5 minute talk: The Collatz Conjecture

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Collatz Conjecture

Co to jest?

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2. also called: $3n + 1$ conjecture, Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, Syracuse problem

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Collatz Conjecture

Idea

1. Consider $n \in \mathbb{N}$:
2. with this number, do the following operation:
   - if n is even, divide it by 2 to get $n/2$
   - if n is odd, multiply it by 3 and add 1 to obtain $3n+1$
3. with the number obtained, go to step 2
4. the conjecture says: no matter what $n \in \mathbb{N}$, you will always eventually reach 1

Collatz Conjecture

More formally

Consider $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \bmod 2 \\ 3n + 1 & \text{if } n \equiv 1 \bmod 2 \end{cases}$$

Consider a $n \in \mathbb{N}$ and define sequence

$$a_i = \begin{cases} n & \text{for } i = 0 \\ f(a_{i-1}) & \text{for } i > 0 \end{cases}$$

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- Collatz conjecture: This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

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- Collatz conjecture: This process will eventually reach the number 1, regardless of which positive integer is chosen initially.
- smallest i such that $a_i = 1$: total stopping time of n.
- so conjecture: every n has a well-defined total stopping time
- if, for some n, such an i doesn't exist, we say that n has infinite total stopping time and the conjecture is false.

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- Collatz conjecture: This process will eventually reach the number 1, regardless of which positive integer is chosen initially.
- smallest i such that $a_i = 1$: total stopping time of n.
- so conjecture: every n has a well-defined total stopping time
- if, for some n, such an i doesn't exist, we say that n has infinite total stopping time and the conjecture is false.
- if conjecture false: there is some starting number which gives rise to a sequence that does not contain 1: Such a sequence might enter a repeating cycle that excludes 1, or increase without bound. No such sequence has been found.

Collatz Conjecture

Examples

- start with $n = 6$; one gets the sequence 6, 3, 10, 5, 16, 8, 4, 2, 1.
- start with $n = 19$; we have 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.

Collatz Conjecture

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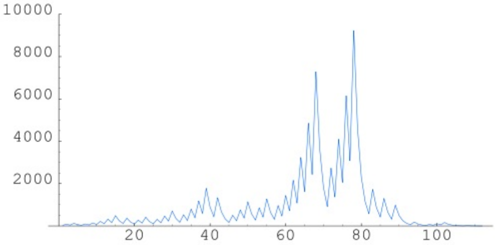
- start with $n = 6$; one gets the sequence 6, 3, 10, 5, 16, 8, 4, 2, 1.
- start with $n = 19$; we have 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.
- The sequence for $n = 27$, takes 111 steps, climbing to 9232 before descending to 1: 27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16,8, 4, 2, 1



Some random facts

1. the longest progression with initial $n < 100$ million is 63,728,127, which has 949 steps.
2. the longest progression with initial $n < 1$ billion is 670,617,279, with 986 steps
3. the longest progression with initial $n < 10$ billion it is 9,780,657,631, with 1132 steps
4. The powers of two converge to one quickly because 2^n is halved n times to reach one, and is never increased

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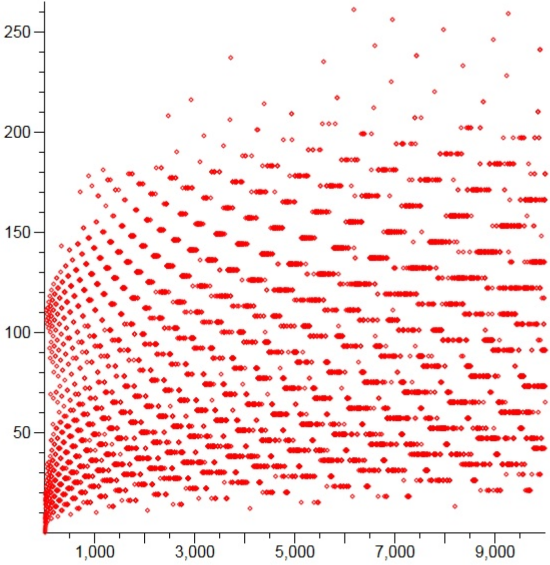
4. The powers of two converge to one quickly because 2^n is halved n times to reach one, and is never increased
5. Experimental evidence: conjecture has been checked by computer for all starting values up to 2^{64}.
6. All initial values tested so far eventually end in the repeating cycle (4; 2; 1), which has only three terms

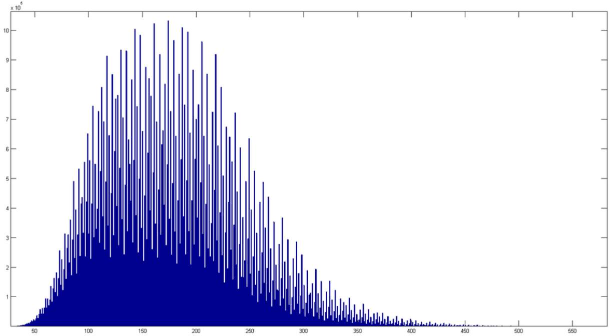
Some random pictures

Figure : Numbers from 1 to 9999 and their corresponding total stopping time



Some random pictures

Figure : Histogram of stopping times for the numbers 1 to 100 million. Stopping time is on the x axis, frequency on the y axis.



Some random pictures

Figure : Directed graph showing the orbits of small numbers under the Collatz map. The Collatz conjecture is equivalent to the statement that all paths eventually lead to 1.

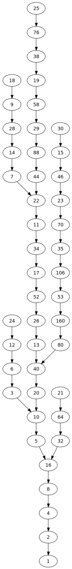


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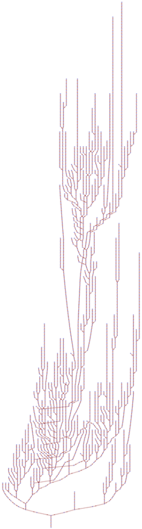
Figure : Directed graph showing the orbits of the first 1000 numbers



Mathematics is not ready for this yet?