Stationary ergodic processes

1 Introduction

Theorem 1 (ergodic theorem) *Let $(X_i)_{i=-\infty}^{\infty}$ be a stationary ergodic process with $\boldsymbol{E}\,|X_0| < \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = \boldsymbol{E}\, X_0$$

holds with probability 1.

Theorem 2 *Let $(X_i)_{i=-\infty}^{\infty}$ be a stationary Markov chain, where $P(X_{i+1} = l|X_i = k) = p_{kl}$, $P(X_i = k) = \pi_k$, and the variables take values in a countable set. These conditions are equivalent:*

1. *Process $(X_i)_{i=-\infty}^{\infty}$ is ergodic.*
2. *There are no two disjoint closed sets of states; a set A of states is called closed if $\sum_{l\in A} p_{kl} = 1$ for each $k \in A$.*
3. *For a given transition matrix (p_{kl}) there exists a unique stationary distribution π_k.*

Theorem 3 *Let $(X_i)_{i=-\infty}^{\infty}$ be a stationary Markov chain with marginal distribution $P(X_i = k) = \pi_k$ and transition probabilities $P(X_{i+1} = l|X_i = k) = p_{kl}$. The entropy rate is*

$$h = -\sum_{kl} \pi_k p_{kl} \log p_{kl}.$$

2 Task

1. Let us consider a few stationary Markov chains defined by the following transition matrices $P(X_{n+1} = j|X_n = i) = p_{ij}$, where

   (a)

   $$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix};$$

   (b)

   $$(p_{ij})_{i,j=1}^{3} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \\ 0 & 3/4 & 1/4 \end{pmatrix};$$

(c)

$$(p_{ij})_{i,j=1}^{4} = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 3/5 & 2/5 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/4 & 3/4 \end{pmatrix}.$$

2. Tell which of these processes are ergodic and which are nonergodic.

3. For the ergodic process(es) estimate numerically the stationary distribution $P(X_1 = i) = \pi_i$, using the fact that for almost any choice of X_1 we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\{X_k = i\} = \pi_i.$$

(To generate values of X_k use a pseudo-random number generator.) Present the resulted probabilities π_i in a report.

4. Using the estimated probabilities π_i compute the entropy rate h of the ergodic process(es).

5. Check numerically how the value of limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}\{X_k = i\}$ depends on the initial value of X_k for the nonergodic process(es).

6. Describe what you have obtained in a report, attach the used scripts, and send it to me (`ldebowsk@ipipan.waw.pl`).