Information Theory and Statistics
Lecture 7: Prefix-free Kolmogorov complexity

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Ph. D. Programme 2013/2014

Project co-financed by the European Union
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Towards algorithmic information theory

- In Shannon's information theory, the amount of information carried by a random variable depends on the ascribed probability distribution.
- Andrey Kolmogorov (1903–1987), the founding father of modern probability theory, remarked that it is extremely hard to imagine a reasonable probability distribution for utterances in natural language but we can still estimate their information content.
- For that reason Kolmogorov proposed to define the information content of a particular string in a purely algorithmic way.
- A similar approach has been proposed a year earlier by Ray Solomonoff (1926–2009), who sought for optimal inductive inference.

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Kolmogorov complexity—analogue of Shannon entropy

- The fundamental object of algorithmic information theory is Kolmogorov complexity of a string, an analogue of Shannon entropy.
- It is defined as the length of the shortest program for a Turing machine such that the machine prints out the string and halts.
- The Turing machine itself is defined as a deterministic finite state automaton which moves along one or more infinite tapes filled with symbols from a fixed finite alphabet and which may read and write individual symbols.
- The concrete value of Kolmogorov complexity depends on the used Turing machine but many properties of Kolmogorov complexity are universal.

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Prefix-free Kolmogorov complexity

- A particular definition of Kolmogorov complexity, called prefix-free Kolmogorov complexity, is convenient to discuss links between Kolmogorov complexity and entropy.
- The fundamental idea is to force that the accepted programs form a prefix-free set.
- Here we will use the construction by Gregory Chaitin (1947–), introduced in 1975.

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Prefix-free Turing machine

We will consider a machine that has two tapes:

1. a bidirectional tape $(\mathbf{X_i})_{i\in\mathbb{Z}}$ filled with symbols $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{B}$,
2. a unidirectional tape $(\mathbf{Y_k})_{k\in\mathbb{N}}$ filled with symbols $\mathbf{0}$ and $\mathbf{1}$.

The machine head moves in both directions along tape $(\mathbf{X_i})_{i\in\mathbb{Z}}$ and only in the direction of growing $\mathbf{k}$ along tape $(\mathbf{Y_k})_{k\in\mathbb{N}}$.
Tape $(\mathbf{X_i})_{i\in\mathbb{Z}}$ is both read and written, tape $(\mathbf{Y_k})_{k\in\mathbb{N}}$ is read only.

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Formal definition

Formally, a Turing machine is defined as a **6**-tuple $\mathbf{T} = (\mathbf{Q}, \mathbf{s}, \mathbf{h}, \mathbf{\Gamma}, \mathbf{B}, \delta)$, where

1. $\mathbf{Q}$ is a finite, nonempty set of states,
2. $\mathbf{s} \in \mathbf{Q}$ is the start state,
3. $\mathbf{h} \in \mathbf{Q}$ is the halt state,
4. $\mathbf{\Gamma} = \{\mathbf{0}, \mathbf{1}\}$ is a finite, nonempty set of symbols,
5. $\mathbf{B} \in \mathbf{\Gamma}$ is the blank symbol,
6. $\delta : \mathbf{Q} \setminus \{\mathbf{h}\} \times \mathbf{\Gamma} \times \mathbf{\Gamma} \cup \{\mathbf{B}\} \to \mathbf{Q} \times \{\mathbf{0}, \mathbf{R}\} \times \mathbf{\Gamma} \cup \{\mathbf{B}\} \times \{\mathbf{L}, \mathbf{0}, \mathbf{R}\}$ is a function called a transition function.

The set of such machines is denoted $\mathcal{S}$.

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From formal description to machine operation

The machine operation is as follows:

- Initially, the machine is in state $\mathbf{s}$ and reads $\mathbf{Y_1}$ and $\mathbf{X_1}$.
- Subsequently, the machine shifts in discrete steps along the tape in the way prescribed by the transition function:

  If the machine is in state $\mathbf{a}$ and reads symbols $\mathbf{y}$ and $\mathbf{x}$ then, for $\delta(\mathbf{a}, \mathbf{y}, \mathbf{x}) = (\mathbf{a'}, \mathbf{M_Y}, \mathbf{x'}, \mathbf{M_X})$, the machine:

  - moves along tape $(\mathbf{Y_i})_{\mathbf{i}\in\mathbb{Z}}$ one symbol to the right or does not move if $\mathbf{M_Y = R}$ or $\mathbf{M_Y = 0}$,
  - writes symbol $\mathbf{x'}$ on tape $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{Z}}$,
  - moves along tape $(\mathbf{X_i})_{\mathbf{i}\in\mathbb{Z}}$ one symbol to the left, to the right or does not move if $\mathbf{M_X = L}$, $\mathbf{M_X = R}$, or $\mathbf{M_X = 0}$,
  - assumes state $\mathbf{a'}$ in the next step.
- This procedure takes place until the machine reaches the halt state $\mathbf{h}$. Then the computation stops.

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Halting on an input

We say that machine $\mathbf{S} \in \mathcal{S}$ halts on input $\mathbf{(p, q)} \in \Gamma^* \times (\Gamma^* \cup \Gamma^{\mathbb{N}})$ and returns $\mathbf{w} \in \Gamma^*$ if:

(A) The head in the start state reads symbols $\mathbf{Y_1}$ and $\mathbf{X_1}$, and the initial state of the tapes is $\mathbf{Y_1^{|p|} = p}$ and $\mathbf{X_1^{|q|+1} = qB}$ or $\mathbf{X_1^{\infty} = q}$ if $\mathbf{q}$ is an infinite sequence. Besides we put $\mathbf{X_i = 0}$ for $\mathbf{i < 1}$ and $\mathbf{i > |q| + 1}$ if $\mathbf{q}$ is finite.

(B) The head in the halt state reads symbols $\mathbf{Y_{|p|}}$ and $\mathbf{X_j}$, and the final state of the tape $\mathbf{(X_i)_{i \in \mathbb{Z}}}$ is $\mathbf{X_j^{|w|+j} = wB}$.

Assuming that condition (A) is satisfied, we write

$$\mathbf{S(p|q)} = \begin{cases} \mathbf{w}, & \text{if condition (B) is satisfied,} \\ \mathbf{\infty}, & \text{if condition (B) is not satisfied for any } \mathbf{w} \in \Gamma^*. \end{cases}$$

It can be easily seen that for a given $\mathbf{q}$ the set of strings $\mathbf{p}$ such that machine $\mathbf{S}$ halts on input $\mathbf{(p, q)}$ is prefix-free. Such strings are called *self-delimiting programs*.

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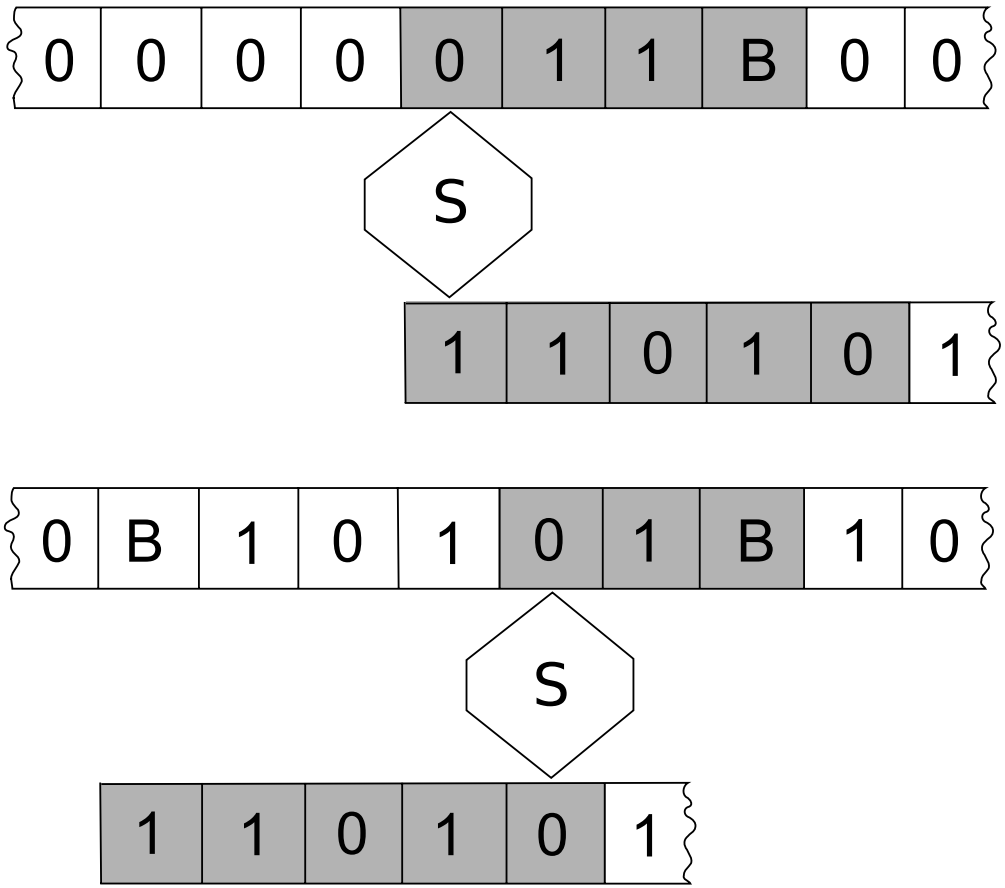
Prefix-free Turing machine (II)



The initial and the final state of machine **S** for $\mathbf{S(11010|011) = 01}$.

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Prefix-free complexity

Definition (prefix-free complexity)

Prefix-free conditional Kolmogorov complexity $\mathbf{K_S(w|q)}$ of a string $\mathbf{w} \in \mathbf{\Gamma^*}$ given string $\mathbf{q} \in \mathbf{\Gamma^*}$ and machine $\mathbf{S} \in \boldsymbol{\mathcal{S}}$ is defined as

$$\mathbf{K_S(w|q)} := \min_{\mathbf{p} \in \mathbf{\Gamma^*}} \{|\mathbf{p}| : \mathbf{S(p|q)} = \mathbf{w}\} .$$

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Universal machines

Definition (universal machine)

Machine $\mathbf{V} \in \boldsymbol{\mathcal{S}}$ is called *universal* if for each machine $\mathbf{S} \in \boldsymbol{\mathcal{S}}$ there exists a string $\mathbf{u} \in \boldsymbol{\Gamma}^*$ such that

$$\mathbf{V(up|q) = S(p|q)}$$

for all $\mathbf{p} \in \boldsymbol{\Gamma}^*$ and $\mathbf{q} \in \boldsymbol{\Gamma}^* \cup \boldsymbol{\Gamma}^{\mathbb{N}}$.

Universal machines exist. The proof is simple but tedious.

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Invariance theorem

Theorem (invariance theorem)

For any two universal machines $\mathbf{V}$ *and* $\mathbf{V'}$ *there exists a constant* $\mathbf{c}$ *such that*

$$|\mathbf{K_V(w|q)} - \mathbf{K_{V'}(w|q)}| \leq \mathbf{c}$$

for any $\mathbf{w} \in \mathbf{\Gamma}^*$ *and* $\mathbf{q} \in \mathbf{\Gamma}^* \cup \mathbf{\Gamma}^{\mathbb{N}}$.

Proof

We have $\mathbf{V(up|q) = V'(p|q)}$ and $\mathbf{V'(u'p|q) = V(p)}$ for certain strings $\mathbf{u}$ and $\mathbf{u'}$. Hence $\mathbf{K_V(w|q) \leq K_{V'}(w|q) + |u|}$ and $\mathbf{K_{V'}(w|q) \leq K_V(w|q) + |u'|}$.

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Prefix-free complexity

Definition (prefix-free complexity II)

Let $\mathbf{V} \in \boldsymbol{\mathcal{S}}$ be a certain universal machine. We put

$$\mathbf{K(w|q) := K_V(w|q).}$$

This quantity is called *prefix-free conditional Kolmogorov complexity*.

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Unconditional complexity

Unconditional complexities are defined as

$$\mathbf{K_S(w) := K_S(w|\lambda)},$$
$$\mathbf{K(w) := K(w|\lambda)},$$

where $\boldsymbol{\lambda}$ is the empty string.

Similarly, we use $\mathbf{V(p) := V(p|\lambda)}$ for any other function $\mathbf{V}$.

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Towards complexity of arbitrary objects

- We want to discuss of objects such as numbers or tuples of numbers and sequences.
- We will define the complexity of such object as the complexity of the corresponding sequence, where the sequence is given by a fixed encoding of the object.

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Complexities of objects defined

- Let $\mathbb{A}$ be a set of discrete objects, such as $\mathbb{A} = \mathbb{Q}^*$, and let $\phi : \mathbb{A} \to \Gamma^*$ be some bijection.
- Additionally, we put $\phi(\infty) := \infty$, where ∞ denotes the indefinite value.
- Let $\mathbb{B}$ be a set of not necessarily discrete objects, such as $\mathbb{B} = \mathbb{R}^{\mathbb{N}}$, and let $\psi : \mathbb{B} \to \Gamma^* \cup \Gamma^{\mathbb{N}}$ be some bijection.
- Additionally, we put $\psi(\infty) := \infty$.
- For a machine $S \in \mathcal{S}$, $a \in \mathbb{A}$, and $b \in \mathbb{B}$, we define

$$S(a|b) := S(\phi(a)|\psi(b)).$$

- The Kolmogorov complexity of objects will be defined as

$$K_S(a|b) := K_S(\phi(a)|\psi(b)),$$
$$K(a|b) := K(\phi(a)|\psi(b)).$$

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Computable discrete functions

The Kolmogorov complexity of a discrete partial function $\mathbf{f} : \mathbb{B} \to \mathbb{A} \cup \{\infty\}$ is defined as

$$\mathbf{K(f)} := \min_{\mathbf{p} \in \Gamma^*} \{|\mathbf{p}| : \forall_{\mathbf{u} \in \mathbb{B}} \mathbf{V(p|u)} = \mathbf{f(u)}\} .$$

Definition (computable discrete function)

A function $\mathbf{f} : \mathbb{B} \to \mathbb{A} \cup \{\infty\}$ is called *computable* if $\mathbf{K(f)} < \infty$.

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Semi-computable real functions

Definition (lower semicomputable real function)

A function $\mathbf{f} : \mathbb{B} \to \mathbb{R}$ is called *lower semicomputable* if there is a computable function $\mathbf{A} : \mathbb{B} \times \mathbb{N} \to \mathbb{Q}$ which satisfies $\mathbf{A(x, k+1)} \geq \mathbf{A(x, k)}$ and

$$\forall_{x \in \mathbb{B}} \lim_{k \to \infty} \mathbf{A(x, k)} = \mathbf{f(x)}.$$

Definition (upper semicomputable real function)

A function $\mathbf{f} : \mathbb{B} \to \mathbb{R}$ is called *upper semicomputable* if there is a computable function $\mathbf{A} : \mathbb{B} \times \mathbb{N} \to \mathbb{Q}$ which satisfies $\mathbf{A(x, k+1)} \leq \mathbf{A(x, k)}$ and

$$\forall_{x \in \mathbb{B}} \lim_{k \to \infty} \mathbf{A(x, k)} = \mathbf{f(x)}.$$

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Computable real functions

Definition (computable real function)

A function $\mathbf{f} : \mathbb{B} \to \mathbb{R}$ is called *computable* if there is a computable function $\mathbf{A} : \mathbb{B} \times \mathbb{N} \to \mathbb{Q}$ which satisfies

$$\forall_{x \in \mathbb{B}} \, |f(x) - A(x, k)| < 1/k.$$

We put

$$K(f) := \min_{A} K(A).$$

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Further notations

We will write $\mathbf{p(x)} \overset{+}{<} \mathbf{q(x)}$ and $\mathbf{p(x)} \overset{+}{>} \mathbf{q(x)}$ if there exists a constant $\mathbf{c}$ such that $\mathbf{p(x)} \leq \mathbf{q(x)} + \mathbf{c}$ and $\mathbf{p(x)} \geq \mathbf{q(x)} + \mathbf{c}$ holds respectively for all $\mathbf{x}$. We will write $\mathbf{p(x)} \overset{+}{=} \mathbf{q(x)}$ when we have both $\mathbf{p(x)} \overset{+}{<} \mathbf{q(x)}$ and $\mathbf{p(x)} \overset{+}{>} \mathbf{q(x)}$.

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A bound for Kolmogorov complexity

Theorem

$$\mathbf{K(w|q)} \stackrel{+}{<} \mathbf{|w| + 2\log|w|}\,.$$

Proof

There is a prefix-free machine **S** which satisfies $\mathbf{S(bw|q) = w}$, where **b** is a prefix-free binary code for $\mathbf{|w|}$. The length of this code can be made as small as $\mathbf{2\log|w|}$. The machine works as follows. First it reads the binary codeword **b** from the unidirectional tape. Then it copies string **w** to the bidirectional tape. After copying the last symbol of **w** the machine halts because it knows its length from reading codeword **b**. Hence we have the desired bound.

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Further inequalities (I)

Theorem

$$\mathbf{K(w|u)} \stackrel{+}{<} \mathbf{K(w)} \stackrel{+}{<} \mathbf{K(\langle u, w\rangle)}.$$

Proof

$\mathbf{K(w|u)} \stackrel{+}{<} \mathbf{K(w)}$ because a certain program that computes $\mathbf{w}$ given $\mathbf{u}$ has form "ignore $\mathbf{u}$ and execute the shortest program that computes $\mathbf{w}$".

$\mathbf{K(w)} \stackrel{+}{<} \mathbf{K(\langle u, w\rangle)}$ because a certain program that computes $\mathbf{w}$ has form "execute the shortest program that computes $\langle \mathbf{u}, \mathbf{w}\rangle$ and compute $\mathbf{w}$ from $\langle \mathbf{u}, \mathbf{w}\rangle$".

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Further inequalities (II)

Theorem

$$\mathbf{K(uw)} \overset{+}{<} \mathbf{K(\langle u, w\rangle)} \overset{+}{<} \mathbf{K(u) + K(w|u)} \overset{+}{<} \mathbf{K(u) + K(w)}.$$

Proof

$\mathbf{K(uw)} \overset{+}{<} \mathbf{K(\langle u, w\rangle)}$ because a certain program that computes $\mathbf{uw}$ has form "execute the shortest program that computes $\langle \mathbf{u}, \mathbf{w}\rangle$ and compute $\mathbf{uw}$ from $\langle \mathbf{u}, \mathbf{w}\rangle$".

Similarly, $\mathbf{K(\langle u, w\rangle)} \overset{+}{<} \mathbf{K(u) + K(w|u)}$ because a certain program that computes $\langle \mathbf{u}, \mathbf{w}\rangle$ has form "execute the shortest program that computes $\mathbf{u}$ and the shortest program that computes $\mathbf{w}$ given $\mathbf{u}$ and from that compute $\langle \mathbf{u}, \mathbf{w}\rangle$".

The last inequality follows from $\mathbf{K(w|u)} \overset{+}{<} \mathbf{K(w)}$.

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Further inequalities (III)

Theorem

$$\mathbf{K(f(w)) \overset{+}{<} K(w) + K(f).}$$

Proof

The inequality follows from the fact that a certain program that computes $\mathbf{f(w)}$ has form "execute the shortest program that computes $\mathbf{w}$ and to the result apply the program that computes $\mathbf{f(w)}$ given $\mathbf{w}$".

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Incompressible strings

A string x_1^n will be called c-*incompressible* if $K(x_1^n) \geq n - c$.
There are infinitely many incompressible strings.

Theorem

There are at least $2^n - 2^{n-c} + 1$ *distinct* c*-incompressible strings of length* n.

Proof

There are at most $2^{n-c} - 1$ programs of length smaller than $n - c$ and there are 2^n strings of length n. Subtracting the latter from the former, we obtain the desired bound.

In particular, there is at least one 0-incompressible string of length n and at least a half of strings of length n is 1-incompressible.

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Halting probability

Lemma

Consider halting probability

$$\Omega = \sum_{\mathbf{p}:\, \mathbf{V}(\mathbf{p}) \neq \infty} 2^{-|\mathbf{p}|}.$$

Let $\Omega_1^{\mathbf{n}}$ *be the first* $\mathbf{n}$ *digits of* Ω *and let* $\mathbf{p}$ *be a string of a length smaller than* $\mathbf{n}$. *Given* $\Omega_1^{\mathbf{n}}$ *we may decide whether machine* $\mathbf{V}$ *stops on input* $\mathbf{p}$.

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Proof of the lemma

We have $\mathbf{0.\Omega_1^n \leq \Omega < 0.\Omega_1^n + 2^{-n}}$. Let us simulate the computation of machine $\mathbf{V}$ on all inputs shorter than $\mathbf{n}$. Namely, in the $\mathbf{i}$-th step we execute the $\mathbf{j}$-th step of computations for all $\mathbf{k}$-th inputs which satisfy $\mathbf{j + k = i}$. In the beginning of the simulations, we set the approximation of $\mathbf{\Omega}$ as $\mathbf{\Omega' := 0}$. When $\mathbf{V}$ halts for a certain input $\mathbf{p}$, we improve the approximation by setting $\mathbf{\Omega' := \Omega' + 2^{-|p|}}$. At a certain instant, $\mathbf{\Omega'}$ becomes equal or greater than $\mathbf{0.\Omega_1^n}$. Then it becomes clear that machine $\mathbf{V}$ will not halt on any other input shorter than $\mathbf{n}$ and we may decide on the halting problem.

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Ω is incompressible

Theorem

We have $\mathbf{K}(\Omega_1^{\mathbf{n}}) \overset{+}{>} \mathbf{n}$.

Proof

By the previous lemma, we can enumerate, given $\Omega_1^{\mathbf{n}}$, all programs shorter than $\mathbf{n}$ for which machine $\mathbf{V}$ halts. For any $\mathbf{w}$ which is not computed by these programs we have $\mathbf{K(w)} > \mathbf{n}$. We can construct a computable function ϕ which computes one of these $\mathbf{w}$ given $\Omega_1^{\mathbf{n}}$. Hence $\mathbf{K}(\phi(\Omega_1^{\mathbf{n}})) \geq \mathbf{n}$, which implies $\mathbf{K}(\Omega_1^{\mathbf{n}}) \geq \mathbf{n} - \mathbf{c}$ for a certain $\mathbf{c}$.

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Incomputability of Kolmogorov complexity

Theorem

Kolmogorov complexity $\mathbf{K(\cdot)}$ *is not a computable function.*

Proof

Assume that there exists a program $\mathbf{q}$ which computes $\mathbf{K(w)}$ for any $\mathbf{w}$. Then there exists a program $\mathbf{p}$ which uses $\mathbf{q}$ as a subroutine to print out the shortest string $\mathbf{w}$ such that $\mathbf{K(w)} > |\mathbf{p}|$. Namely, such a program $\mathbf{p}$ inspects strings $\mathbf{w}$ sorted according to their length, computes $\mathbf{K(w)}$ using subroutine $\mathbf{q}$ and checks whether $\mathbf{K(w)} > |\mathbf{p}|$. It is obvious that this inequality will hold for a certain $\mathbf{w}$ because $\mathbf{K(w)}$ is unbounded. But by the definition of Kolmogorov complexity, we have $\mathbf{K(w)} \leq |\mathbf{p}|$ for the same string. Hence our assumption about the existence of program $\mathbf{q}$ was false.

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Towards information-theoretic Gödel theorem

- A similar search for the shortest element appears in the proof of the information-theoretic Gödel theorem, another fundamental result in the algorithmic information theory.
- A formal inference system is a finite collection of axioms and inference rules. The system is called *consistent* if it is not possible to prove both a statement and its negation, whereas the system is called *sound* if only true propositions can be proved. (Thus a sound system is consistent.)
- According to the information-theoretic Gödel theorem, in any sound formal inference system it is not possible to prove incompressibility of any string which is substantially longer than the definition of this formal system.

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Information-theoretic Gödel theorem

Theorem (information-theoretic Gödel theorem)

For a sound formal inference system, there exists a constant $\mathbf{M}$ *such that propositions "*$\mathbf{K(w) > M}$*" are unprovable in the system.*

Proof

Suppose that for any number $\mathbf{M}$ there exists a proof of proposition "$\mathbf{K(w) > M}$". Then we may construct a program of length of $\mathbf{L}$ which searches all proofs of the formal system to find the first proof that a certain string $\mathbf{w}$ has the complexity greater than $\mathbf{M}$ and then prints out that string. Then we have $\mathbf{K(w) \leq L}$. Since $\mathbf{L} \overset{+}{<} \mathbf{2 \log M}$, we obtain contradiction for sufficiently large $\mathbf{M}$.

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