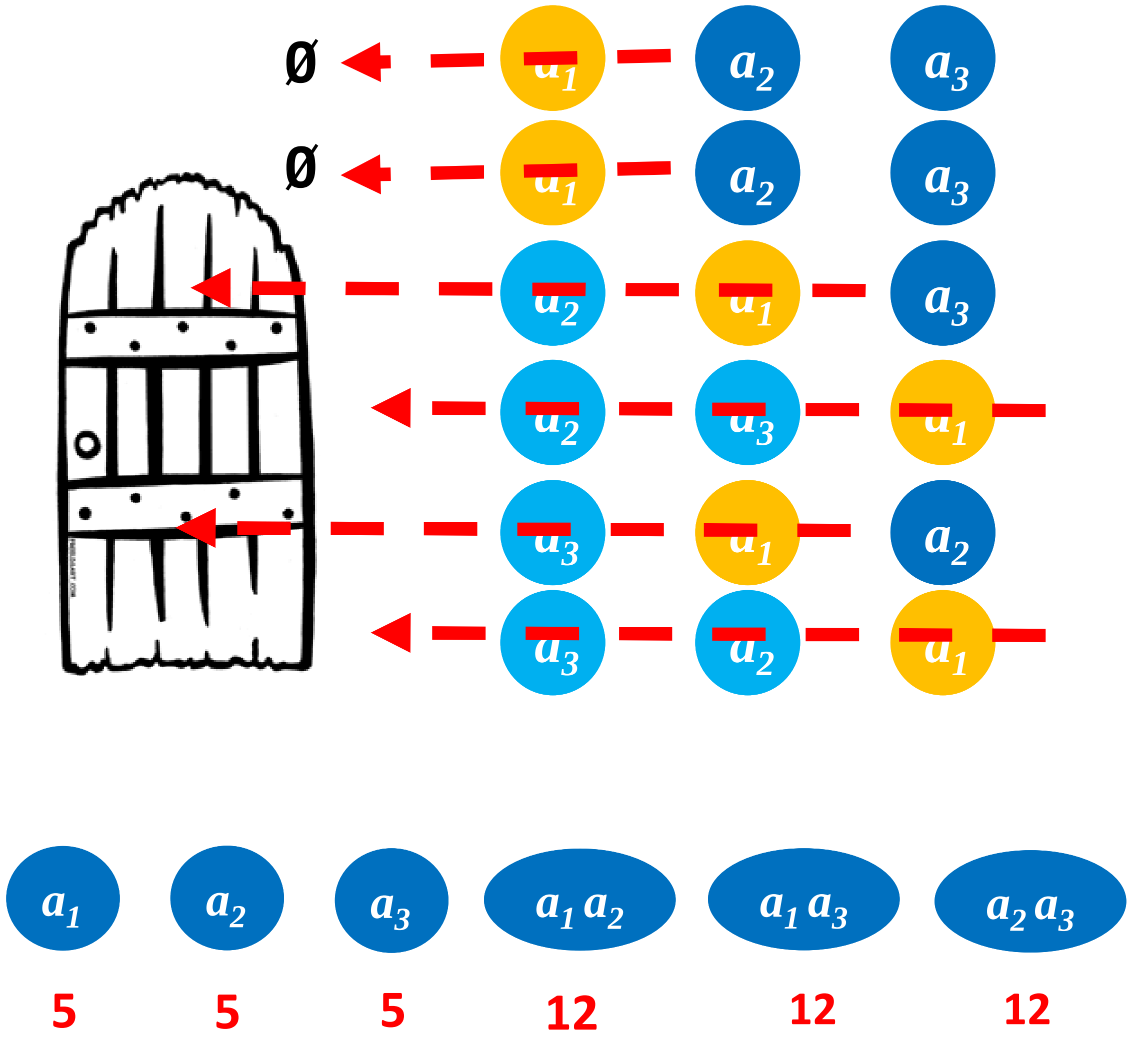


MC(a_1)

+5

+5

+7

+12

+7

+12

48/6 = 8

= $SV_1(v)$

Shapley Value – Formulas

Marginal contribution of a_i to coalition made of agents in the left part of the permutation

n!

$$SV_i(v) = \frac{1}{n!} \sum_{\text{all } \pi} [v(C_\pi(i) \cup \{a_i\}) - v(C_\pi(i))]$$

The part of the permutation before agent a_i (left part of permutation)

Shapley Value – Formulas

n!

$$SV_i(v) = \frac{1}{n!} \sum_{\text{all } \pi} [v(C_\pi(i) \cup \{a_i\}) - v(C_\pi(i))]$$

2^n

$$SV_i(v) = \sum_{C \subseteq N \setminus \{a_i\}} \frac{|C|!\,(n - |C| - 1)!}{n!} [v(C \cup \{a_i\}) - v(C)]$$

➔ **Computational Challenge** ➔

Circumventing intractability of the Characteristic Function

New, more concise representations of coalitional games:

General idea:

Find a new model of a coalitional game. That is:

- ✓ concise
- ✓ expressive
- ✓ effective
- ✓ simple

the computational problem …

… is solved on the model level

Circumventing intractability of the Characteristic Function

New, more concise representations of coalitional games:

- Induced Subgraph Representation
- Marginal Contribution Nets
- Algebraic Decision Diagrams

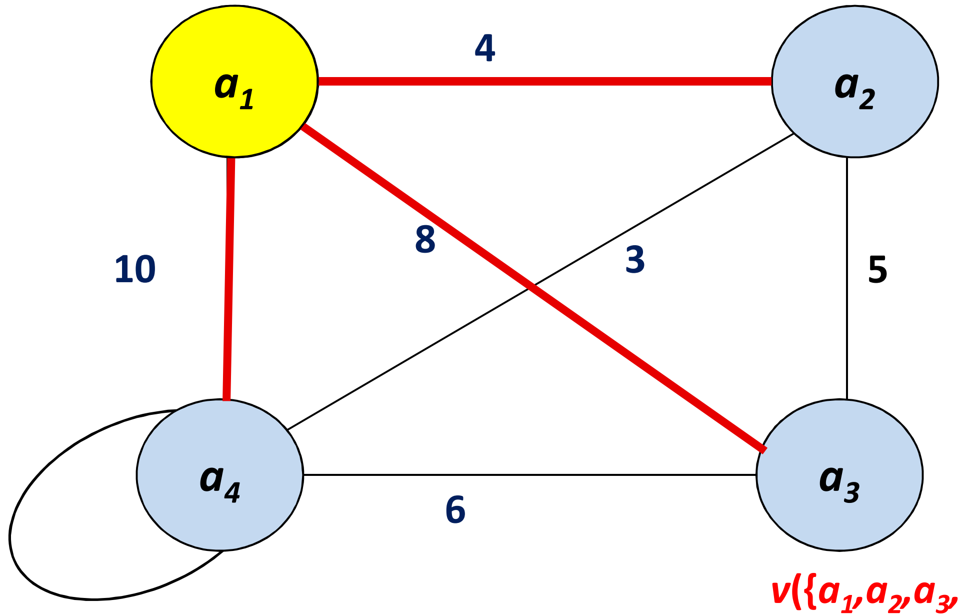
Always concise but not fully expressive

Fully expressive but not always concise

Note: There are, of course, other representations – for specific types of games
See more G. Chalkiadakis, E. Elkind, and M. Wooldrdidge. *Computational Aspects of Cooperative Game Theory*. Morgan & Claypool Publishers, 2011

Induced Subgraph Representation
Deng and Papadimitriu (1994)

agents are nodes of a coalition edges is represented the value of the cooperation subgraph by this coalition



$v(\{a_1\}) = v(\{a_2\}) = v(\{a_1\}) = 0$

$v(\{a_4\}) = 1$

$v(\{a_1,a_2\}) = 4$

$v(\{a_1,a_3\}) = 8$

:

$v(\{a_3,a_4\}) = 1+6$

$v(\{a_1,a_2,a_3\}) = 4+5+8$

:

$v(\{a_2,a_3,a_4\}) = 1+5+3+6$

$v(\{a_1,a_2,a_3,a_4\}) = 1+4+5+6+10+3+8$

(1) Expressivity ?

(2) Conciseness ✓

(3) Simplicity ✓

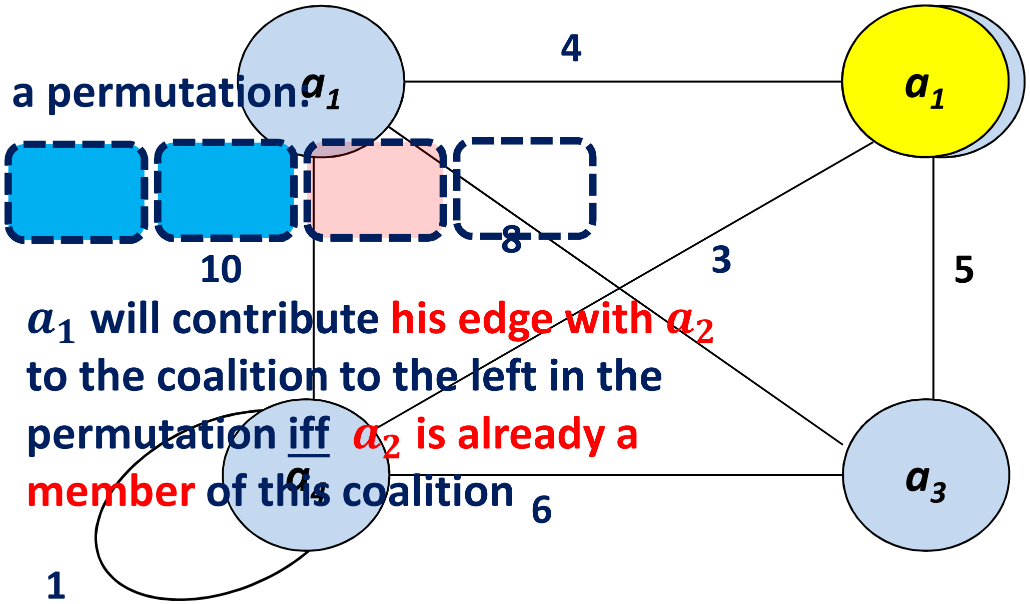
(4) Effectiveness ✓

$$SV_1 = \frac{4+8+10}{2}$$

representation is not fully expressive ✗

Induced Subgraph Representation
Deng and Papadimitriu (1994)

Let us consider the following intuition for the Shapley value formula under this representation



a permutation?

a_1 will contribute his edge with a_2 to the coalition to the left in the permutation <u>iff</u> a_2 is already a member of this coalition

The probability of this event is $\frac{1}{2}$

Marginal Contribution Nets
Ieong and Shoham (2005)

$Logical\ Pattern \rightarrow Value$

Fully Expressive
Often Concise
Efficient?

a_1	5			
a_2	4			
a_3	1			
$a_1 a_2$	5	+ 4	+ 3	
$a_1 a_3$	5	+ 1		
$a_2 a_3$	4	+ 1		
$a_1 a_2 a_3$	5	+ 4	+ 1	+ 3

$a_1 \rightarrow 5$

$a_2 \rightarrow 4$

$a_3 \rightarrow 1$

$a_1 \wedge a_2 \rightarrow 3$

$Sv_1 = 5 + 3/2$

$Sv_2 = 4 + 3/2$

$Sv_3 = 1$

Symmetry axiom

Additivity axiom

Marginal Contribution Nets
Ieong and Shoham (2005)

$Logical\ Pattern \rightarrow Value$

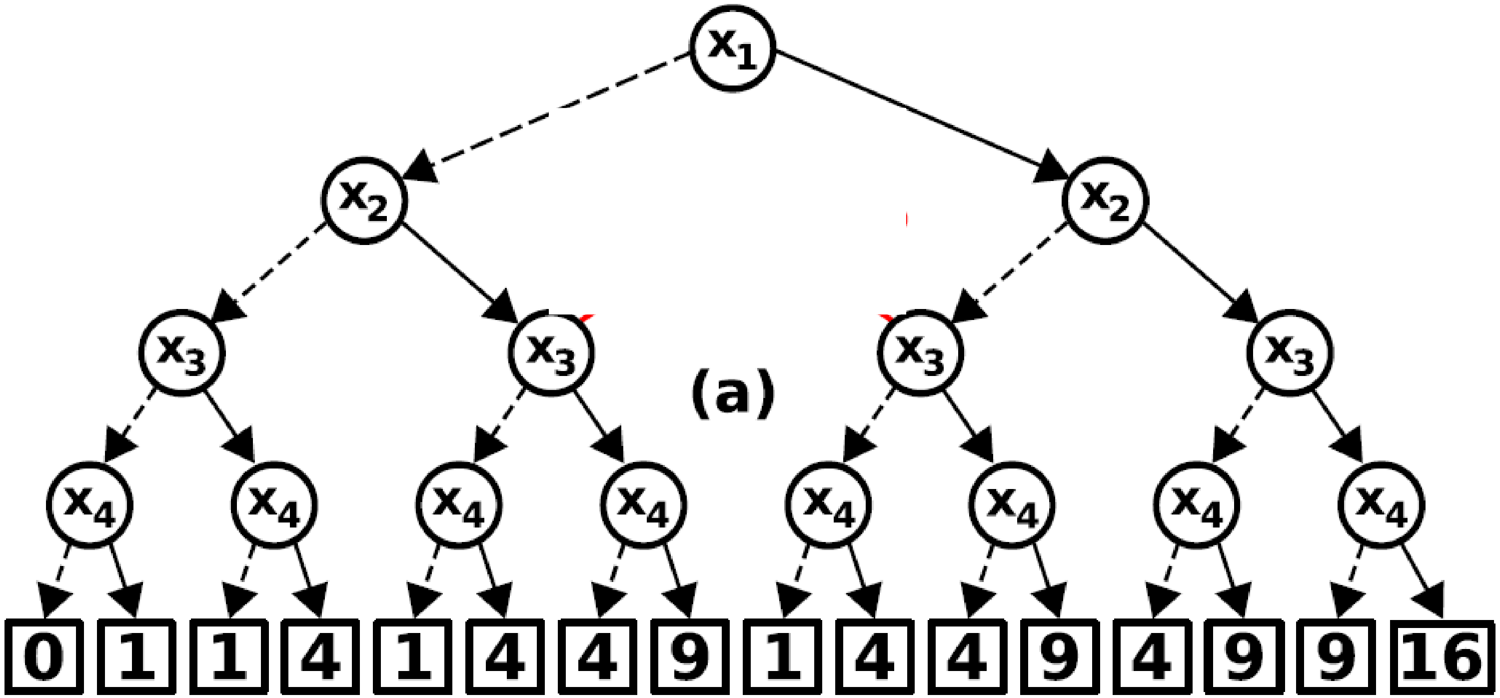
- ❑ Such spectacular computational properties were initially shown for very simple rules, where only ^ and ¬ are allowed.

- ❑ Such representation is called simple MC-Nets.

- ❑ But what about more complex rules?

- ❑ Elkind, Wooldridge, Goldberg and Goldberg (2009) proposed MC-Nets with arbitrary logical connectives but which are read-once. Still, polynomial computation of the Shapley value.

Algebraic Decision Diagrams
Aadithya Michalak Jennings (2011)

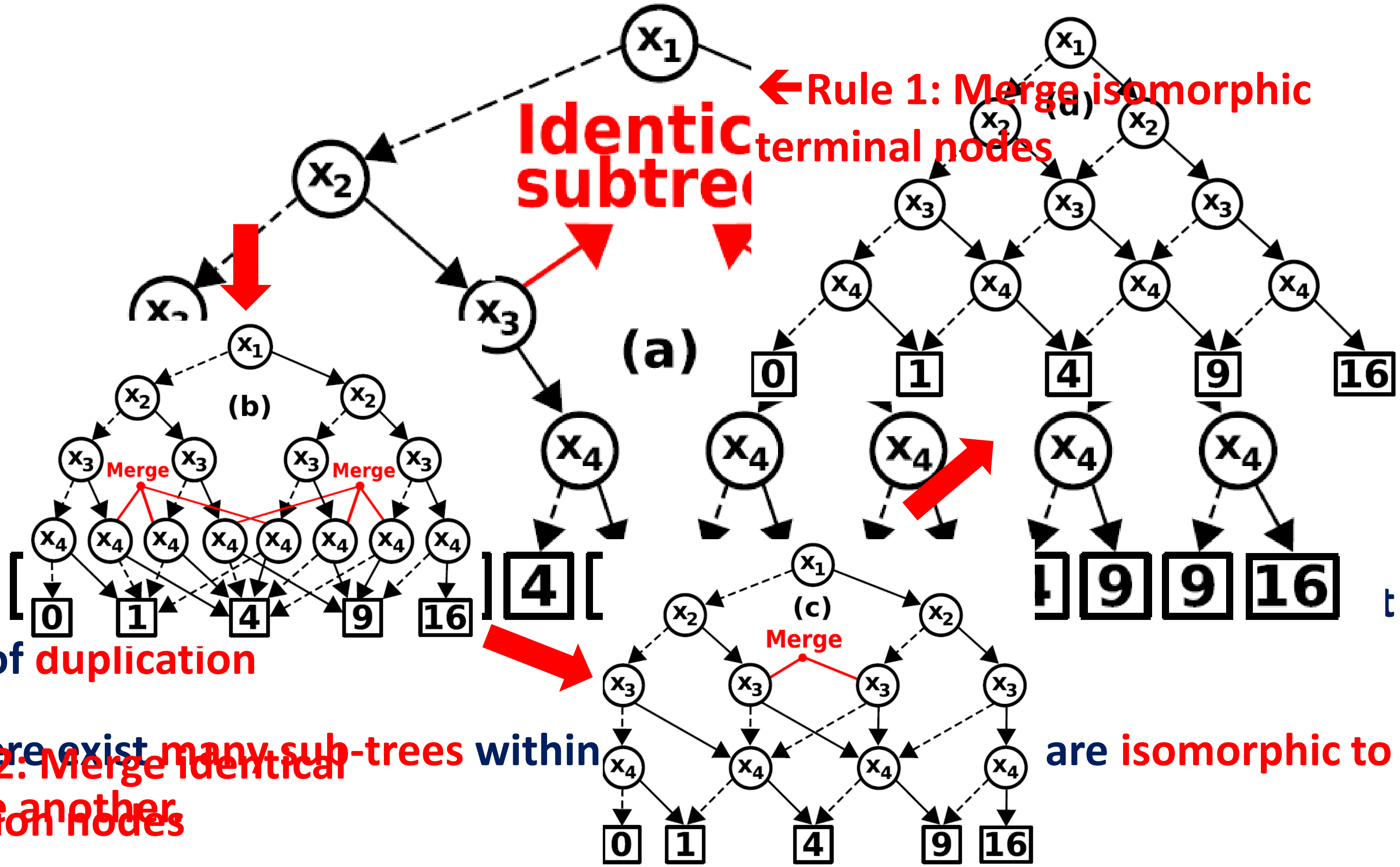


In general, a decision tree is of size exponential in the number of decision variables.

ADDs are, in essence, highly optimized representations for **ordered decision trees on boolean decision variables**.

However...

Algebraic Decision Diagrams



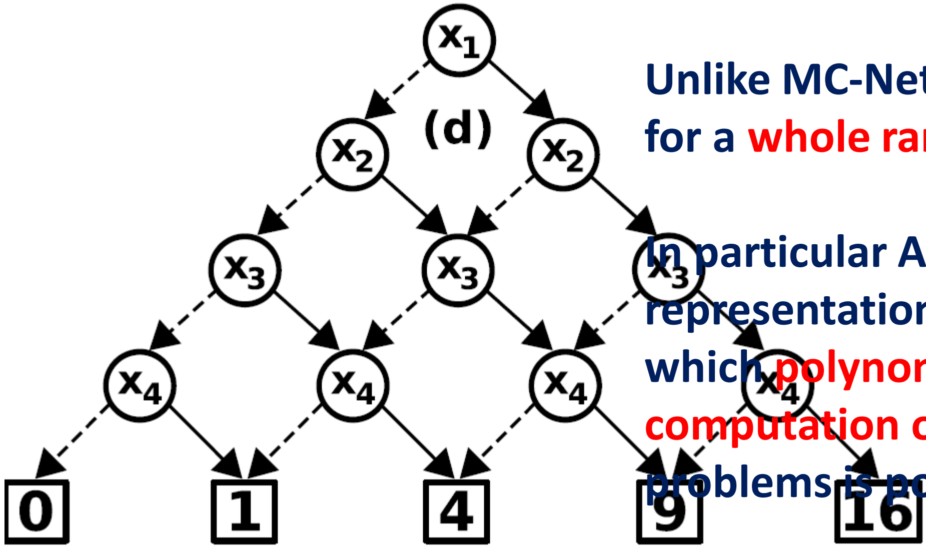
Rule 1: Merge isomorphic terminal nodes

In r ... t of duplication

There exist many sub-trees within ... are isomorphic to one another

Rule 2: Merge identical decision nodes

Algebraic Decision Diagrams



Unlike MC-Nets, ADDs can be used for a whole range of problems

In particular ADDs are the only representation formalism under which polynomial time computation of the core related problems is possible.

Problem	Induced subgraph	Unrestricted MC-Net	Basic MC-Net	Read-once MC-Net	ADD
$\nu(C)$ given C	✓	✓	✓	✓	✓
TEST-CORE	×	×	×	×	✓
EMPTY-CORE	×	×	×	×	✓
ϵ-CORE	×	×	×	×	✓
CoS	×	×	×	×	✓
BI	✓	×	✓	✓	✓
SV	✓	×	✓	✓	✓

ZDD
zero-supressed decision diagrams
Sakurai, Ueda, Iwasaki, Minato, and Yokoo (2011)

Not only the Shapley value...

the Shapley value:

2^n

$$SV_i(v) = \sum_{C \subseteq N \setminus \{a_i\}} \frac{|C|!\,(n - |C| - 1)!}{n!} [v(C \cup \{a_i\}) - v(C)]$$

the Banzhaf index

2^n

$$SV_i(v) = \sum_{C \subseteq N \setminus \{a_i\}} \frac{1}{2^n} [v(C \cup \{a_i\}) - v(C)]$$

Semivalues = {Shapley, Banzhaf, ...}

2^n

$$SV_i(v) = \sum_{C \subseteq N \setminus \{a_i\}} \frac{\beta(k)}{2^n} [v(C \cup \{a_i\}) - v(C)]$$

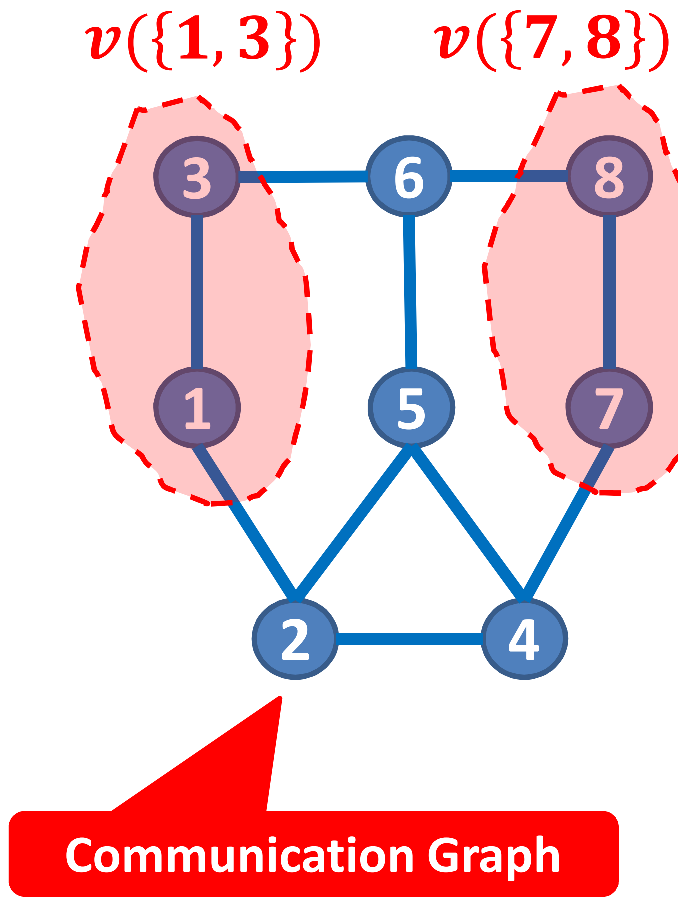
the Nowak & Radzik value:

Generalized characteristic function

n!

$$NRV_i(v) = \frac{1}{n!} \sum_{\text{all } \pi} \left[v(\overrightarrow{C_\pi}(i) \cup \{a_i\}) - v(\overrightarrow{C_\pi}(i)) \right]$$

Myerson's game



What if the cooperation is restricted by a graph?

If a coalition C is connected then players in C can communicate and create an arbitrary value added

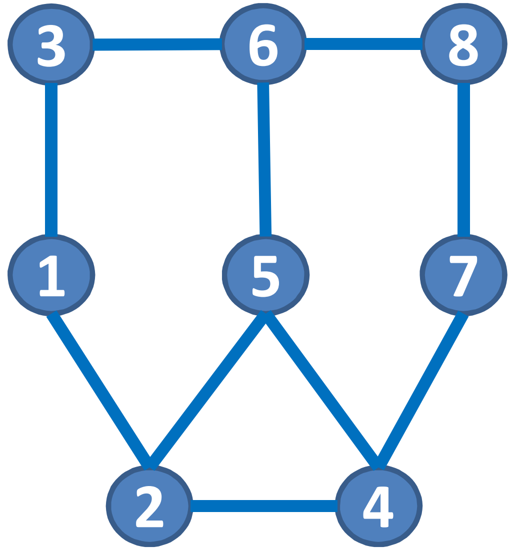
If a coalition C is disconnected then players in C cannot communicate; hence, creating value added is restricted to connected components

$$v(\{1,3\}) + v(\{7,8\})$$

$$v/G(C) = \begin{cases} v(C) & \text{if } C \text{ is connected} \\ \sum_{K \in C} v(K) & \text{otherwise} \end{cases}$$

Myerson's graph-restricted game

The Myerson value



There exist the unique value that satisfies:

- Axiom 1: *fairness* - any two agents connexted with an edge profit from this connection equally
- Axiom 2: *efficiency* - the value of any connected component is distributed among the agents within this components

$$MV_i(v, G) = SV_i(v/G)$$

the Myerson value

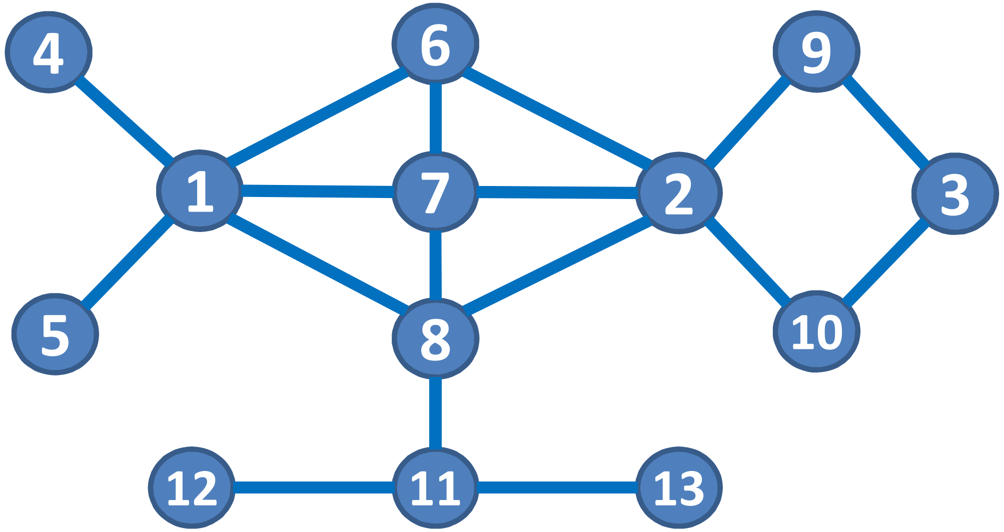
$$v/G(C) = \begin{cases} v(C) & \text{if } C \text{ is connected} \\ \sum_{K \in C} v(K) & \text{otherwise} \end{cases}$$

2

Game-theoretic Network Centrality

Centrality Measures

Which node is the most important in this network ?

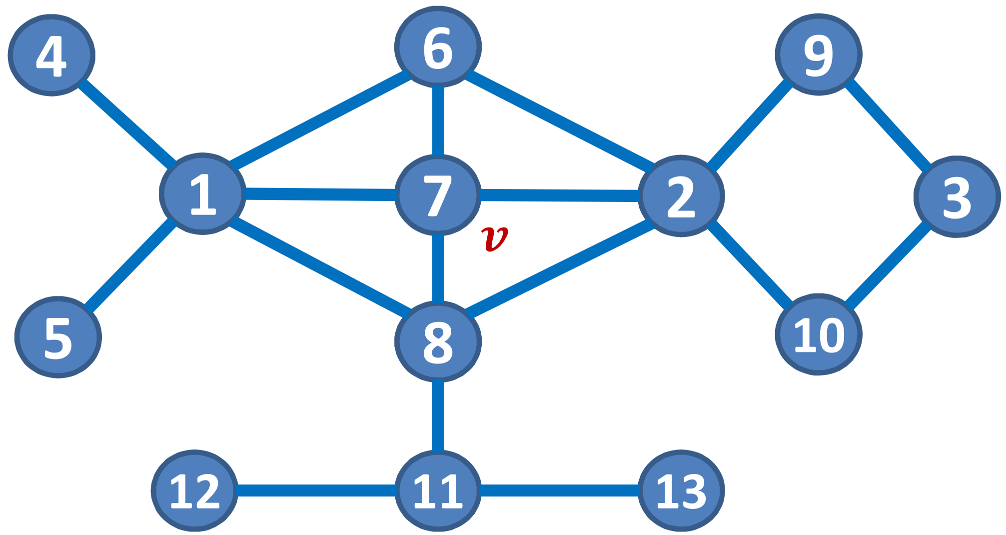


Informal definition: **methods to determine the role played by a node in the network.**

They differ depending on the application. Three, mostly used are:

Centrality Measures

Which node is the most important in this network ?



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1. **Degree centrality** – how many adjacent edges node v has 4

Centrality Measures

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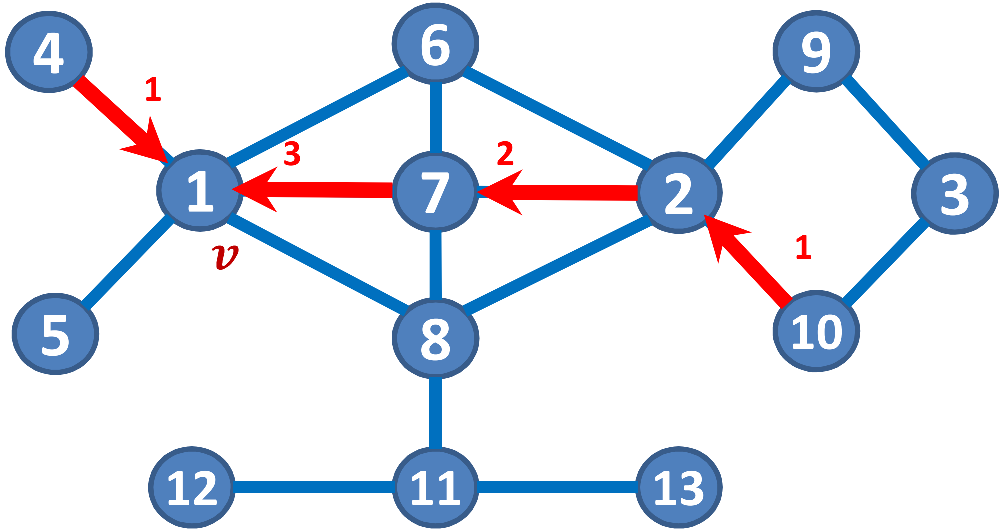


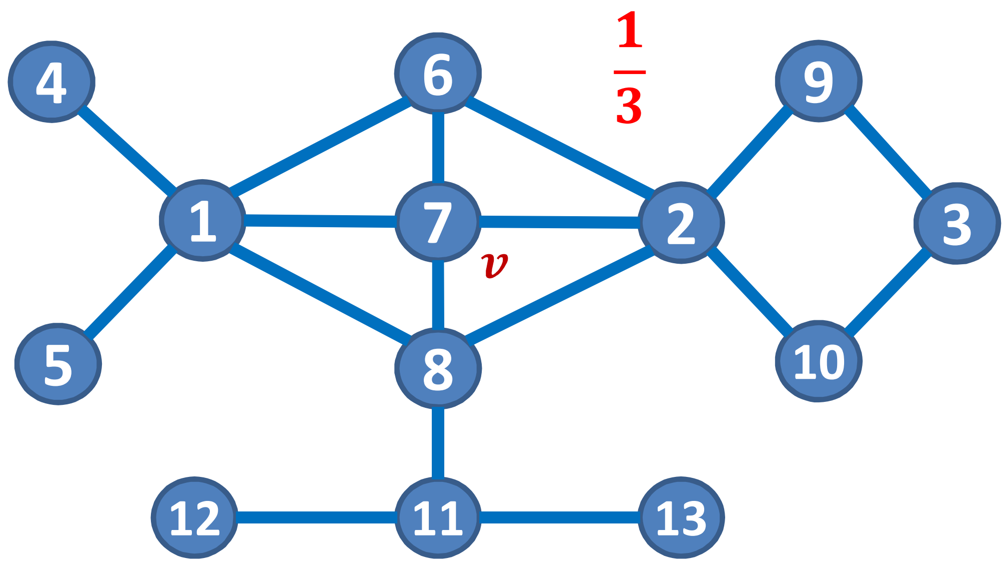
Informal definition: methods to determine the role played by a node in the network.

They differ depending on the application. Three, mostly used are:

1. **Degree centrality** – how many adjacent edges node v has
2. **Closeness centrality** – how many edges, on overage, one needs to traverse to reach v from other nodes in the network $\frac{1}{1} + \frac{1}{3} + \dots$

Centrality Measures

Which node is the most important in this network ?



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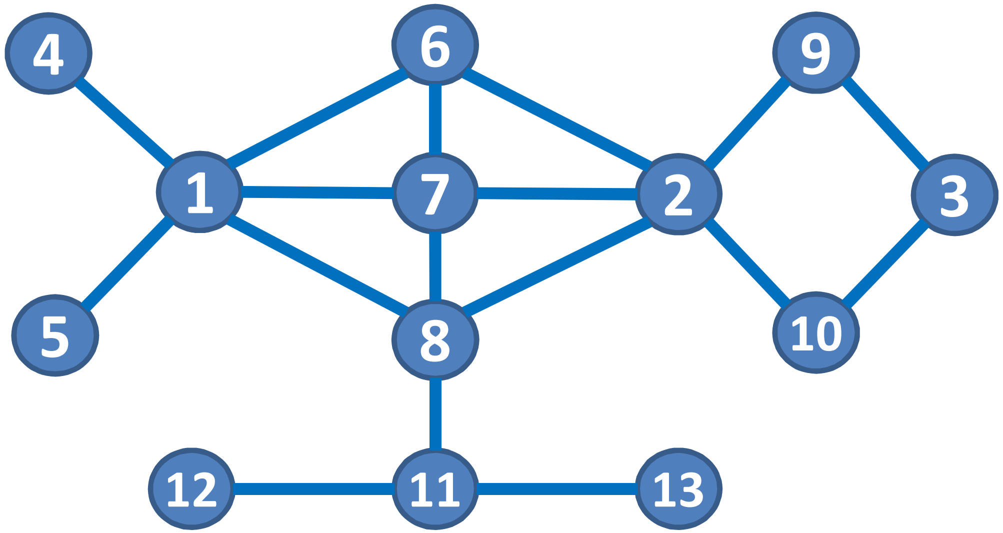
1. **Degree centrality** – how many adjacent edges node v has
2. **Closeness centrality** – how many edges, on overage, one needs to traverse to reach v from other nodes in the network
3. **Betweenness centrality** – what proportion of the shortest paths between any two nodes traverse through node v

A Problem with Standard Measures

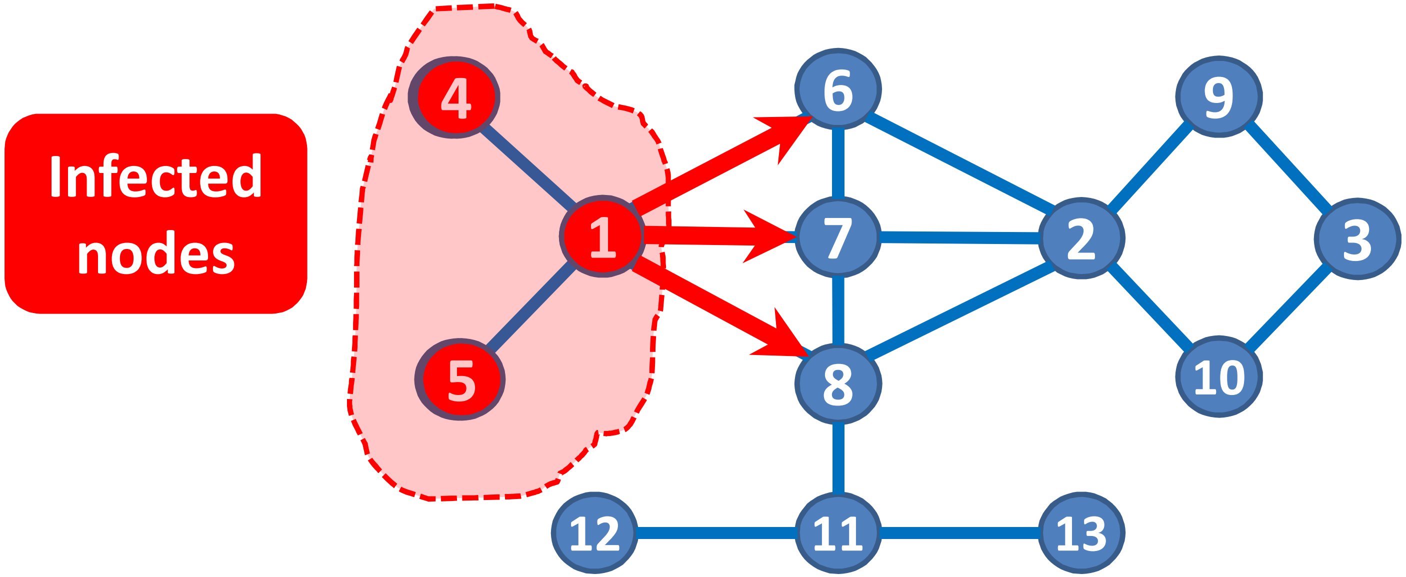


The common feature of all standard centrality measures is that they assess the importance of a node = the role that a node plays by itself

However, they may exist synergies if functioning of the nodes is considered in groups

Epidemiology: who to vaccinate in the society in case of epidemics?

Motivation



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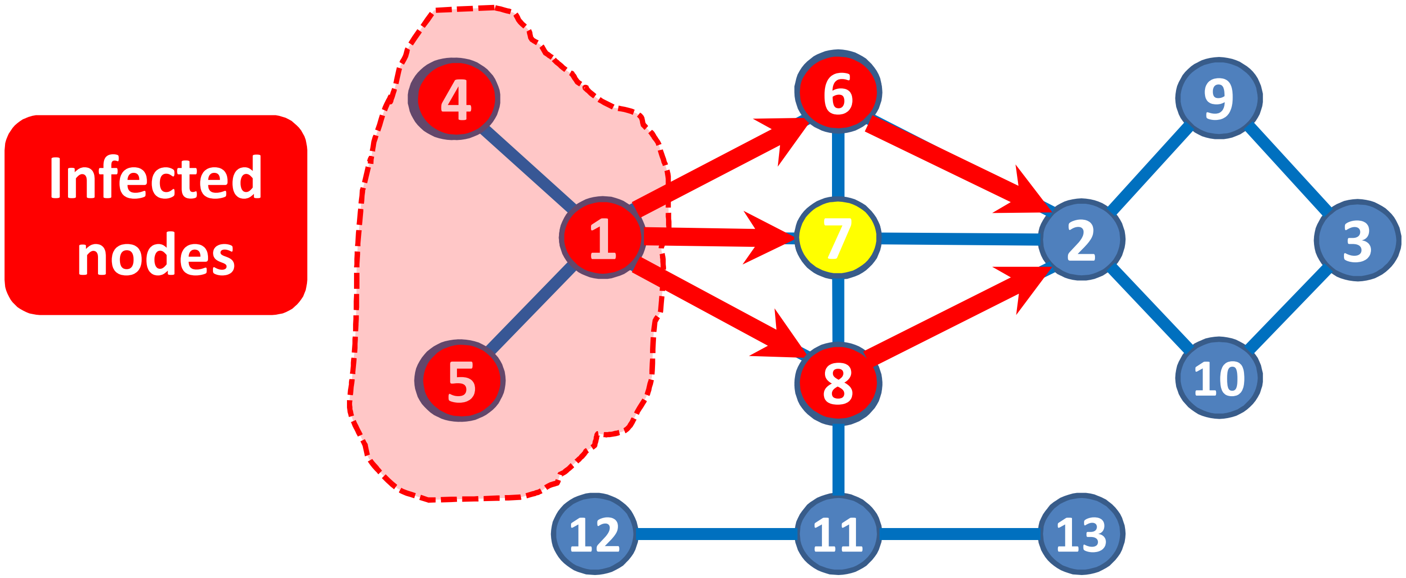

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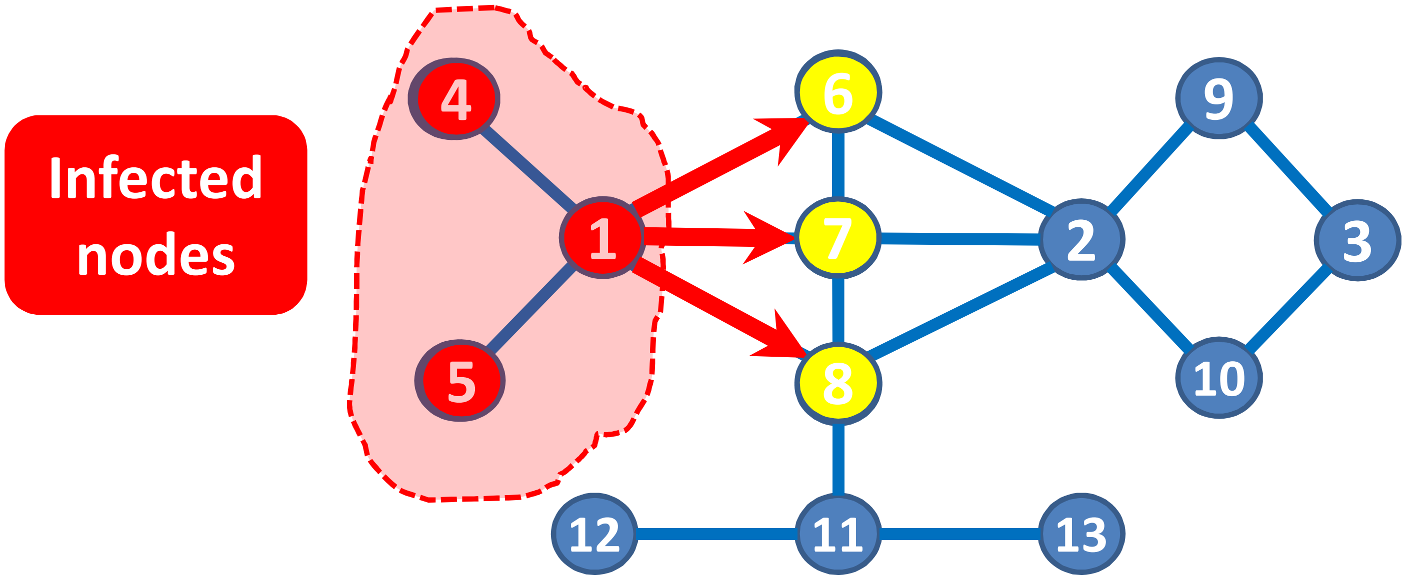
Epidemiology: who to vaccinate in the society in case of epidemics?

If we ask: who can we individually vaccinate to stop the epidemics, we may fail? Vaccinataing v_6 or v_7 or v_8 individually <u>cannot</u> stop the epidemics!

Motivation



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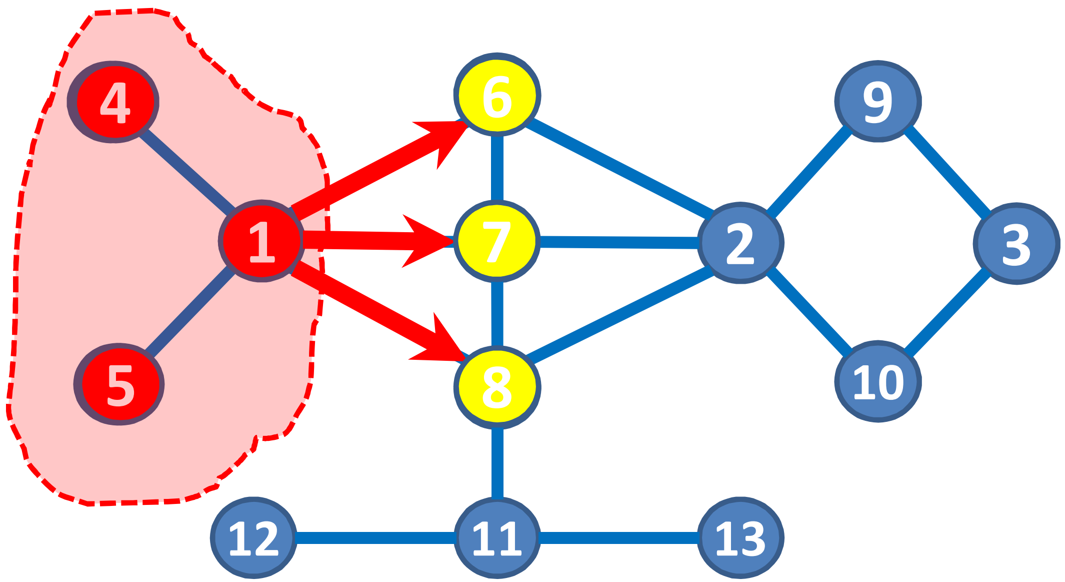
However, they may exist synergies if functioning of the nodes is considered in groups

Epidemiology: who to vaccinate in the society in case of epidemics?

But vaccinating v_6, v_7 and v_8 together can achieve our goal!

Thus, in terms of spread of epidemics these three nodes individually has no value but together they do! ➔ Group Centrality

Group Centrality



Introduced by Everett and Borgatti (1999)

Intuitively, these centralities measure the role player in the network by a given group of nodes (group degree, closeness, betweenness)

It is a nice solution, but it has disadvantages:

- How can we know on which group of nodes we should focus?

- Even if we study all groups of nodes, how can we derive a ranking of individual nodes based on this information?

Game-theoretic centrality: bird's-eye view

Graph Theory

Problem:
how important is a <u>node</u> in the network?

Answer: Centrality Metrics

Game Theory

Problem:
how important is a <u>player</u> in the game?

Answer: A Solution of the Coalitional Game

Game-theoretic centrality metrics

Seminal paper: **Grofman & Owen (1982), A game-theoretic approach to measuring degree of centrality in social networks. Social Networks, 4, 213–224 ➔ Banzhaf index**

Somewhat forgotten…

Key advantages of Game-Theoretic Centrality

1. Game-theoretic centrality takes into account group performance of nodes in a structured way (using extensively studied solution concepts from game theory)

2. The approach is very flexible and can be adapted to particular application
   - ❑ by choosing a game (characteristic function, generalized char. fun., games with externalities, etc.)
   - ❑ by choosing a value function
   - ❑ by choosing a solution concept (SV, BI, Semivalues, MV, etc.)

3. Potential drawback ➔ computation?

Literature Overview

Year	Authors	Features
1982	Grofman & Owen	Banzhaf Index, characteristic function games, all coalitions are feasible

3 Sample Application

Top k-node Problem

Introduced by Domingos and Richardson (2001), ACM SIGKDD.

How to find a set of nodes with an a-priori given cardinality k that can maximize the infor-mation cascade in a viral marketing campaign

The authors proposed some predictive models to show that selecting the right set of users for a marketing campaign can make a big difference.

In an influential paper, Kempe, Kleinberg and Tardos (2003), ACM SIGKDD, showed that the problem is NP-Hard and they proposed greedy approximation algorithm (which is now a standard approach in the literature).

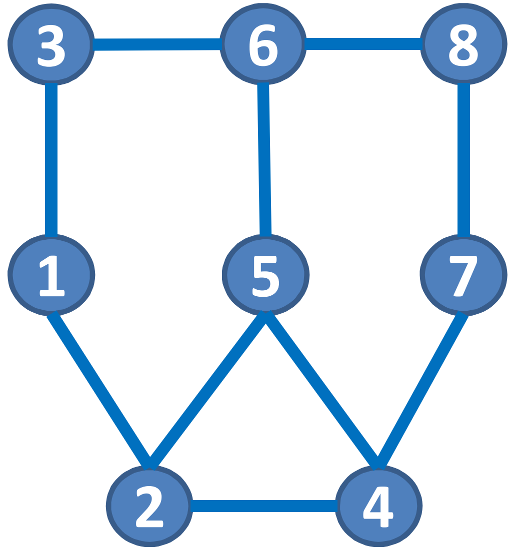
Suri and Narahari (2008,2010) proposed to use the Shapley-value based centrality to more efficiently approximate the k-node problem

We will call the game proposed by them: Game 1

Game 1: #agents at most 1 degree away

Let *C* be an arbitrary coalition of nodes in the graph

The nodes in the coalition do <u>not</u> have to be connected



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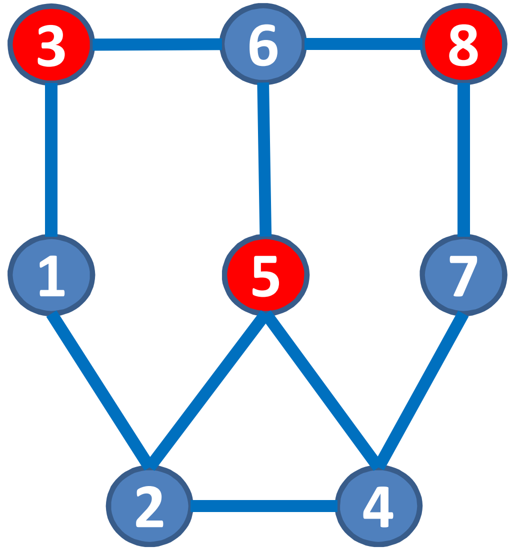
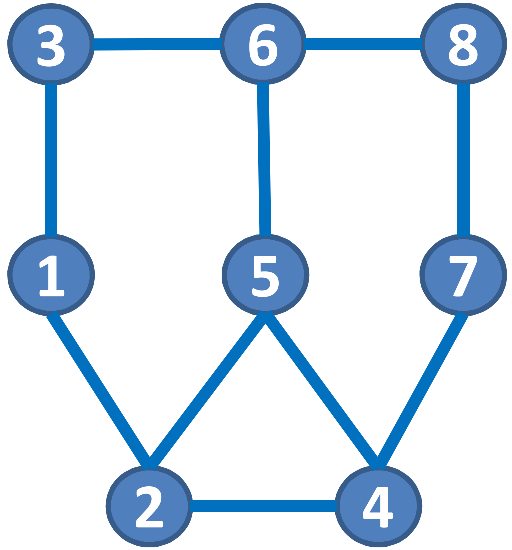


Let *C* be an arbitrary coalition of nodes in the graph
The nodes in the coalition do <u>not</u> have to be connected

$$C = \{v_3, v_5, v_8\}$$

is a valid coalition

Game 1: #agents at most 1 degree away



Let C be an arbitrary coalition of nodes in the graph
The nodes in the coalition do <u>not</u> have to be connected

Definition of the **characteristic function:**

$$v(C) = \begin{cases} 0 & if\ C = \emptyset \\ fringe(C) & otherwise \end{cases}$$

Number of nodes in C
+
number of all their neighbours

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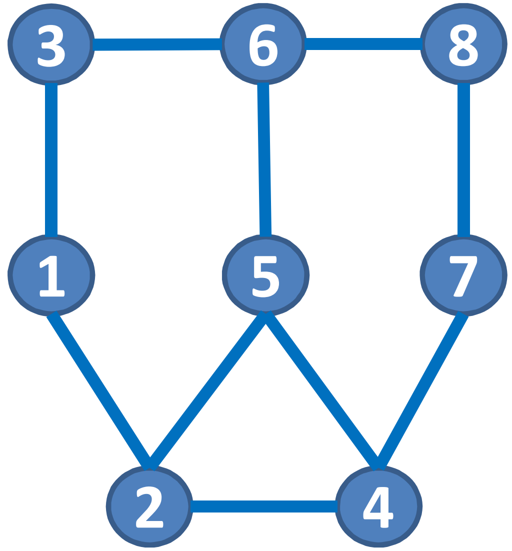
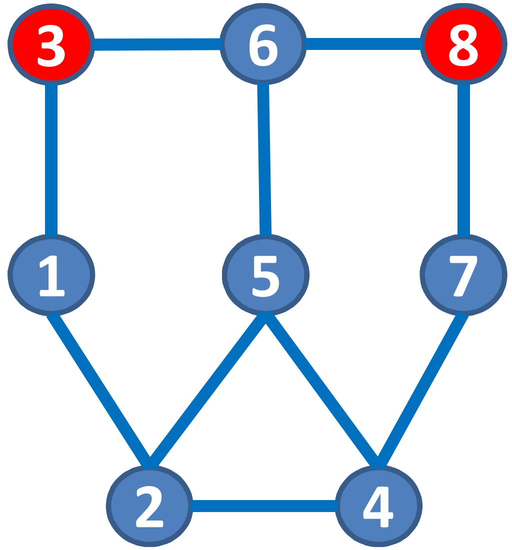


$C = \{v_3, v_8\}$

Game 1: #agents at most 1 degree away



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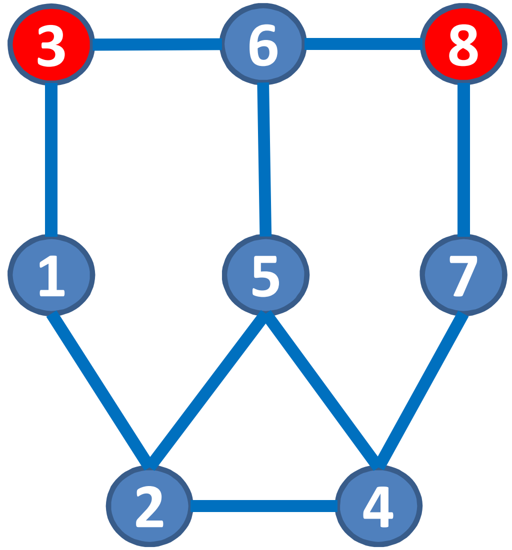
Game 1: #agents at most 1 degree away



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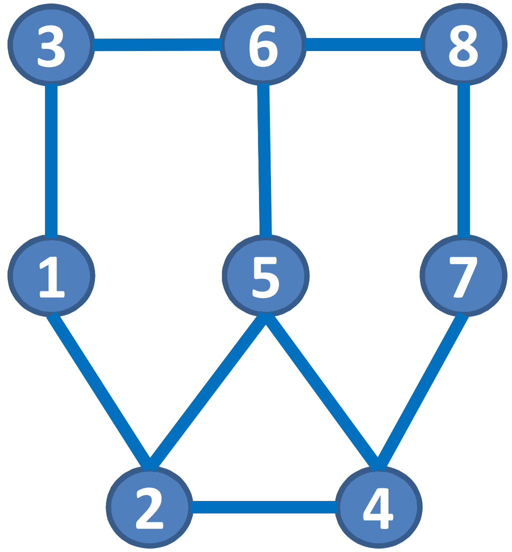
Number of nodes in *C*
+
number of all their neighbours

$$C = \{v_3, v_8\}$$

$$v(C) = |\{v_3, v_8\}| + $$
$$+ |N(\{v_3, v_8\})|$$

Game 1: #agents at most 1 degree away



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Definition of the characteristic function:

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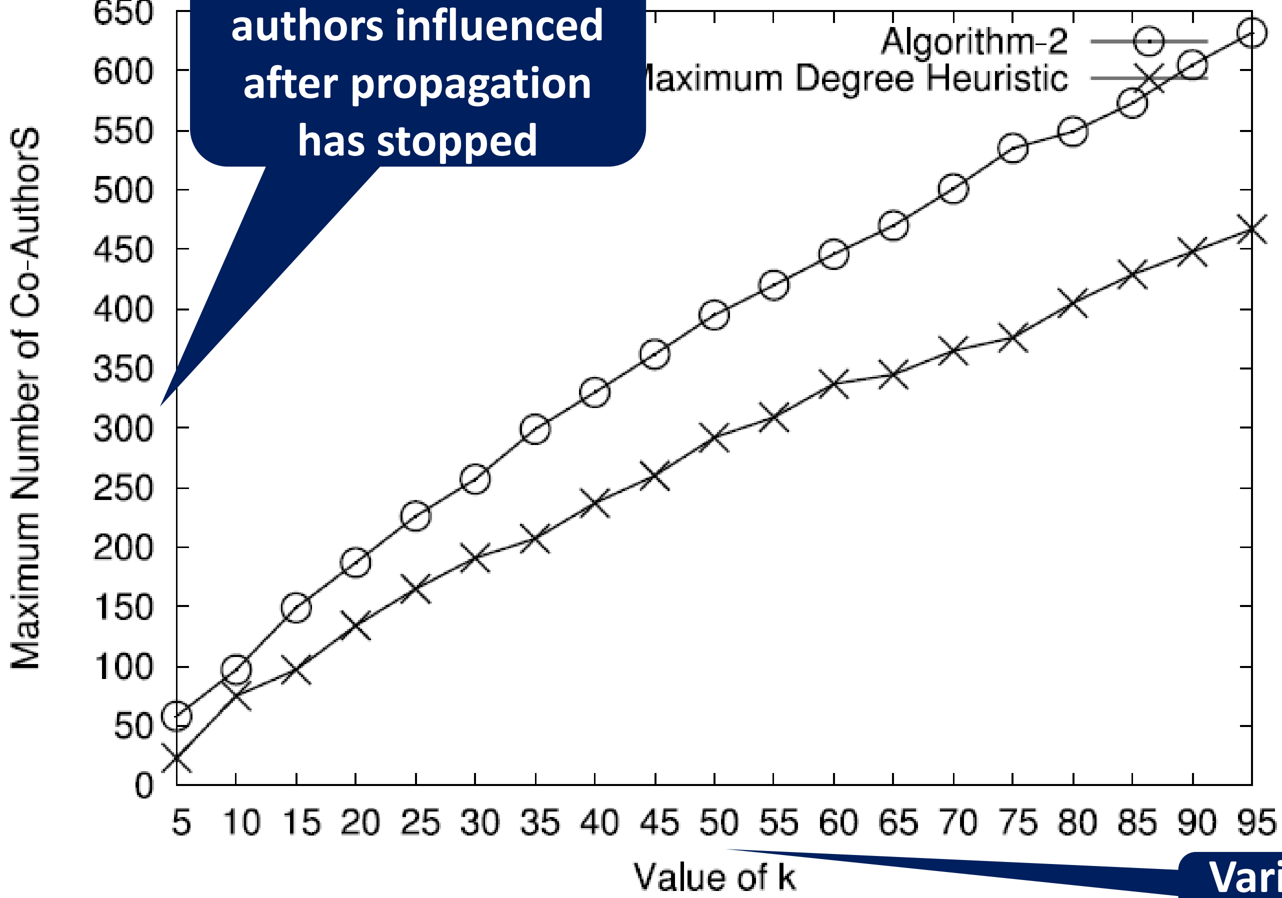
Number of nodes in C
+
number of all their neighbours

$$C = \{v_3, v_8\}$$

$$v(C) = |\{v_3, v_8\}| + \\ +|N(\{v_3,\ v_8\})|$$

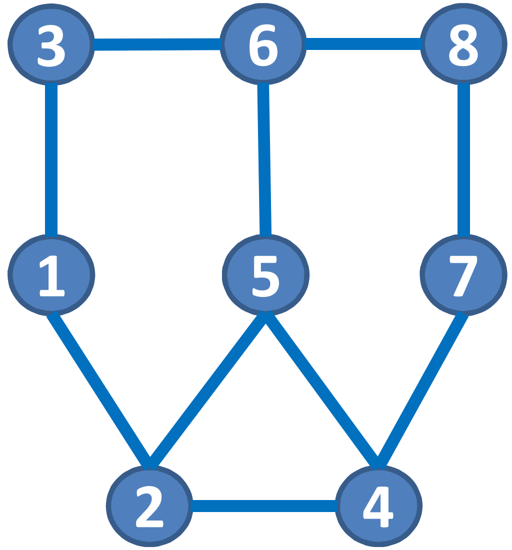
Game 1: #agents at most 1 degree away

Suri and Narahari showed that SV-based approach is superior to well-known Maximum Degree Heuristics



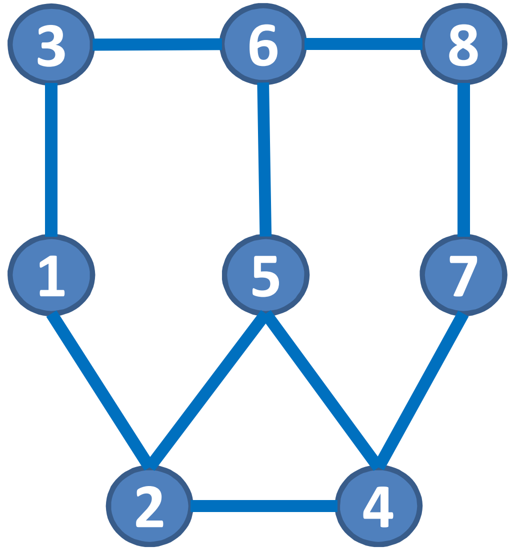
Number of co-authors influenced after propagation has stopped

Various pre-defined values of k



Game 1: #agents at most 1 degree away

How to compute the Shapley value in our game?



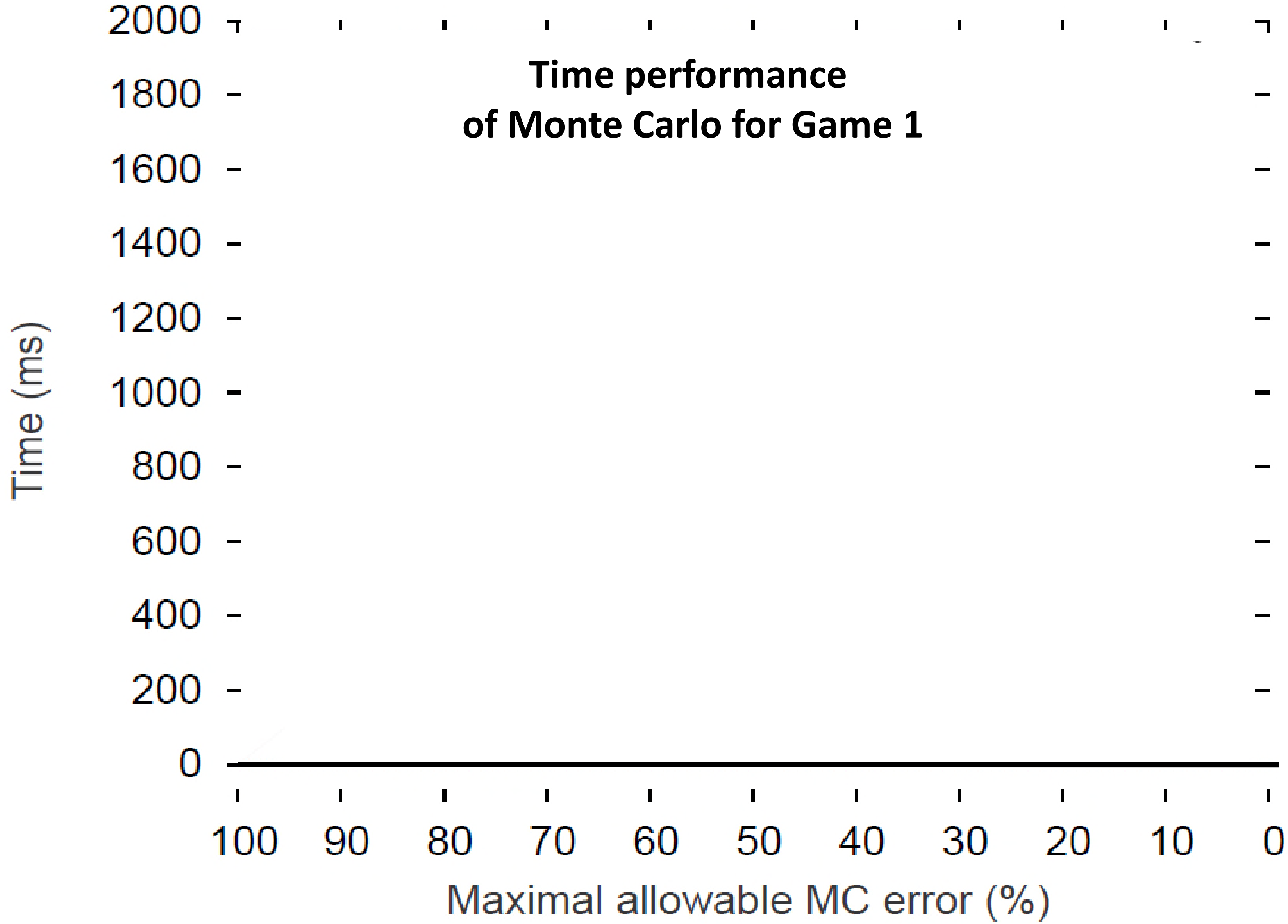
Suri and Narahari (2008, 2010) proposed to use Monte Carlo technique.

How does it perform?

Data for Monte Carlo simulations:

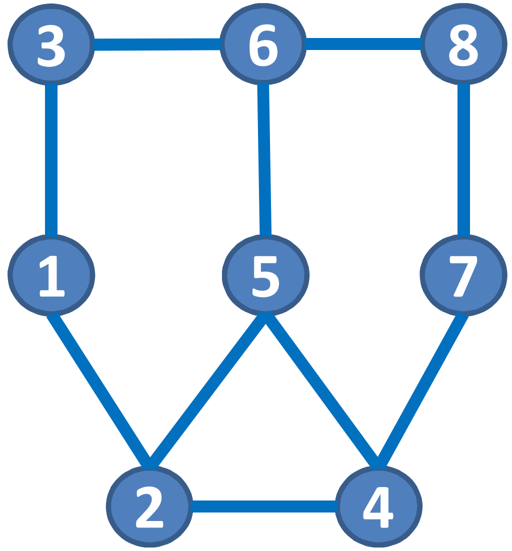
- Western States Power Grid
- 4940 nodes
- 6954 edges

Game 1: #agents at most 1 degree away



Game 1: #agents at most 1 degree away

Can we do any better than the Monte Carlo sampling?



Game 1 is the first game out of 5 considered in Michalak et al. (2013), JAIR (Earlier version Aadithya et al. (2010), WINE)

This games are all about the influence in the network

Before we proceed let us compare computational challenge to representations of coalitional games

Computation of SV-based centrality vs. concise representations

Concise representations:

General idea:

Find a new model of a coalitional game. That is:

- ✓ concise
- ✓ expressive
- ✓ effective
- ✓ simple

vs.

SV-based centrality:

General idea:

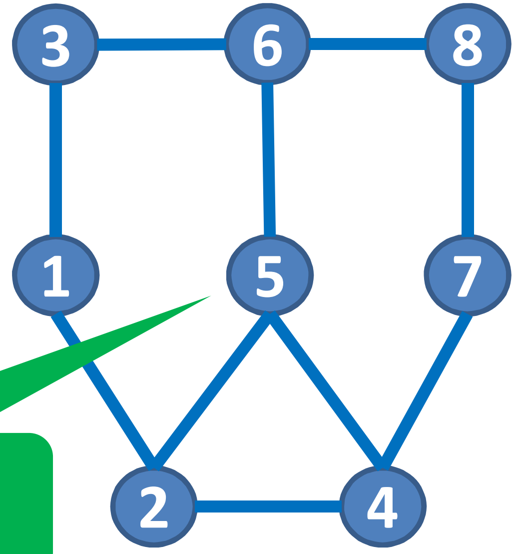
Given
- ✓ the network (a concise model)
- ✓ the definition of the coalitional game = definition of the characteristic function

Find an algorithm to compute the SV

in general, less freedom here

Game 1: #agents at most 1 degree away

Can we do any better than the Monte Carlo sampling?



Let us focus on v_5

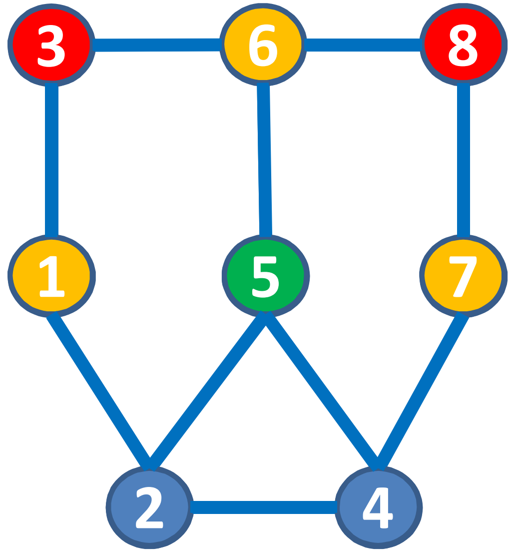
Game 1: #agents at most 1 degree away

The key <u>question</u> to ask is:

What is the necessary and sufficient condition for node v_5 to "marginally contribute" node $v_j \in N(v_5)$ to $fringe(\{v_3, v_8\})$"?

Clearly, this happens if and only if neither v_j nor any of its neighbours are present in *C*.

Thus, v_5 will contribute v_2 and v_4, if he joins $\{v_3, v_8\}$



$C = \{v_3, v_8\}$

Game 1: #agents at most 1 degree away

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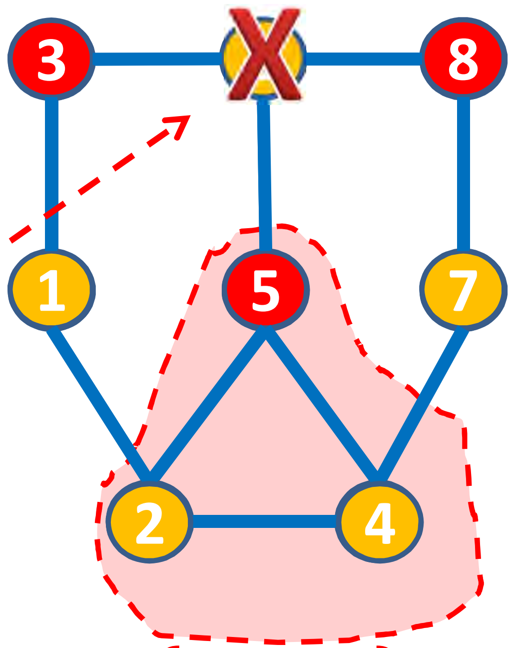
Thus, v_5 will contribute v_2 and v_4, if he joins $\{\boldsymbol{v}_3, \boldsymbol{v}_8\}$

But v_5 does not contribute v_6!



$C = \{\boldsymbol{v}_3, \boldsymbol{v}_8\}$

Let us now find a permutation in which v_5 contributes to fringe of a coalition with v_2

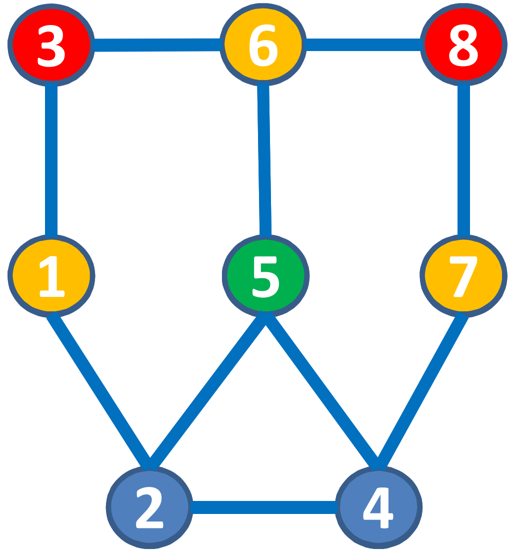
Game 1: #agents at most 1 degree away

Let us consider the following permutation:

Is this one of the permutations we are looking for?
i.e. where v_5 contributes to fringe of *C* (here C $= \{v_3, v_8\}$)
with v_2

YES

Because v_2 and all its neighbours are in the permutation
after v_5 (thus, they are not members of C)

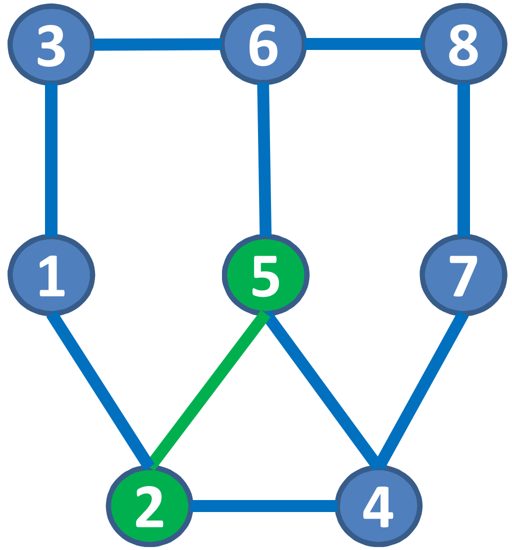


$C = \{v_3, v_8\}$

Let us now compute the number of permutations in which v_5 contributes to any *C* with v_2, i.e. such permutations where v_2 and all its neighbours are after v_5

Game 1: #agents at most 1 degree away

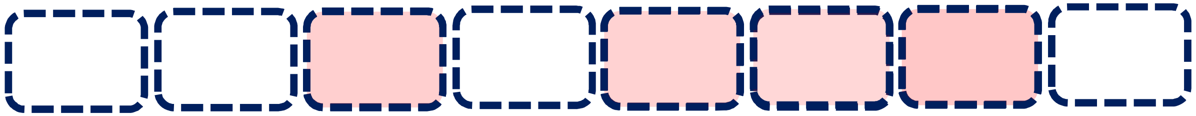
AIM: number of permutations where v_2 and all its neighbours are after v_5



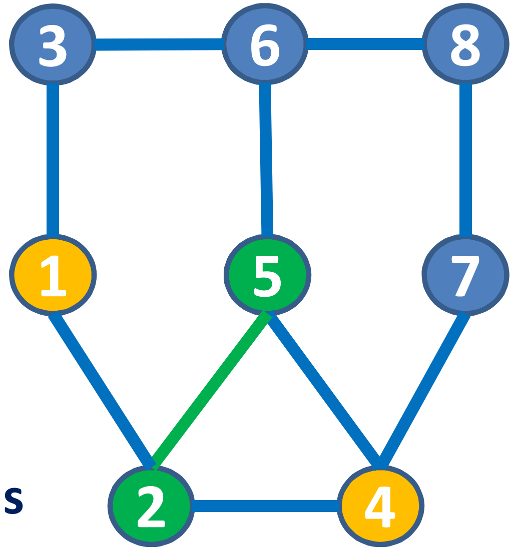
Game 1: #agents at most 1 degree away

AIM: number of permutations where v_2 and all its neighbours are after v_5

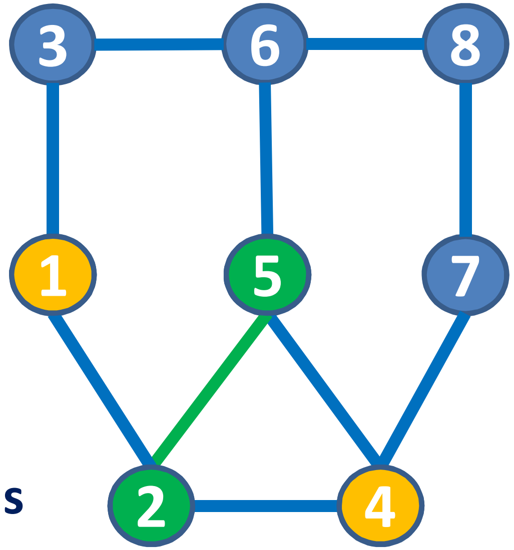
We have 8 agents in any random permutation:



For agents v_5, v_2, v_1, and v_4 we choose randomly 4 positions in the permutation ➔ this can be done in $\binom{8}{4}$ ways



Game 1: #agents at most 1 degree away



AIM: number of permutations where v_2 and all its neighbours are after v_5

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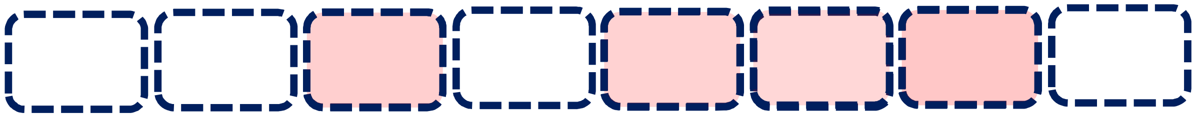


For agents v_5, v_2, v_1, and v_4 we choose randomly 4 positions in the permutation ➔ this can be done in $\binom{8}{4}$ ways

Node v_5 is places first in the selection

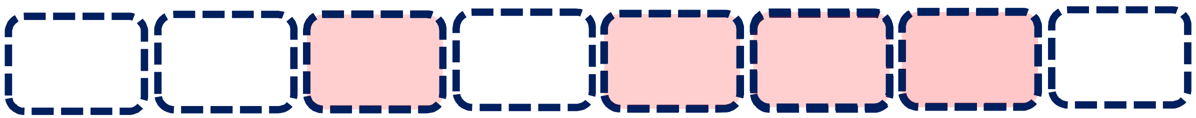
Then we place v_2 and all its neighbours randomly after v_5
➔ this can be done in 3! ways

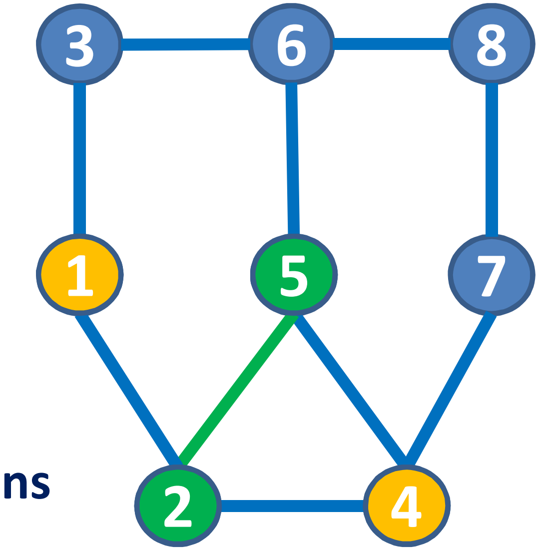
The remaining players can be places in 4! ways

Game 1: #agents at most 1 degree away

AIM: number of permutations where v_2 and all its neighbours are after v_5

We have 8 agents in any random permutation:





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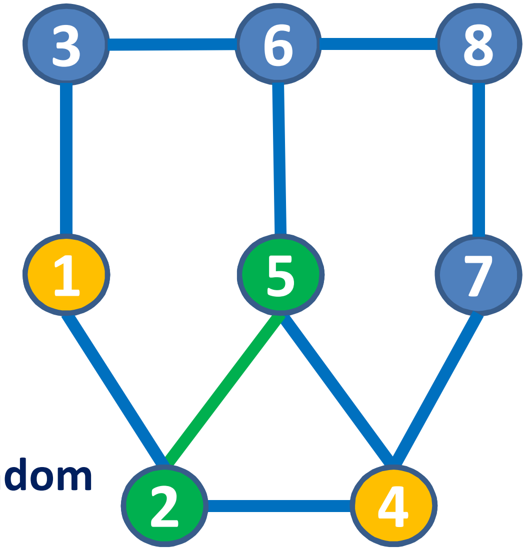
Game 1: #agents at most 1 degree away



General formula:

$\binom{n}{1+\deg(v_j)}$ - for v_j and all its neighbours we choose $1+\ deg(v_j)$ random places among n

$(\deg(v_j))!$ – we place v_i at the first position and v_j with his neighbours randomly later on

$(n-(1+\deg(v_j)))!$ – we arrange the remaining agents at random

Overall, the number of permutations, where v_i contributes to any C with v_j, is:

$$\frac{n!}{1+\deg(v_j)}$$

Thus, the probability that one of such permutations is randomly chosen is:

$$\frac{1}{1+\deg(v_j)} = E(B_{v_i,v_j})$$

Bernoulli random variable that v_i marginally contributes v_j

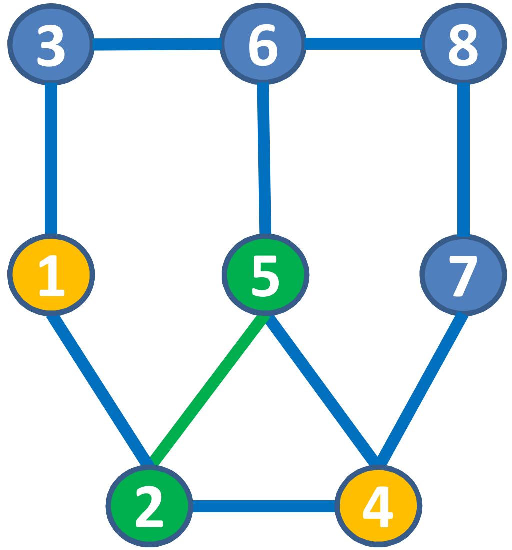
Game 1: #agents at most 1 degree away

Since the Shapley value is the expected marginal contribution of v_i, we have:

$$SV_i(\text{Game 1}) = \sum_{v_j \in \{v_i\} \cup N(v_i)} E\left(B_{v_i, v_j}\right)$$

$$SV_i(\text{Game 1}) = \sum_{v_j \in \{v_i\} \cup N(v_i)} \frac{1}{1 + \deg(v_j)}$$

Running time: $O(|V| + |E|)$



Game 1: #agents at most 1 degree away

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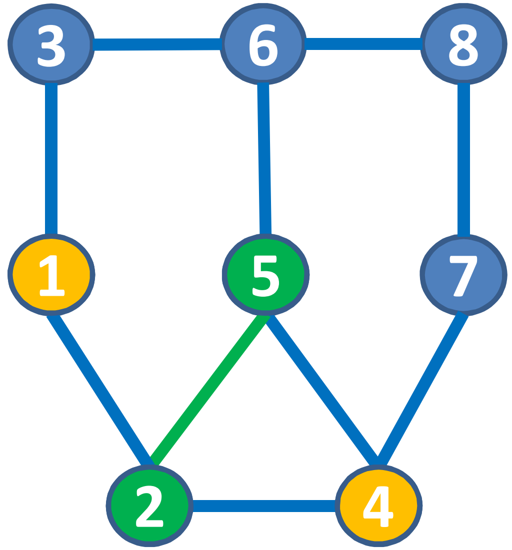

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$$SV_i(\text{Game 1}) = \sum_{v_j \in \{v_i\} \cup N(v_i)} \frac{1}{1 + \deg(v_j)}$$

- It is possible to derive some intuition from the above formula.
- If a node has a high degree the number of terms in $\sum(.)$ above is also high.
- But the terms themselves will be inversely related to the degree of neighboring nodes.
- This gives the intuition that a node will have high centrality not only when its degree is high, but also whenever its degree tends to be higher in comparison to the degree of its neighboring nodes.
- In other words, power comes from being connected to those who are powerless, a fact that is well-recognized by the centrality literature (e.g., Bonacich, 1987).