Information Theory and Statistics
Lecture 3: Stationary ergodic processes

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Measurable space

Definition (measurable space)

Measurable space $(\Omega, \mathcal{J})$ is a pair where Ω is a certain set (called the *event space*) and $\mathcal{J} \subset 2^{\Omega}$ is a σ*-field*. The σ-field $\mathcal{J}$ is an algebra of subsets of Ω which satisfies

- $\Omega \in \mathcal{J}$,
- $A \in \mathcal{J}$ implies $A^c \in \mathcal{J}$, where $A^c := \Omega \setminus A$,
- $A, B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$,
- $A_1, A_2, A_3, \ldots \in \mathcal{J}$ implies $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{J}$.

The elements of $\mathcal{J}$ are called *events*, whereas the elements of Ω are called *elementary events*.

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Probability measure

Definition (probability measure)

Probability measure $P : \mathcal{J} \to [0, 1]$ is a normalized measure, i.e., a function of events that satisfies

- $P(\Omega) = 1$,
- $P(A) \geq 0$ for $A \in \mathcal{J}$,
- $P\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} P(A_n)$ for disjoint events $A_1, A_2, A_3, \ldots \in \mathcal{J}$,

The triple $(\Omega, \mathcal{J}, P)$ is called a *probability space*.

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Invariant measures and dynamical systems

Definition (invariant measure)

Consider a probability space $(\Omega, \mathcal{J}, P)$ and an invertible operation $T : \Omega \to \Omega$ such that $T^{-1}A \in \mathcal{J}$ for $A \in \mathcal{J}$. Measure P is called T-*invariant* and T is called P-*preserving* if

$$P(T^{-1}A) = P(A)$$

for any event $A \in \mathcal{J}$.

Definition (dynamical system)

A dynamical system $(\Omega, \mathcal{J}, P, T)$ is a quadruple that consists of a probability space $(\Omega, \mathcal{J}, P)$ and a P-preserving operation T.

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Stationary processes

Definition (stationary process)

A stochastic process $(X_i)_{i=-\infty}^{\infty}$, where $X_i : \Omega \to \mathbb{X}$ are discrete random variables, is called *stationary* if there exists a distribution of blocks $p : \mathbb{X}^* \to [0, 1]$ such that

$$P(X_{i+1} = x_1, ..., X_{i+n} = x_n) = p(x_1 ... x_n)$$

for each i and n.

Example (IID process)

If variables X_i are independent and have identical distribution $P(X_i = x) = p(x)$ then $(X_i)_{i=-\infty}^{\infty}$ is stationary.

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Invariant measures and stationary processes

Example

Let measure $\mathbf{P}$ be $\mathbf{T}$-invariant and let $\mathbf{X_0} : \mathbf{\Omega} \to \mathbb{X}$ be a random variable on $(\mathbf{\Omega}, \mathcal{J}, \mathbf{P})$. Define random variables $\mathbf{X_i}(\omega) = \mathbf{X_0}(\mathbf{T}^i\omega)$. For

$$\mathbf{A} = (\mathbf{X}_{i+1} = x_1, \dots, \mathbf{X}_{i+n} = x_n)$$

we have

$$\begin{aligned}
\mathbf{T}^{-1}\mathbf{A} &= \left\{\mathbf{T}^{-1}\omega : \mathbf{X}_0(\mathbf{T}^{i+1}\omega) = x_1, \dots, \mathbf{X}_0(\mathbf{T}^{i+n}\omega) = x_n\right\} \\
&= \left\{\omega : \mathbf{X}_0(\mathbf{T}^{i+2}\omega) = x_1, \dots, \mathbf{X}_0(\mathbf{T}^{i+n+1}\omega) = x_n\right\} \\
&= (\mathbf{X}_{i+2} = x_1, \dots, \mathbf{X}_{i+n+1} = x_n).
\end{aligned}$$

Because $\mathbf{P}(\mathbf{T}^{-1}\mathbf{A}) = \mathbf{P}(\mathbf{A})$, process $(\mathbf{X}_i)_{i=-\infty}^{\infty}$ is stationary.

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Process theorem

Theorem (process theorem)

Let distribution of blocks $\mathbf{p} : \mathbb{X}^* \to [0,1]$ *satisfy conditions*

$$\sum_{x \in \mathbb{X}} p(xw) = p(w) = \sum_{x \in \mathbb{X}} p(wx)$$

and $p(\lambda) = 1$, *where* λ *is the empty word. Let event space*

$$\Omega = \left\{ \omega = (\omega_i)_{i=-\infty}^{\infty} : \omega_i \in \mathbb{X} \right\}$$

consist of infinite sequences and introduce random variables $X_i(\omega) = \omega_i$. *Let* $\mathcal{J}$ *be the* σ*-field generated by all cylinder sets* $(X_i = s) = \{\omega \in \Omega : X_i(\omega) = s\}$. *Then there exists a unique probability measure* P *on* $\mathcal{J}$ *that satisfies*

$$P(X_{i+1} = x_1, \dots, X_{i+n} = x_n) = p(x_1 \dots x_n).$$

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Invariant measures and stationary processes

Theorem

Let $(\Omega, \mathcal{J}, P)$ be the probability space constructed in the process theorem. Measure P is T-invariant for the operation

$$(T\omega)_i := \omega_{i+1},$$

called shift. Moreover, we have $X_i(\omega) = X_0(T^i\omega)$.

Proof

By the π-λ theorem it suffices to prove $P(T^{-1}A) = P(A)$ for $A = (X_{i+1} = x_1, ..., X_{i+n} = x_n)$. But $X_i(\omega) = X_0(T^i\omega)$. Hence $T^{-1}A = (X_{i+2} = x_1, ..., X_{i+n+1} = x_n)$. In consequence, we obtain $P(T^{-1}A) = P(A)$ by stationarity.

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Generated dynamical system

Definition

The triple $(\Omega, \mathcal{J}, P)$ and the quadruple $(\Omega, \mathcal{J}, P, T)$ constructed in the previous two theorems will be called the *probability space* and the *dynamical system generated* by a stationary process $(X_i)_{i=-\infty}^{\infty}$ (with a given block distribution $p : \mathbb{X}^* \to [0, 1]$).

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Markov processes

Theorem

A Markov chain $(X_i)_{i=-\infty}^{\infty}$ *is stationary if and only if it has marginal distribution* $P(X_i = k) = \pi_k$ *and transition probabilities* $P(X_{i+1} = l|X_i = k) = p_{kl}$ *which satisfy*

$$\pi_l = \sum_k \pi_k p_{kl}.$$

Matrix (p_{kl}) is called the transition matrix.

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Proof

If $(X_i)_{i=-\infty}^{\infty}$ is stationary then marginal distribution $P(X_i = k)$ and transition probabilities $P(X_{i+1} = l|X_i = k)$ may not depend on i. We also have

$$P(X_{i+1} = k_1, ..., X_{i+n} = k_n) = p(k_1...k_n) := \pi_{k_1} p_{k_1 k_2} ... p_{k_{n-1} k_n}$$

Function $p(k_1...k_n)$ satisfies $\sum_x p(xw) = p(w) = \sum_x p(wx)$. Hence we obtain $\pi_l = \sum_k \pi_k p_{kl}$. On the other hand, if $P(X_i = k) = \pi_k$ and $P(X_{i+1} = l|X_i = k) = p_{kl}$ hold with $\pi_l = \sum_k \pi_k p_{kl}$ then function $p(k_1...k_n)$ satisfies $\sum_x p(xw) = p(w) = \sum_x p(wx)$ and the process is stationary.

For a given transition matrix the stationary distribution may not exist or there may be many stationary distributions.

Example

Let variables X_i assume values in natural numbers and let $P(X_{i+1} = k + 1 | X_i = k) = 1$. Then the process $(X_i)_{i=1}^{\infty}$ is not stationary. Indeed, assume that there is a stationary distribution $P(X_i = k) = \pi_k$. Then we obtain $\pi_{k+1} = \pi_k$ for any k. Such distribution does not exist if there are infinitely many k.

Example

For the transition matrix

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we may choose

$$\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} = \begin{pmatrix} a & 1-a \end{pmatrix}, \qquad a \in [0, 1].$$

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Block entropy

Definition (block)

Blocks of variables are written as $X_k^l = (X_i)_{k \le i \le l}$.

Definition (block entropy)

The entropy of the block of n variables drawn from a stationary process will be denoted as

$$H(n) := H(X_1^n) = H(X_1, ..., X_n) = H(X_{i+1}, ..., X_{i+n}).$$

For convenience, we also put $H(0) = 0$.

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Block entropy (continued)

Theorem

Let Δ *be the difference operator,* $\Delta F(n) := F(n) - F(n-1)$. *Block entropy satisfies*

$$\Delta H(n) = H(X_n|X_1^{n-1}),$$
$$\Delta^2 H(n) = -I(X_1; X_n|X_2^{n-1}),$$

where $H(X_1|X_1^0) := H(X_1)$ *and* $I(X_1; X_2|X_2^1) := I(X_1; X_2)$.

Remark: Hence, for any stationary process, block entropy $H(n)$ is nonnegative ($H(n) \geq 0$), nondecreasing ($\Delta H(n) \geq 0$) and concave ($\Delta^2 H(n) \leq 0$).

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Proof

We have

$$
\begin{aligned}
H(X_n|X_1^{n-1}) &= H(X_1^n) - H(X_1^{n-1}) \\
&= H(n) - H(n-1) = \Delta H(n), \\
-I(X_1; X_n|X_2^{n-1}) &= H(X_1^n) - H(X_1^{n-1}) - H(X_2^n) + H(X_2^{n-1}) \\
&= H(n) - 2H(n-1) + H(n-2) = \Delta^2 H(n).
\end{aligned}
$$

Entropy rate

Definition (entropy rate)

The *entropy rate* of a stationary process will be defined as

$$\mathbf{h} = \lim_{n\to\infty} \Delta H(n) = H(1) + \sum_{n=2}^{\infty} \Delta^2 H(n).$$

By the previous theorem, we have $0 \le h \le H(1)$.

Example

Let $(X_i)_{i=-\infty}^{\infty}$ be a stationary Markov chain with marginal distribution $P(X_i = k) = \pi_k$ and transition probabilities $P(X_{i+1} = l | X_i = k) = p_{kl}$. We have $\Delta H(n) = H(X_n | X_1^{n-1}) = H(X_n | X_{n-1})$, so

$$h = -\sum_{kl} \pi_k p_{kl} \log p_{kl}.$$

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Entropy rate (continued)

Theorem

Entropy rate satisfies equality

$$h = \lim_{n\to\infty} \frac{H(n)}{n}.$$

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Proof

Difference $\Delta H(\cdot)$ is nonincreasing. Hence block entropy $H(n) = H(m) + \sum_{k=m+1}^{n} \Delta H(k)$ satisfies inequalities

$$H(m) + (n - m) \cdot \Delta H(n) \le H(n) \le H(m) + (n - m) \cdot \Delta H(m). \qquad (1)$$

Putting $m = 0$ in the left inequality in (1), we obtain

$$\Delta H(n) \le H(n)/n \qquad (2)$$

Putting $m = n - 1$ in the right inequality in (1), we hence obtain $H(n) \le H(n-1) + \Delta H(n-1) \le H(n-1) + H(n-1)/(n-1)$. Thus $H(n)/n \le H(n-1)/(n-1)$. Because function $H(n)/n$ is nonincreasing, the limit $h' := \lim_{n\to\infty} H(n)/n$ exists. By (2), we have $h' \ge h$. Now we will prove the converse. Putting $n = 2m$ in the right inequality in (1) and dividing both sides by m we obtain $2h' \le h' + h$ in the limit. Hence $h' \le h$.

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Concepts for the definition of expectation

Let us write $\mathbf{1}\{\phi\} = \mathbf{1}$ if proposition ϕ is true and $\mathbf{1}\{\phi\} = \mathbf{0}$ if proposition ϕ is false. The characteristic function of a set $\mathbf{A}$ is defined as

$$\mathbf{I}_{\mathbf{A}}(\omega) := \mathbf{1}\{\omega \in \mathbf{A}\}.$$

The supremum $\mathbf{sup}_{\mathbf{a}\in\mathbf{A}}\, \mathbf{a}$ is defined as the least real number $\mathbf{r}$ such that $\mathbf{r} \geq \mathbf{a}$ for all $\mathbf{a} \in \mathbf{A}$. On the other hand, infimum $\mathbf{inf}_{\mathbf{a}\in\mathbf{A}}\, \mathbf{a}$ is the largest real number $\mathbf{r}$ such that $\mathbf{r} \leq \mathbf{a}$ for all $\mathbf{a} \in \mathbf{A}$.

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Expectation

Definition (expectation)

Let $\mathbf{P}$ be a probability measure. For a discrete random variable $\mathbf{X} \geq \mathbf{0}$, the *expectation* (integral, or average) is defined as

$$\int \mathbf{X} d\mathbf{P} := \sum_{\mathbf{x}:\mathbf{P}(\mathbf{X}=\mathbf{x})>0} \mathbf{P}(\mathbf{X} = \mathbf{x}) \cdot \mathbf{x}.$$

For a real random variable $\mathbf{X} \geq \mathbf{0}$, we define

$$\int \mathbf{X} d\mathbf{P} := \sup_{\mathbf{Y} \leq \mathbf{X}} \int \mathbf{Y} d\mathbf{P},$$

where the supremum is taken over all discrete variables $\mathbf{Y}$ that satisfy $\mathbf{Y} \leq \mathbf{X}$.

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Expectation II

Definition (expectation)

Integrals over subsets are defined as

$$\int_A X dP := \int X I_A dP.$$

For random variables that assume negative values, we put

$$\int X dP := \int_{X>0} X dP - \int_{X<0} (-X) dP,$$

unless both terms are infinite. A more frequent notation is

$$E\, X \equiv E_P X \equiv \int X dP,$$

where we suppress the index P in $E_P X$ for probability measure P.

Invariant algebra

Definition (invariant algebra)

Let $(\Omega, \mathcal{J}, P, T)$ be a dynamical system. The set of events which are invariant with respect to operation T,

$$\mathcal{I} := \left\{A \in \mathcal{J} : A = T^{-1}A\right\},$$

will be called the *invariant algebra*.

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Examples of invariant events

Example

Let $(\Omega, \mathcal{J}, P, T)$ be the dynamical system generated by a stationary process $(X_i)_{i=-\infty}^{\infty}$, where $X_i : \Omega \to \{0, 1\}$. The operation T results in shifting variables X_i, i.e., $X_i(\omega) = X_0(T^i\omega)$. Thus these events belong to the invariant algebra $\mathcal{I}$:

$$(X_i = 1 \text{ for all } i) = \bigcap_{i=-\infty}^{\infty} (X_i = 1),$$

$$(X_i = 1 \text{ for infinitely many } i \geq 1) = \bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty}(X_j = 1),$$

$$\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = a\right) = \bigcap_{p=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - a\right| \leq \frac{1}{p}\right).$$

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Ergodic processes

If the process $(X_i)_{i=-\infty}^{\infty}$ is a sequence of independent identically distributed variables then the probability of the mentioned events is 0 or 1. Following our intuition for independent variables, we may think that a stationary process is well-behaved if the probability of invariant events is 0 or 1.

Definition (ergodicity)

A dynamical system $(\Omega, \mathcal{J}, P, T)$ is called *ergodic* if any event from the invariant algebra has probability 0 or 1, i.e.,

$$A \in \mathcal{I} \implies P(A) \in \{0, 1\}.$$

Analogously, we call a stationary process $(X_i)_{i=-\infty}^{\infty}$ *ergodic* if the dynamical system generated by this process is ergodic.

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Ergodic theorem

We say that Φ holds with probability 1 if $P(\{\omega : \Phi(\omega) \text{ is true}\}) = 1$.

In 1931, Georg David Birkhoff (1884–1944) showed this fact:

Theorem (ergodic theorem)

Let $(\Omega, \mathcal{J}, P, T)$ be a dynamical system and define stationary process $X_i(\omega) := X_0(T^i\omega)$ for a real random variable X_0 on the probability space $(\Omega, \mathcal{J}, P)$. The dynamical system is ergodic if and only if for any real random variable X_0 where $E\,|X_0| < \infty$ equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = E\,X_0$$

holds with probability 1.

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Ergodicity in information theory

In information theory, we often invoke the ergodic theorem in the following way. Namely, for a stationary ergodic process $(X_i)_{i=-\infty}^{\infty}$, with probability 1 we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[-\log P(X_k|X_{k-m}^{k-1}) \right] = E \left[-\log P(X_0|X_{-m}^{-1}) \right]$$
$$= H(X_0|X_{-m}^{-1}).$$

This equality holds since $P(X_k|X_{k-m}^{k-1})$ is a random variable on the probability space generated by process $(X_i)_{i=-\infty}^{\infty}$.

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An example of a nonergodic process

Example (nonergodic process)

Let $(U_i)_{i=-\infty}^{\infty}$ and $(W_i)_{i=-\infty}^{\infty}$ be independent stationary ergodic processes having different distributions and let an independent variable Z have distribution $P(Z = 0) = p \in (0, 1)$ and $P(Z = 1) = 1 - p$. We will consider process $(X_i)_{i=-\infty}^{\infty}$, where

$$X_i = 1\{Z = 0\}U_i + 1\{Z = 1\}W_i.$$

Assume that $P(U_1^p = w) \neq P(W_1^p = w)$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1\left\{X_k^{k+p-1} = w\right\} = 1\{Z = 0\}P(U_1^p = w) + 1\{Z = 1\}P(W_1^p = w),$$

which is not constant. Hence $(X_i)_{i=-\infty}^{\infty}$ is not ergodic.

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Ergodic Markov chains

Theorem

Let $(X_i)_{i=-\infty}^{\infty}$ *be a stationary Markov chain, where* $P(X_{i+1} = l|X_i = k) = p_{kl}$, $P(X_i = k) = \pi_k$, *and the variables take values in a countable set. These conditions are equivalent:*

1. *Process* $(X_i)_{i=-\infty}^{\infty}$ *is ergodic.*
2. *There are no two disjoint closed sets of states; a set* A *of states is called closed if* $\sum_{l \in A} p_{kl} = 1$ *for each* $k \in A$.
3. *For a given transition matrix* (p_{kl}) *there exists a unique stationary distribution* π_k.

Proof (1)⇒(2): Suppose that there are two disjoint closed sets of states A and B. Then we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1\{X_k \in A\} = 1\{X_1 \in A\} \neq \text{const} .$$

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Examples of ergodic processes

Example

A sequence of independent identically distributed random variables is an ergodic process.

Example

Consider a stationary Markov chain $(X_i)_{i=-\infty}^{\infty}$ where $P(X_{n+1} = j|X_n = i) = p_{ij}$ and $P(X_1 = i) = \pi_i$. It is ergodic for

$$\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}, \qquad \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and nonergodic for

$$\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} = \begin{pmatrix} a & 1-a \end{pmatrix}, \qquad \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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