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PROBABILISTIC MODELS OF RANDOM BEHAVIOURS OF CONCURRENT SYSTEMS [1]

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PROBABILISTIC MODELS
OF RANDOM BEHAVIOURS OF CONCURRENT SYSTEMS

Abstract

The paper presents a theoretical basis for describing and analysing random behaviours of concurrent systems of a broad class.

Keywords

Concurrent system, state, process, composition, labelled poset, pomset, directed complete partial order, Scott topology, Borel σ-algebra of sets, probability measure.

MODELE PROBABILISTYCZNE
LOSOWYCH ZACHOWAŃ SYSTEMÓW WSPÓŁBIEŻNYCH

Streszczenie

Praca zawiera podstawy teoretyczne opisu i analizy losowych zachowań systemów współbieżnych dowolnej natury.

Słowa kluczowe

System współbieżny, stan, proces, składanie, etykietowany poset (zbiór częściowo uporządkowany), częściowo uporządkowany wielozbiór (pomset), skierowany zupełny porządek częściowy, topologia Scotta, Borelowska σ-algebra zbiorów, miara probabilistyczna.

1 Introduction

Faulty computer systems, some production systems controlled by automata, some communication systems, and the like, may show random behaviours. In order to characterize such behaviours it is necessary to define for each system an adequate probability space.

The definition of probability spaces characterizing random behaviours is relatively obvious for sequential systems since runs of such systems and segments of runs can be identified with paths of the corresponding transition systems, and branching of paths at states represents always a choice. It is less obvious for concurrent systems since in such systems branching paths may represent segments of the same run and, consequently, branching at states does not necessarily represent a choice.

1.1. Example. Consider two independent machines M_1 and M_2, each machine working as shown in figure 1.1, the machine M_1 executing each of actions α and β with probability 0.5 and synchronizing with the machine M_2 by executing together with it the action represented as γ. These machines form together a system M represented by the transition system in figure 1.2.

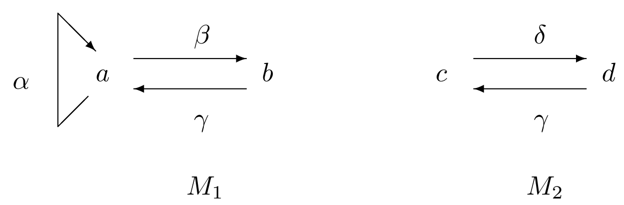
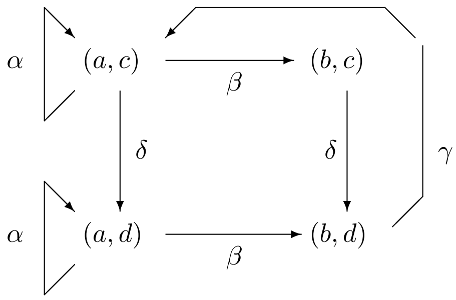


Figure 1.1



M

Figure 1.2

In the system M the paths $(a, c) \xrightarrow{\alpha} (a, c) \xrightarrow{\delta} (a, d)$ and $(a, c) \xrightarrow{\delta} (a, d) \xrightarrow{\alpha} (a, d)$ represent the same initial segment of a run of this system. Consequently, branching at (a, c) does not represent a choice. Similarly, the paths $(a, c) \xrightarrow{\beta} (b, c) \xrightarrow{\delta} (b, d)$ and $(a, c) \xrightarrow{\delta} (a, d) \xrightarrow{\beta} (b, d)$ represent the same initial segment of a run. Consequently, branching at (a, c) does not represent a choice. In particular, the probabilities of transitions from this state to other states need not to sum up to 1, as it really happens. $\sharp$

Sometimes the difficulties of this type can be overcome by representing a concurrent system as collection of sequential modules, each module with its own probabilistic choice of transitions, and by identifying each run of entire system as a sequence of interleaved transitions of its modules (see [3], [5], [6], [9]). However, this is possible only for discrete systems.

An idea of defining probability spaces characterizing random behaviours of concurrent systems in terms of runs rather than in terms of paths of the respective transition systems has been described in [4]. It exploits the fact that a probability measure on a continuous directed complete poset is determined uniquely by its values on Scott open subsets (see [1] for details). In [14] this idea have been applied to concurrent systems whose behaviours can be represented by event structures, each event structure consisting of a set of events with relations of causal dependency and conflicts. In this case system runs are represented by sets of events which have occurred, called configurations, and they are ordered by inclusion of representing configurations. Consequently, the required probability space can easily be defined if the behaviours thus represented are continuous directed complete posets.

In the present paper we try to develop a basis as universal as possible for describing and studying random behaviours of concurrent systems, a basis that would allow us to describe in a uniform way behaviours of systems of various kinds (discrete, continuous, partially discrete and partially continuous), including behaviours that need not to be continuous directed complete posets. As in [4], we represent behaviours of concurrent systems as directed complete posets of their runs.

The paper is organized as follows. In section 2 we introduce a model of a run. In section 3 we introduce operations of composing runs. In section 4 we describe a partial order of runs. In section 5 we define behaviours of systems as directed complete partially ordered sets of runs. In section 6 we present set theoretical probabilistic models of system behaviours. In section 7 we present topological probabilistic models of system behaviours. All the auxiliary notions and proofs exploited in the paper are presented in appendices (Appendix A: Posets and their cross-sections, Appendix B: Directed complete posets, Appendix C: Probability spaces).

2 A representation of system runs

Partial and complete runs of a concurrent system can be regarded as activities in a universe of objects, each object with a set of possible instances corresponding to its states, each activity changing states of some objects. They can be defined as processes in [15].

2.1. Definition. By a *universe of objects* we mean a structure $\mathbf{U} = (W, V, ob)$, where V is a set of *objects*, W is a set of *instances* of objects from V (a set of *object instances*), and ob is a mapping that assigns the respective object to each of its instances. $\sharp$

2.2. Example. Consider machines M_1 and M_2 as in Example 1.1. Let $V_1 = \{M_1, M_2\}$, $W_1 = \{a, b, c, d\}$, $ob_1(a) = ob_1(b) = M_1$, $ob_1(c) = ob_1(d) = M_2$. Then $\mathbf{U}_1 = (W_1, V_1, ob_1)$ is a universe of objects. $\sharp$

2.3. Example. Suppose that a producer p produces some material and delivers from time to time an amount of this material to a distributor d, . Define an instance of p to be a pair (p, q), where $q \geq 0$ is the amount of material at disposal of p. Define an instance of d to be a pair (d, r), where $r \geq 0$ is the amount of material at disposal of d. Define $V_2 = \{p, d\}$, $W_2 = W_p \cup W_d$, where $W_p = \{(p, q) : q \geq 0\}$, $W_d = \{(d, r) : r \geq 0\}$. Define $ob_2(w) = p$ for $w = (p, q) \in W_p$ and $ob_2(w) = d$ for $w = (d, r) \in W_d$. Then $\mathbf{U}_2 = (W_2, V_2, ob_2)$ is a universe of objects. $\sharp$

Potential runs of a system can be defined without specifying the system. It suffices to define them as runs in the respective universe of objects. The notions used in the definition can be found in Appendix A (see Definitions A.1 and A.5).

2.4. Definition. A *concrete run* in a universe $\mathbf{U} = (W, V, ob)$ of objects is a labelled partially ordered set (lposet) $E = (X, \leq, ins)$, where X is a set (of *occurrences* in E of (instances of) objects), $ins : X \to W$ is a mapping (a *labelling* that assigns the respective object instance to each occurrence of this object instance), and $\leq$ is a partial order (the *causal dependency relation* of E) such that

(1) for every object $v \in V$, the set $X|v = \{x \in X : ob(ins(x)) = v\}$ is either empty or it is a maximal chain and has an element in every cross-secton of $(X, \leq)$,

(2) every element of X belongs to a cross-section of $(X, \leq)$,

(3) no bounded segment of E is isomorphic to its proper bounded subsegment,

(4) the set of minimal elements of $(X, \leq)$ is a cross-section. $\sharp$ $\sharp$

Put in another way, E is a partially ordered set of occurrences of object instances. Each object may have many instances and each of these instances may have in E many occurrences. Condition (1) says that the occurrences of instances of every object which takes part in E form a maximal chain, that E contains all information on such an object, and that every state of E contains the respective part of this information. Condition (2) says that every occurrence of an object in E belongs to a state of E. Condition (3) guarantees that the progress of E is a purely intrinsic property of E that is fully reflected by what happens to the involved objects. It generalizes a natural property of finite runs. Condition (4) says that E has an initial state.

As concrete runs are lposets, their morphisms are defined as morphisms of lposets, that is as mappings that preserve the ordering and the labelling (see Appendix A).

2.5. Example. Let $\mathbf{U}_1 = (W_1, V_1, ob_1)$ be the universe of objects described in example 2.2.

An execution of action α by the machine M_1 is a concrete run $A = (X_A, \leq_A, ins_A)$ in $\mathbf{U}_1$, where
$X_A = \{x_1, x_2\}$,
$x_1 <_A x_2$,
$ins_A(x_1) = ins_A(x_2) = a$.

An execution of action β by the machine M_1 is a concrete run $B = (X_B, \leq_B, ins_B)$ in $\mathbf{U}_1$, where
$X_B = \{x_1, x_2\}$,
$x_1 <_B x_2$,
$ins_B(x_1) = a$, $ins_B(x_2) = b$.

Joint execution of action γ by the machines M_1 and M_2 is a concrete run $C = (X_C, \leq_C, ins_C)$ in $\mathbf{U}_1$, where
$X_C = \{x_1, x_2, x_3, x_4\}$,
$x_1 <_C x_3, \quad x_1 <_C x_4, \quad x_2 <_C x_3, \quad x_2 <_C x_4$,
$ins_C(x_1) = b$, $ins_C(x_2) = d$, $ins_C(x_3) = a$, $ins_C(x_4) = c$.

An execution of action δ by the machine M_2 is a concrete run $D = (X_D, \leq_D, ins_D)$ in $\mathbf{U}_1$, where
$X_D = \{x_1, x_2\}$,
$x_1 <_D x_2$,
$ins_D(x_1) = c$, $ins_D(x_2) = d$.

Independent execution of α and δ followed by an execution of α is a concrete run $E = (X_E, \leq_E, ins_E)$ in $\mathbf{U}_1$, where
$X_E = X_{A'} \cup X_{D'} \cup X_{A''}$,
$\leq_E$ is the transitive closure of $\leq_{A'} \cup \leq_{D'} \cup \leq_{A''}$,
$ins_E = ins_{A'} \cup ins_{D'} \cup ins_{A''}$,
for variants A' and A'' of A and a variant D' of D such that the maximal element of $X_{A'}$ coincides with the minimal element of $X_{A''}$, and these are the

only common elements of pairs of sets from among $X_{A'}$, $X_{D'}$, $X_{A''}$.

Independent execution of β and δ followed by an execution of γ is a concrete run $F = (X_F, \leq_F, ins_F)$ in $\mathbf{U}_1$, where
$X_F = X_{B'} \cup X_{D'} \cup X_{C'}$,
$\leq_F$ is the transitive closure of $\leq_{B'} \cup \leq_{D'} \cup \leq_{C'}$,
$ins_F = ins_{B'} \cup ins_{D'} \cup ins_{C'}$,
for a variant B' of B, a variant D' of D, and a variant C' of C such that the maximal element of $X_{B'}$ coincides with the minimal element of $X_{C'}$ with the same label, the maximal element of $X_{D'}$ coincides with the minimal element of $X_{C'}$ with the same label, and these are the only common elements of pairs of sets from among $X_{B'}$, $X_{D'}$, $X_{C'}$.

The lposets representing the concrete runs A, B,C, D, E, F are represented graphically in figure 2.1.

The isomorphism classes of lposets corresponding to the concrete runs A, B, C, D are represented graphically in figure 2.2 as α, β, γ, δ, respectively. The isomorphism classes of lposets corresponding to the concrete runs E and F are represented graphically in figure 2.3 as ε and φ, respectively. $\sharp$

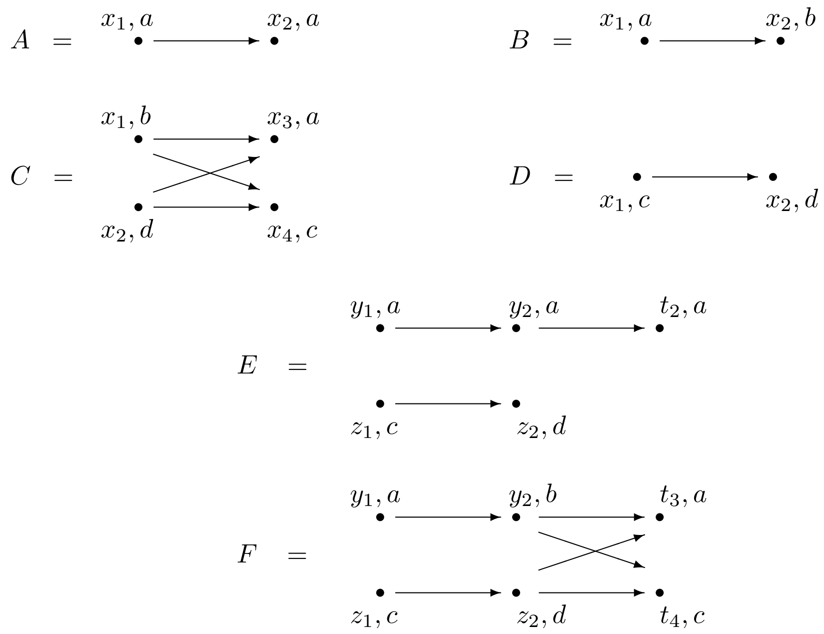


Figure 2.1

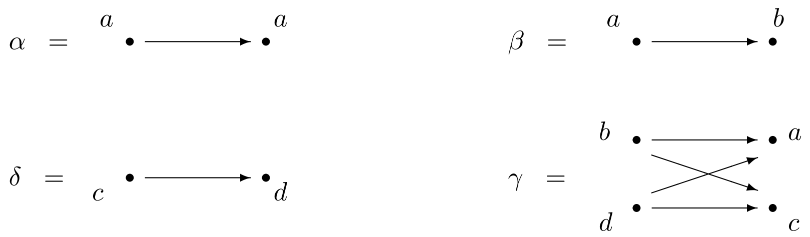


Figure 2.2

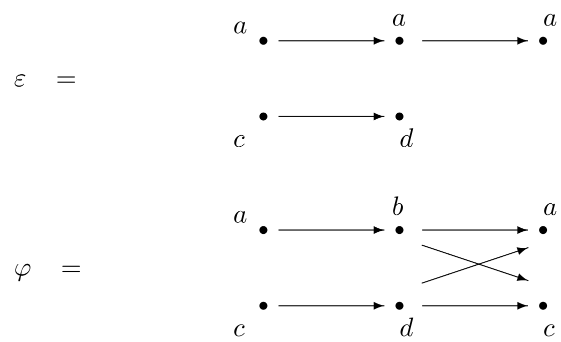


Figure 2.3

2.6. Example. Let $\mathbf{U}_2 = (W_2, V_2, ob_2)$ be the universe of objects described in example 2.3.

Undisturbed production of material by the producer p in an interval $[t', t'']$ of global time is a concrete run $Q = (X_Q, \leq_Q, ins_Q)$ in $\mathbf{U}_2$, where X_Q is the set of values of variations $var(t \mapsto q(t); t', t)$ in intervals $[t', t] \subseteq [t', t'']$ of the real valued function $t \mapsto q(t)$ which specifies the amount of material at disposal of p at every moment of $[t', t'']$, $\leq_Q$ is the restriction of the usual order of numbers to X_Q, and $ins_Q(x) = (p, q(t))$ for $x = var(t \mapsto q(t); t', t)$. The number $var(t \mapsto q(t); t', t'')$, written as $length(Q)$, is called the length of Q. The set X_Q with the order $\leq_Q$ represents the intrinsic local time of the producer. If the material is produced in a continuous way than the function $t \mapsto q(t)$ is continuous and X_Q is a closed interval. Otherwise it may consist of a set of disjoint intervals. If there is no uncontrolled lose of the material then the function $t \mapsto q(t)$ is increasing and $q(t'') - q(t') = length(Q)$. Otherwise $q(t'') - q(t') < length(Q)$. (We remind that the variation of a real-valued function f on an interval $[a, b]$, written as $var(f; a, b)$, is the least upper bound of the set of numbers $|f(a_1) - f(a_0)| + ... + |f(a_n) - f(a_{n-1})|$ corresponding to subdivisions $a = a_0 < a_1 < ... < a_n = b$ of $[a, b]$. In the case of more than one real-valued function the concept of variation turns into the concept of the length of the curve defined by these functions.)

Undisturbed distribution of material by the distributor d in an interval $[t', t'']$ of global time is a concrete run $R = (X_R, \leq_R, ins_R)$ in $\mathbf{U}_2$, where X_R

is the set of values of variations $var(t \mapsto r(t); t', t)$ in intervals $[t', t] \subseteq [t', t'']$ of the real valued function $t \mapsto r(t)$ which specifies the amount of material at disposal of d at every moment of $[t', t'']$, $\leq_R$ is the restriction of the usual order of numbers to X_R, and $ins_R(x) = (d, q(t))$ for $x = var(t \mapsto q(t); t', t)$. The number $var(t \mapsto r(t); t', t'')$, written as $length(R)$, is called the length of R. The set X_R with the order $\leq_R$ represents the intrinsic local time of the distributor. If the material is distributed in a continuous way than the function $t \mapsto r(t)$ is continuous and X_R is a closed interval. Otherwise it may consist of a set of disjoint intervals. If there is no uncontrolled supply of the material then the function $t \mapsto r(t)$ is decreasing and $r(t') - r(t'') = length(R)$. Otherwise $r(t') - r(t'') < length(R)$.

Transfer of an amount m of material from the producer p to the distributor d is a concrete run $S = (X_S, \leq_S, ins_S)$ in $\mathbf{U}_2$, where $X_S = \{x_1, x_2, x_3, x_4\}$, $x_1 <_S x_3$, $x_1 <_S x_4$, $x_2 <_S x_3$, $x_2 <_S x_4$, $ins_S(x_1) = (d, r)$, $ins_S(x_2) = (p, q)$, $ins_S(x_3) = (d, r + m)$, $ins_S(x_4) = (p, q - m)$. The set X_R with the order $\leq_R$ represents the intrinsic global time of the system consisting of the producer and the distributor.

Transfer of an amount of material from the producer p to the distributor d followed by independent behaviour of p and d and by another transfer of material from p to d is a concrete run $T = (X_T, \leq_T, ins_T)$ in $\mathbf{U}_2$, where $X_T = X_{Q'} \cup X_{R'} \cup X_{S'} \cup X_{S''}$, $\leq_T$ is the transitive closure of $\leq_{Q'} \cup \leq_{R'} \cup \leq_{S'} \cup \leq_{S''}$, $ins_T = ins_{Q'} \cup ins_{R'} \cup ins_{S'} \cup ins_{S''}$, for a variant Q' of Q, a variant R' of R, and variants S' and S'' of S, such that one maximal element of $X_{S'}$ coincides with the minimal element of $X_{Q'}$ with the same label and the other maximal element coincides with the minimal element of $X_{R'}$ with the same label, one minimal element of $X_{S''}$ coincides the maximal element of $X_{Q'}$ with the same label and the other minimal element coincides with the maximal element of $X_{R'}$ with the same label, and these are the only common elements of pairs of sets from among $X_{Q'}$, $X_{R'}$, $X_{S'}$, $X_{S''}$.

The isomorphism classes of lposets corresponding to the concrete runs Q, R, S, T, are represented graphically in figure 2.4. $\sharp$

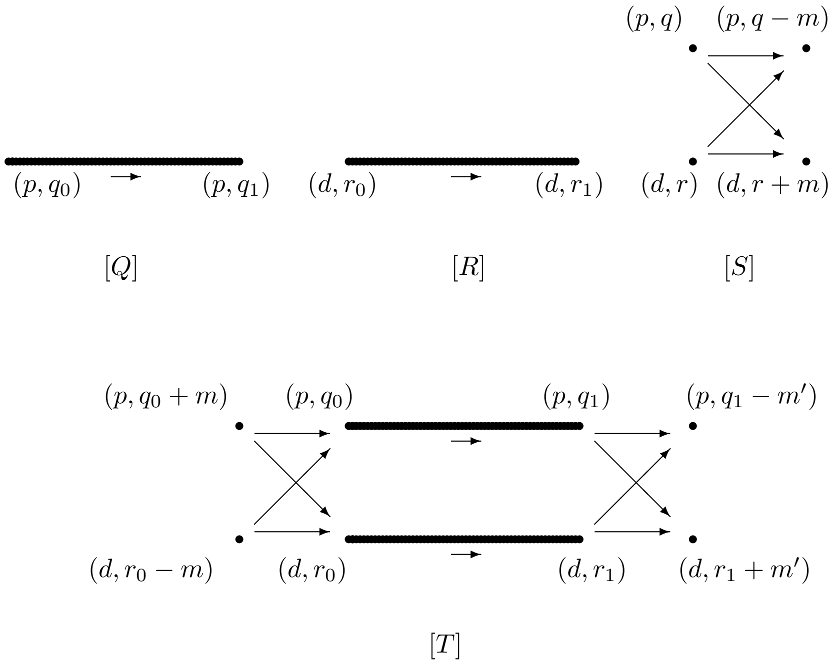


Figure 2.4: $[Q]$, $[R]$, $[S]$, $[T]$

Let $\mathbf{U} = (W, V, ob)$ be a universe of objects.

Let $E = (X, \leq, ins)$ be a concrete run in $\mathbf{U}$.

Every cross-section of $(X, \leq)$ contains an occurrence of an instance of each object v with nonempty $X|v$, and it is called a *cross-section* of E. By *csections*(E) we denote the set of cross-sections of E. This set is partially ordered by the relation $\preceq$ defined in Appendix A and, according to Proposition A.4, for every two cross-sections Z' and Z'' from *csections*(E) there exist in *csections*(E) the greatest lower bound $Z' \wedge Z''$ and the least upper bound $Z' \vee Z''$ of Z' and Z'' with respect to $\preceq$. It follows from (1) and (2) in Definition 2.4 that the set of objects with instances occurring in a cross-section is the same for all cross-sections of E. We call it the *range* of E and write as *objects*(E).

The set of elements of E that are minimal with respect to $\leq$ is a cross-section of E. We call it the *origin* of E and write as $origin(E)$. If the set of elements of E that are are maximal with respect to $\leq$ is also a cross-section then we call it the *end* of E and write as $end(E)$, and we say that E is *bounded*.

The following propositions are direct consequences of definition.

2.7. Proposition. Every segment of E is a concrete run. $\sharp$

2.8. Proposition. For each cross-section c of E, the restrictions of E to the subsets $X^{-}(c) = \{x \in X : x \leq z \text{ for some } z \in c\}$ and $X^{+}(c) = \{x \in X : z \leq x \text{ for some } z \in c\}$ are concrete runs, called respectively the *head* and the *tail* of E with respect to c, and written respectively as $head(E,c)$ and $tail(E,c)$. $\sharp$

For example, for the concrete run T in example 2.6 and its cross-section c that consists of the maximal element x of X_Q and the minimal element y of X_R, $head(T,c)$ is the restriction of T to $X_Q \times \{y\}$, and $tail(T,c)$ is the restriction of T to $\{x\} \times X_R$.

2.9. Proposition. For every decomposition $s = (X^F, X^S)$ of the underlying set X of E into two disjoint subsets X^F and X^S such that $x' \leq x''$ only if x' and x'' are both in one of these subset, called a *splitting* of E, the restrictions of E to the subsets X^F and X^S are concrete runs, called respectively the *first part* and the *second part* of E with respect to s, and written respectively as $first(E,s)$ and $second(E,s)$. Each concrete run E' such that $E' = first(E,s)$ or $E' = second(E,s)$ for some s is called an *independent component* of E. $\sharp$

For example, for the concrete run T in Example 2.6 and its splitting s that consists of X_Q and X_R, $first(T,s)$ is the restriction of T to X_Q and $first(T,s) = Q$, and $second(T,s)$ is the restriction of T to X_R and $second(T,s) = R$. Moreover, Q and R are independent components of T.

The following proposition reflects an important property of concrete runs.

2.10. Proposition. For every cross-section c of E, every isomorphism between bounded initial segments of $tail(E,c)$ (resp.: between bounded final segments of $head(E,c)$) is an identity. $\sharp$

Proof. Let Q be the restriction of E to $X^+(c)$ and let R and S be two initial segments of Q. Suppose that $f : R \to S$ is an isomorphism that it is not an identity. Then there exists an initial subsegment T of R such that the image of T under f, say T', is different from T. By (3) of definition 2.4 neither T' is a subsegment of T nor T is a subsegment of T'. Define T'' to be the least segment containing both T and T', and consider $f' : T \to T''$, where $f'(x) = f(x)$ for $x \le f(x)$ and $f'(x) = x$ for $f(x) < x$. In order to derive a contradiction, and thus to prove that f is an identity, it suffices to verify, that f' is an isomorphism. It can be done as follows.

For injectivity suppose that $f'(x) = f'(y)$. If $x \le f(x)$ and $y \le f(y)$ then $f(x) = f'(x) = f'(y) = f(y)$ and thus $x = y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) = f'(y) = y$. The case $x \le f(x)$ and $f(y) < y$ is excluded by $f'(x) = f'(y)$ since $x \le f(x) = f'(x) = f'(y) = y$ and, on the other hand, $f(y) < y = f(x)$ implies $y < x$. Similarly, the case $f(x) < x$ and $y \le f(y)$ is excluded. Consequently, f' is injective.

For surjectivity suppose that y is in T''. If $y \le f(y)$ then, by surjectivity of f and condition (1) of Definition 2.4, there exists $t \le y$ such that $y = f(t)$ and thus $y = f(t) = f'(t)$ since $t \le y = f(t)$. If $f(y) < y$ then $y = f'(y)$. Consequently, f' is surjective.

For monotonicity suppose that $x \le y$. If $x \le f(x)$ and $y \le f(y)$ then $f'(x) = f(x) \le f(y) = f'(y)$. If $f(x) < x$ and $f(y) < y$ then $f'(x) = x \le y = f'(y)$. If $x \le f(x)$ and $f(y) < y$ then $f'(x) = f(x) \le f(y) < y = f'(y)$. If $f(x) < x$ and $y \le f(y)$ then $f'(x) = x \le y \le f(y) = f'(y)$. Consequently, f' is monotonic.

For monotonicity of the inverse suppose that $f'(x) < f'(y)$. If $x \le f(x)$ and $y \le f(y)$ then $f(x) = f'(x) < f'(y) = f(y)$ and thus $x < y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) < f'(y) = y$. If $x \le f(x)$ and $f(y) < y$ then $x \le f(x) = f'(x) < f'(y) = y$. If $f(x) < x$ and $y \le f(y)$ then $f(x) < x = f'(x) < f'(y) = f(y)$ and thus $x < y$. Consequently, the inverse of f' is monotonic.

A proof for final subsegments of the restriction of E to $X^-(c)$ is similar. ♯

2.11. Corollary. For every bounded segment Q of E, every automorphism of Q is an identity. ♯

2.12. Corollary. For every bounded concrete run E' there exists at most one isomorphism from E' to an initial segment of E. $\sharp$

2.13. Corollary. If E is bounded then for every bounded concrete run E' there may be at most one isomorphism from E to E'. $\sharp$

Concrete runs with the same instances of objects and the same transformations of these instances are isomorphic lposets. Consequently, they are members of the same isomorphism class of lposets, that is members of the same pomset. This is reflected in the following definition.

2.14. Definition. An *abstract run* in $\mathbf{U}$ is an isomorphism class ξ of concrete runs. Each member E of such a class ξ is called an instance of this class and ξ is written as $[E]$. $\sharp$

Collecting concrete runs into isomorphism classes, i.e. making abstract runs, is convenient because it allows one to define some natural operations on the latter (see section 3).

2.15. Example. The isomorphism classes $[Q]$, $[R]$, $[S]$, $[T]$ in figure 2.4 of lposets corresponding to the concrete runss Q, R, S, T in example 2.4 are abstract executions. $\sharp$

For every concrete run E' such that E and E' are isomorphic we have $objects(E') = objects(E)$. Consequently, for the abstract run $[E]$ that corresponds to a concrete run E we define $objects([E]) = objects(E)$.

We say that an abstract run is *bounded* if the instances of this run are bounded.

By $RUNS(\mathbf{U})$ we denote the set of runs in $\mathbf{U}$.

In the set $RUNS(\mathbf{U})$ there exists the run with the empty underlying set of its instance, called the *empty run*, and written as 0. For each bounded run α from $RUNS(\mathbf{U})$ with an instance $E \in \alpha$ and its cross-section $origin(E)$ there exists the unique run $[origin(E)]$, called the *initial state* or the *source* or the *domain* of α and written as $dom(\alpha)$. For each bounded run α from $RUNS(\mathbf{U})$ with an instance $E \in \alpha$ and its cross-section $end(E)$ there exists the unique run $[end(E)]$, called the *final state* or the *target* or the *codomain* of α and written as $cod(\alpha)$.

3 A representation of operations on system runs

In what follows, the word "run" means "abstract run".

Let $\mathbf{U} = (W, V, ob)$ be a universe of objects. In the set $RUNS(\mathbf{U})$ of runs in $\mathbf{U}$ there are two partial operations: a parallel composition and a sequential composition.

3.1. Definition. A run α is said to *consist* of a run α_1 *followed* by a run α_2 iff an instance L of α has a cross-section c such that $head(L, c)$ is an instance of α_1 and $tail(L, c)$ is an instance of α_2. $\sharp$

3.2. Proposition. For every two runs α_1 and α_2 such that $cod(\alpha_1)$ is defined and $cod(\alpha_1) = dom(\alpha_2)$ there exists a unique run, written as $\alpha_1;\alpha_2$, or as $\alpha_1\alpha_2$, that consists of α_1 followed by α_2. $\sharp$

Proof. Take $E_1 = (X_1, \leq_1, ins_1) \in \alpha_1$ and $E_2 = (X_2, \leq_2, ins_2) \in \alpha_2$ with $X_1 \cap X_2 = end(E_1) = origin(E_2)$ and with the restriction of E_1 to $end(E_1)$ identical with the restriction of E_2 to $origin(E_2)$, and provide $X = X_1 \cup X_2$ with the least common extension of the causal dependency relations and labellings of E_1 and E_2. Let E be the lposet thus obtained. It suffices to prove that E is a run and notice that $head(E, c) = E_1$ and $tail(E, c) = E_2$. In order to prove that E is a run it suffices to show that E does not contain a segment with isomorphic proper subsegment. To this end suppose the contrary. Suppose that $f : Q \rightarrow R$ is an isomorphism from a segment Q of E to a proper subsegment R of Q, where Q consists of a part Q_1 contained in E_1 and a part Q_2 contained in E_2. By applying twice the method described in the proof of proposition 2.10 we can modify f to an isomorphism $f' : Q \rightarrow R$ such that the image of Q_1 under f', say R_1, is contained in Q_1, and the image of Q_2 under f', say R_2, is contained in Q_2. As R is a proper subsegment of Q, one of these images, say R_1, is a proper part of the respective Q_i. By taking the greatest lower bounds and the least upper bounds of appropriate cross-sections we can extend Q_1 and R_1 to segments Q'_1 and R'_1 of P_1 such that R'_1 is a proper subsegment of Q'_1 and there exists an isomorphism from Q'_1 to R'_1. This is in a contradiction with the fact that E_1 is a run. Consequently, E is a run. $\sharp$

3.3. Definition. The operation $(\alpha_1, \alpha_2) \mapsto \alpha_1\alpha_2$ is called the *sequential composition* of runs. $\sharp$

Each run which is a source or a target of a run is an identity, i.e. a run ι such that $\iota\phi = \phi$ whenever $\iota\phi$ is defined and $\psi\iota = \psi$ whenever $\psi\iota$ is defined. Moreover, $dom(\alpha)$ is the unique identity ι such that $\iota\alpha$ is defined, and if $cod(\alpha)$ is defined then it is the unique identity κ such that $\alpha\kappa$ is defined. Consequently, $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable partial operations on runs.

Identities are bounded runs with causal dependency relations reducing to identity relations. They are called *states*, or *identities*, and we can identify them with the sets of occurring instances of objects.

3.4. Definition. A run α is said to *consist* of two *parallel* runs α_1 and α_2 iff an instance E of α has a splitting s such that $first(E, s)$ is an instance of α_1 and $second(E, s)$ is an instance of α_2. $\sharp$

3.5. Proposition. For every two runs α_1 and α_2 such that $objects(\alpha_1) \cap objects(\alpha_2) = \emptyset$ there exists a run α with an instance E that has a splitting s such that $first(E, s)$ is an instance of α_1 and $second(E, s)$ is an instance of α_2. If such a run α exists then it is unique, we write it as $\alpha_1 + \alpha_2$, and we say that the executions α_1 and α_2 are *parallel*. $\sharp$

For a proof it suffices to take $E_1 = (X_1, \leq_1, ins_1) \in \alpha_1$ and $E_2 = (X_2, \leq_2, ins_2) \in \alpha_2$ with $X_1 \cap X_2 = \emptyset$, and to provide $X_1 \cup X_2$ with the least common extension of the causal dependency relations and labellings of E_1 and E_2.

3.6. Definition. The operation $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ is called the *parallel composition* of runs. $\sharp$

The operations on runs allow one to represent complex runs in terms of their components.

3.7. Example. In the case of runs in example 2.4 we can represent $[T]$ as $[S']([Q'] + [R'])[S'']$. $\sharp$

The operations of composing runs allow one to turn the set $RUNS(\mathbf{U})$ into a partial algebra.

3.8. Definition. The partial algebra $\mathbf{RUNS}(\mathbf{U}) = (RUNS(\mathbf{U}), ;, +)$ and its subalgebras are called *algebras of runs* in $\mathbf{U}$. $\sharp$

The restriction of an algebra of runs to the subset of bounded runs is an arrows-only category in the sense of [7] with the additional operation $+$.

4 The partial order of runs

Let $\mathbf{A} = (A, ;, +)$ be an algebra of runs. The operations of the algebra $\mathbf{A}$ of can be used to define in this algebra a partial order.

4.1. Proposition. The relation *pref*, where α *pref* β iff $\beta = (\alpha + \gamma)\delta$ for some γ and δ, is a partial order on A. If α and β are such that α *pref* β then we say that α is a *prefix* of β. $\sharp$

Proof. For transitivity suppose that $\beta = (\alpha + \gamma)\delta$ and $\beta' = (\beta + \gamma')\delta'$. If $E_{\beta'}$ is an instance of β' then there exists c such that $head(E_{\beta'}, c)$ is an instance $E_{\beta+\gamma'}$ of $\beta+\gamma'$ and $head(first(E_{\beta+\gamma'}, s), c_1)$ is an instance E_β ofβ for some s and a part c_1 of c. Moreover, there exists d such that $head(E_\beta, d)$ is an instance $E_{\alpha+\gamma}$ of $\alpha+\gamma$ and $head(first(E_{\alpha+\gamma}, t), d_1)$ is an instance of E_α for some t and a part d_1 og d. Consequently, $head(E_{\beta'}, c')$ is an instance of $\alpha+\gamma+\gamma'$ for c' consisting of d and of the complement of c_1 to c, and $\beta' = (\alpha+\gamma+\gamma')\delta''$ for $\delta'' = tail(E_{\beta'}, c')$. For antisymmetry suppose that $\beta = (\alpha + \gamma)\delta$ and $\alpha = (\beta + \gamma')\delta'$. As objects with instances occurring in α cannot occur in γ and objects with instances occurring in β cannot occur in γ', there must be $\gamma = \gamma' = 0$. Consequently, $\alpha = \alpha\delta\delta'$ and, by Corollary 2.12, δ and δ' must be identities. $\sharp$

4.2. Proposition. The extention $\sqsubseteq$ of the relation *pref*, where $\alpha \sqsubseteq \beta$ iff every prefix of α is a prefix of β, is a partial order on A. The poset $(A, \sqsubseteq)$ is a DCPO. Every element of $\mathbf{A}$ is the least upper bound of the directed set of its prefixes. $\sharp$

Proof. Given a directed subset D of the poset $(A, \sqsubseteq)$, the pefixes of elements of D form a directed set D'. For every element of D' we choose a concrete instance, and we consider α and $\beta = (\alpha + \gamma)\delta$ such that E is the chosen instance of α, E_1 is the chosen instance of β, E_2 is the chosen instance of

$\alpha + \gamma$ and $E_3 = head(E_1, c)$ is an instance of $\alpha + \gamma$. Then there exists a unique isomorphism f from E_2 to E_3 since otherwise there would be another isomorphism g and the correspondence $f(x) \mapsto g(x)$ would be different from identity isomorphism between two initial segments of E_1. On the other hand, f determines a unique isomorphism between E and $first(E_2, s)$ with a splitting s due to the fact that the first part of E_2 is determined uniquely by the set of objects which occur in it. Consequently, we can construct a direct system of instances of elements of D' such that the colimit of this system in the category **LPOSETS** is an instance of the least upper bound of D' and of D.

The last part of the proposition is a simple consequence of the condition (2) of Definition 2.4. $\sharp$

4.3. Definition. The relation $\sqsubseteq$ on A is called the *prefix order*. The least upper bound of a directed subset D of the partially ordered set $(A, \sqsubseteq)$ is called the *limit* of D. $\sharp$

Note that the least upper bounds of directed subsets of the poset $(A, \sqsubseteq)$ are limits of the corresponding filters in $\mathbf{A}$ with the Scott topology induced by the partial order $\sqsubseteq$.

What has been said about the prefix order in $\mathbf{A}$ applies to every algebra of runs. In some algebras of runs the corresponding DCPOs are continuous. This is true for a class of such algebras defined below.

4.4. Definition. An algebra $\mathbf{A}(\mathbf{U}) = (A, ;, +)$ of runs is said to be *locally complete* if every run in A is locally complete in the sense that its every bounded segment is a complete lattice. $\sharp$

The following property of operations of composing runs imply that certain subalgebras of algebras of runs are locally complete.

4.5. Proposition. The result of sequential or parallel composition of locally complete runs is locally complete. $\sharp$

Proof. In the case of parallel composition the proposition is obvious. In order to prove that $\alpha_1\alpha_2$ is locally complete if α_1 and α_2 are locally complete suppose that E is an instance of $\alpha_1\alpha_2$ with a cross-section c such that $head(E, c)$ is an instance of α_1 and $tail(E, c)$ is an instance of α_2. Given a segment Q of E and a subset S of cross-sections of E contained in Q, let c^- be the least upper bound of the set of cross-sections $s \wedge c$ with $s \in S$ and c^+ the least upper bound of cross-sections $s \vee c$ with $s \in S$. Then for every $v \in V$ define x_v as the greater of the two elements of $X|v$ in c^- and in c^+, and define d as the set of all x_v. As c^- and c^+ are cross-sections, d does not contain comparable elements and is an antichain. As all $v \in V$ have in d occurrences, d is a maximal antichain. It is also straightforward to verify that d is a cross-section and the least upper bound of S. In a similar way we can define a cross-section that is the greatest lower bound of S. $\sharp$

Locally complete algebras of runs enjoy the following property.

4.6. Proposition. If $\mathbf{A}$ is a locally complete algebra of runs then $(A, \sqsubseteq)$ is a continuous DCPO. $\sharp$

Proof. Suppose that $\alpha \in B$ is a bounded run with an instance E such that $E = head(E', c)$ for a concrete run E' with $[E'] \in A$ and for c being the least upper bound of cross-sections c' of E' with the underlying sets of $head(E', c')$ containing occurrences $x_1, ..., x_n$ of instances of objects $v_1, ..., v_n$ from a finite subset of V. Then α is a compact element of A. Indeed, suppose that $\alpha \sqsubseteq \bigsqcup S$ for a directed subset S of A. Then all $s \in S$ and $\bigsqcup S$ have instances E_s and E_S that are initial segments of E' such that the underlying set of E_S is the union of the underlying sets of all E_s and it contains the underlying set of E. Consequently, for every $i \in \{1, ..., n\}$ there must be $s_i \in S$ such that the underlying set of E_{s_i} contains x_i. Consequently, $x_1, ..., x_n$ belong to the underlying set of E_s for an upper bound s of $s_1, ..., s_n$ that belongs to S. Consequently, c must be a cross-section of E_s and $\alpha \sqsubseteq s \in S$, as required.

In order to prove that A with the prefix order is algebraic domain, consider any $\alpha \in A$ and its instance E. As every run is an inductive limit of a direct system of its bounded segments, it suffices to consider the case when α is bounded. Then for every finite set $f = \{x_1, ..., x_n\}$ of occurrences of instances of objects $v_1, ..., v_n$ in the underlying set of E there exists the least cross-section c_f of E such that $x_1, ..., x_n$ belong to the underlyin set of $head(E, c_f)$. Then $s_f = [head(E, c_f)]$ is a compact element of A. On the other hand, processes s_f form a directed set S and $\alpha = \bigsqcup S$, as required. $\sharp$

The following theorem gives sufficient conditions of local completeness of runs and of algebras of runs.

4.7. Theorem. A concrete run $E = (X, \leq, ins)$ in a universe $\mathbf{U} = (W, V, ob)$ of objects is locally complete if the following conditions are satisfied:

(1) For every object v that occurs in L the set $X|v$ of its occurrences in E is a locally complete chain.

(2) The relation of incomparability with respect to the flow order $\leq$ is a closed subset of the product $X \times X$ for X provided with the interval topology, i.e., the weakest topology in which all intervals $\{x \in X : a < x < b\}$ are open sets. $\sharp$

Proof. Let Z_1 and Z_2 be cross-sections of E such that $Z_1 \preceq Z_2$ and let S be the set of cross-sections of E such that $Z_1 \preceq s \preceq Z_2$. Due to (1) for every $v \in V$ that occurs in L there exists the least upper bound x_v of those elements of $X|v$ which belong to some $s \in S$. Due to (2) the set Z of all such elements is an antichain. This set is a maximal antichain of E and it is easy to verify that it is also a cross-section of E. $\sharp$

5 Behaviours

The behaviour of a concurrent system can be represented by the set of its potential runs. The system may be *reactive* in the sense that it may communicate with the environment, behave depending on the data it receives, and act jointly with the environment (cf. [13]).

A behaviour is potential rather than actual. What has happened up to a certain stage of its potential run is a prefix of this run. What may happen next depends on the presence of suitable instances of objects taking part in the behaviour. Moreover, it is natural to assume that a behaviour contains the existing least upper bound of its subsets. Consequently, a behaviour is a specific set of runs. It automatically posesses the structure of partial order given by the prefix relation, and is a directed complete poset (a DCPO).

In order to define behaviours formally it is convenient to fix an algebra of runs, and think of this algebra as of a framework for the respective definitions.

Let $\mathbf{A} = (A, ;, +)$ be an algebra of runs.

5.1. Definition. A *behaviour* represented in **A**, or a behaviour in **A**, or simply a behaviour, if **A** is known from the context, is a subset B of the set A of runs of **A** such that:

(1) B is downward closed with respect to $\sqsubseteq$,

(2) if α and β are initial segments of runs which are maximal elements of B then $\alpha(\gamma + s) \in B$ iff $\beta(\gamma + t) \in B$ for every γ such that $dom(\gamma) + s = cod(\alpha)$ and $dom(\gamma) + t = cod(\beta)$,

(3) $\bigsqcup D \in B$ for every subset D of B such that $\bigsqcup D$ exists. ♯

5.2. Example. The underlying set of any algebra of runs is a behaviour represented in this algebra. Note that such a behaviour contains all the sources of maximal elements of **A** with respect to the prefix order. This reflects the indeterministic choice of the initial state of the behaviour from among all the sources of maximal elements of **A**. ♯

5.3. Example. Consider the machines M_1 and M_2 and their system M in example 1.1.

The behaviour of the machine M_1 alone can be represented in $\mathbf{RUNS}(\mathbf{U}_{\mathit{1}})$ as the set of runs a, b, α, α^2, ... , α^ω, β, $\alpha\beta$, $\alpha^2\beta$, ...

The behaviour of the machine M_2 alone can be represented in $\mathbf{RUNS}(\mathbf{U}_{\mathit{1}})$ as the set of runs c, d, δ.

The behaviour of the system M can be represented in $\mathbf{RUNS}(\mathbf{U}_{\mathit{1}})$ as the set B_1 of runs of the subalgebra $\mathbf{A}_1$ of the algebra $\mathbf{RUNS}(\mathbf{U}_{\mathit{1}})$ that can be obtained by combining $a, b, c, d, \alpha, \beta, \gamma, \delta$ with the aid of compositions and construction of limits.

It is clear that $\mathbf{A}_1$ is an algebra of runs and that B_1 is also a behaviour in $\mathbf{A}_1$. In this behaviour runs which have not in $\mathbf{A}_1$ a common extension (i.e., a runs of which they are predecessors relative to the prefix order) cannot represent initial segments of the same full run of M. Note that the lack of such a common extension can be decided without a reference to full runs of M.

An initial part of B_1 is depicted in figure 5.1, where the prefix order is indicated by directed edges. ♯

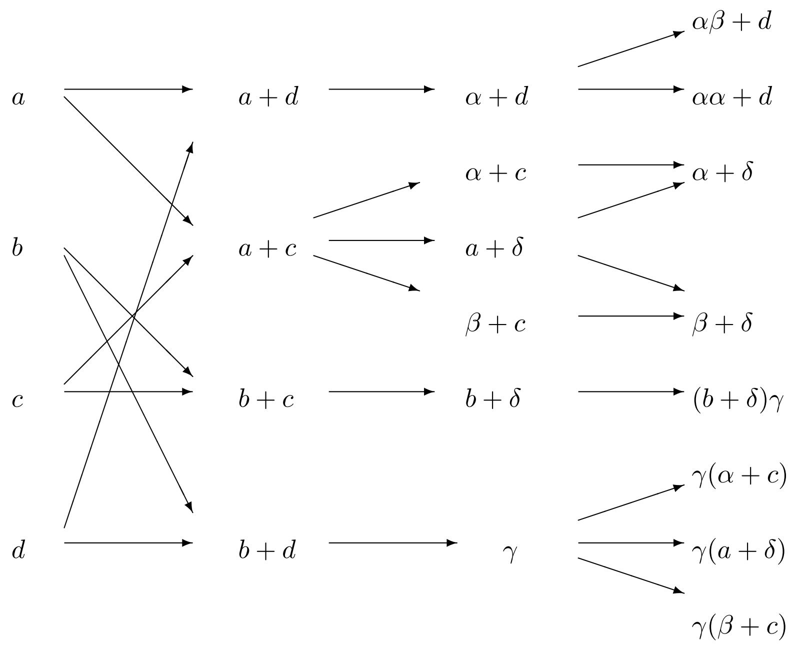


Figure 5.1: An initial part of B_1

5.4. Example. Consider a producer p and a distributor d in example 2.6. By combining the abstract runs corresponding to the possible variants of concrete runs Q and R of the producer and the distributor with the aid of compositions and construction of limits, we obtain a subalgebra $\mathbf{A}_2 = (A_2, ;)$ of $\mathbf{RUNS}(\mathbf{U}_2)$. This subalgebra is an algebra of runs in the universe $\mathbf{U}_2$. The set B_2 of runs of this algebra is a behaviour represented in $\mathbf{A}_2$. It reflects an independent activity of the producer and the distributor.

By combining the abstract runs corresponding to the possible variants of concrete runs Q, R, S with the aid of compositions and construction of limits, we obtain a subalgebra $\mathbf{A}_3 = (A_3, ;)$ of $\mathbf{RUNS}(\mathbf{U}_2)$. This subalgebra is an algebra of runs in the universe $\mathbf{U}_2$. The set B_3 of runs of this algebra is a behaviour represented in $\mathbf{A}_3$. It reflects an activity of the producer p and the distributor d that is mainly independent, but from time to time is interrupted by transfer of some material from the producer to the distributor. $\sharp$

6 Set-theoretical models

A way of defining a probability space representing a random behaviour is to define it as a projective limit of a projective system consisting of a directed family of probability spaces characterizing initial parts of the represented behaviour, each such a space obtained by endowing a set of processes with a suitable σ-algebra of subsets and with a suitable probability measure defined on this σ-algebra. It can be done as follows.

Let B be a behaviour in an algebra of runs $\mathbf{A} = (A, ;, +)$ in the sense of definition 3.8, and let $\Omega(B)$ be the set of maximal elements of B with respect to the prefix order $\sqsubseteq$.

Our aim is to show how to provide $\Omega(B)$ with a suitable probability measure μ on a given σ-algebra $\mathcal{F}$ of subsets of $\Omega(B)$. Our idea is to define μ with the aid of probability distributions on the sets of maximal elements of initial parts of the considered behaviour, called *sections*.

First of all, we define a directed partially ordered set of sections of the behaviour. This can be done as follows.

6.1. Definition. Two elements of B are said to be *confluent* iff they are predecessors of an element of B relative to the prefix order. $\sharp$

6.2. Definition. A set I of elements of B is said to be *confluence-free* iff it does not contain different elements that are confluent. $\sharp$

Note that the set of maximal elements of every subset of B which contains all the least upper bounds of its finite subsets is a confluence-free set.

From Kuratowski - Zorn Lemma we obtain the following property.

6.3. Proposition. Each confluence-free set of elements of B is contained in a maximal confluence-free set. $\sharp$

Note that the set of all sources of maximal elements of the behaviour B is a maximal confluence-free set.

6.4. Definition. Each maximal confluence-free set of bounded initial segments of maximal elements of the behaviour B is said to be a *section* of B. $\sharp$

6.5. Example. The following sets of runs of the behaviour B_1 defined in example 5.3 are sections of this behaviour (see figure 5.1):
$I = \{a + c, a + d, b + c, b + d\}$
$J = \{a + d, b + c, b + d, a + \delta\}$
$K = \{a + d, b + c, b + d, \alpha + c, \beta + c\}$
$L = \{a + d, b + c, b + d, \alpha + \delta, \beta + c\}$ $\sharp$

6.6. Example. Let B_2 be the behaviour of a producer p and a distributor d as in example 5.4. For every real $s \geq 0$ there exists a variant Q' of the run Q of the producer that has the length s. Similarly, for every real $t \geq 0$ there exists a variant R' of the run R of the distributor that has the length t. Consequently, for every real $s \geq 0$ and $t \geq 0$, the set of runs of B_2 of the form $\varphi + \psi$ such that φ is a run of the producer of the length s and ψ is a run of the distributor of the length t is a non-empty set $I(s, t)$. As two different members of $I(s, t)$ cannot be prefixes of a run in B_2, the set $I(s, t)$ is a section of B_2.

Let B_3 be the behaviour of a producer p and a distributor d as in example 5.4. For every integer $n \geq 1$, let $J(n)$ be the set of processes of B_3 of the form $(\varphi_1 + \psi_1)\sigma_1 ... (\varphi_n + \psi_n)\sigma_n$, where φ_i, ψ_i, σ_i represent variants of abstract runs $[Q]$, $[R]$, $[S]$, respectively. As two different members of $J(n)$ cannot be prefixes of a run of B_3, the set $J(n)$ is a section of B_3. $\sharp$

6.7. Definition. We say that a section I of B *precedes* another such a section J, and we write $I \ll J$, iff each element of J has a prefix in I. $\sharp$

6.8. Proposition. The set of all sections of B with the partial order $\ll$ is a directed set $\mathcal{T}(B)$. $\sharp$

For a proof it suffices to consider two arbitrary sections of B, say I and J, and to notice that the set K of maximal elements of the union of the downward closures of I and J is a section of B.

Now, taking into account the directed set $\mathcal{T}(B)$, we may think of defining the required probability space as a limit in the category **PSPACES** of a projective system of simpler probability spaces.

For $I \in \mathcal{T}(B)$, let $\mathbf{\Gamma}_I = (\Gamma_I, \mathcal{F}_I, \mu_I)$ be probability spaces such that

(1) $\Gamma_I = I$,

(2) $\mathcal{F}_I$ is a σ-algebra of subsets of I.

For $I, J \in \mathcal{T}(B)$ such that $I \ll J$, let $\pi_{IJ} : \Gamma_J \to \Gamma_I$ be the mappings assigning to each $j \in J$ its predecessor $i \in I$. Due to $I \ll J$ there exists such a predecessor and due to the fact that I is confluence-free it is unique.

The following facts follow easily from definitions.

6.9. Proposition. If $\pi_{IJ}(F) \in \mathcal{F}_I$ for all $F \in \mathcal{F}_J$ and $\mu_J(\pi_{IJ}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{F}_I$ then $\pi_{IJ} : \Gamma_I \leftarrow \Gamma_J$ is a morphism $\pi_{IJ} : \mathbf{\Gamma}_I \leftarrow \mathbf{\Gamma}_J$ $\sharp$

6.10. Proposition. If $\pi_{IJ}(F) \in \mathcal{F}_I$ for all $F \in \mathcal{F}_J$ and $\mu_J(\pi_{IJ}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{F}_I$ then $(\mathbf{\Gamma}_I \stackrel{\pi_{IJ}}{\leftarrow} \mathbf{\Gamma}_J : I, J \in \mathcal{T}(B), I \ll J)$ is a projective system in **PSPACES**. $\sharp$

Let $\mathbf{\Gamma} = (\Omega(B), \mathcal{F}, \mu)$ be a probability space such that $\mathcal{F}$ is the σ-algebra of subsets of $\Omega(B)$ generated by the σ-algebras $\mathcal{G}_I$, $I \in \mathcal{T}(B)$, where every $G \in \mathcal{G}_I$ is an I-cylinder in the sense that together with an element with a prefix belonging to I it contains also all the elements with this prefix, and where $\mathcal{G}_I \subseteq \mathcal{G}_J$ for $I \ll J$. Let π_{I*} be the mapping that assigns to each element of $\Omega(B)$ its unique prefix in I.

6.11. Theorem. The probability space $\mathbf{\Gamma} = (\Omega(B), \mathcal{F}, \mu)$ is a limit of the projective system $(\mathbf{\Gamma}_I \stackrel{\pi_{IJ}}{\leftarrow} \mathbf{\Gamma}_J : I, J \in \mathcal{T}(B), I \ll J)$, where each $\mathbf{\Gamma}_I = (\Gamma_I, \mathcal{F}_I, \mu_I)$ is the probability space such that

(1) $\Gamma_I = I$,

(2) $\mathcal{F}_I$ is the σ-algebra of those subsets of I whose inverse-images under π_{I*} belong to $\mathcal{G}_I$,

(3) $\mu(\pi_{I*}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{F}_I$,

and every $\pi_{IJ} : \Gamma_I \leftarrow \Gamma_J$ is the morphism assigning to each $j \in J$ its unique predecessor $i \in I$. $\sharp$

6.12. Example. Consider the following probability measures on the sections I, J, K, L defined in example 6.5 of the behaviour B_1 of the system M of machines M_1 and M_2 in example 5.3:

$\mu_I(\{a+c\}) = 1$, $\mu_I(\{a+d\}) = \mu_I(\{b+c\}) = \mu_I(\{b+d\}) = 0$
$\mu_J(\{a+\delta\}) = 1$, $\mu_J(\{a+d\}) = \mu_J(\{b+c\}) = \mu_J(\{b+d\}) = 0$
$\mu_K(\{\alpha+c\}) = \mu_K(\{\beta+c\}) = 0.5$
$\mu_K(\{a+d\}) = \mu_K(\{b+c\}) = \mu_K(\{b+d\}) = 0$
$\mu_L(\{\alpha+\delta\}) = \mu_L(\{\beta+c\}) = 0.5$
$\mu_L(\{a+d\}) = \mu_L(\{b+c\}) = \mu_L(\{b+d\}) = 0.$

Then $I \ll J \ll L$, $I \ll K \ll L$, and it is easy to verify that the probability spaces corresponding to these measures satisfy the conditions of Proposition 6.10. For example, we have

$\mu_K(\{\alpha+c\}) = \mu_L(\pi_{KL}^{-1}(\{\alpha+c\})) = \mu_L(\{\alpha+\delta\}) = 0.5$
$\mu_I(\{a+c\}) = \mu_K(\pi_{IK}^{-1}(\{\alpha+c\})) = \mu_K(\{\alpha+c, \beta+c\}) =$
$= \mu_K(\{\alpha+c\}) + \mu_K(\{\beta+c\}) = 0.5 + 0.5 = 1.$ $\sharp$

Random behaviours as described in this paper are similar to classical stochastic processes as defined in [2], [8], and [10]. In order to define them we have to solve the problem of defining the respective projective systems of probability spaces and the problem of the defining for such systems the respective limits.

In the case of the second problem the main point is to guarantee the existence of the required extension of given probability measures. For some behaviours the spaces of their runs are simple enough to exploit the known results on the existence of stochastic processes. For instance, with such a situation we have to do in the case of the behaviour of the system in example 5.3 where the space of runs is contained in the product of finite sets. However, in general we need universal results on the existence of limits of projective systems of probability spaces. One of them can be the result that the respective limit exists if the probability measures of system components are regular in the sense that they can be approximated by their values on members of a compact family of measurable subsets, where compactness means that every subfamily with nonempty intersections of all finite subfamilies has a nonempty intersection (see [8] for detailed notions and results which can easily be adapted).

In the case of defining for the considered behaviour B a projective system of probability spaces representing initial segments of this behaviour it is sometimes possible to assume a limited dependence of runs of this behaviour on the past, as in Markov processes.

To see this let us consider a random behaviour $\mathbf{\Gamma} = (\Omega(B), \mathcal{F}, \mu)$ which is a limit of a projective system $(\mathbf{\Gamma}_I \stackrel{\pi_{IJ}}{\leftarrow} \mathbf{\Gamma}_J : I, J \in \mathcal{T}(B), I \ll J)$ of probability spaces $\mathbf{\Gamma}_I = (\Gamma_I, \mathcal{F}_I, \mu_I)$, and sections I and J such that $I \ll J$.

For every $\beta \in J$ there exists in I a unique prefix $\alpha = \pi_{IJ}(\beta)$, and a unique ξ, written as $\mathit{link}_{IJ}(\beta)$, such that $\alpha\xi = \beta$. We say that the set of ξ such that $\xi = \mathit{link}_{IJ}(\beta)$ for some $\beta \in J$, written as $[I, J]$, is a *segment* of B.

It is clear that the mapping $\pi_{IJ} : J \to I$ is surjective. We call it the *projection* of J on I.

Similarly, it is clear that the mapping $\mathit{link}_{IJ} : J \to [I, J]$ is bijective. We call it the *reduction* of J to $[I, J]$.

Moreover, for every $\xi \in [I, J]$ there exists a unique $\alpha \in I$ such that $\alpha\xi \in J$, written as $\mathit{pred}_{IJ}(\xi)$, and that $\pi_{IJ}(\beta) = \mathit{pred}_{IJ}(\mathit{link}_{IJ}(\beta))$.

Finally, by $\mathcal{F}_{[IJ]}$ we denote the σ-algebra of those $F \subseteq [I, J]$ for which $\mathit{link}_{IJ}^{-1}(F) \in \mathcal{F}_J$.

For every $E \in \mathcal{F}_I$ we have $\mathit{pred}_{IJ}^{-1}(E) \in \mathcal{F}_{IJ}$.

For every $E \in \mathcal{F}_I$ and for $\mu_J\pi_{IJ}^{-1}(E)$ defined as $\mu_J(\pi_{IJ}^{-1}(E))$ we have $\mu_J\pi_{IJ}^{-1}(E) = \mu_I(E)$.

For every $\xi \in \Gamma_I$ and every $F \in \mathcal{F}_J$ we have a conditional probability $\mu_{IJ}(F|\xi)$, where

$$\mu_J(F \cap \pi_{IJ}^{-1}(E)) = \int_E \mu_{IJ}(F|\xi) d\mu_J\pi_{IJ}^{-1}(\xi) \text{ for every } E \in \mathcal{F}_I$$

or, equivalently,

$$\mu_J(F \cap \pi_{IJ}^{-1}(E)) = \int_E \mu_{IJ}(F|\xi) d\mu_I(\xi) \text{ for every } E \in \mathcal{F}_I.$$

Now suppose that the choice of a run in a state does not depend on the past in the sense that $\mu_{IJ}(F|\xi) = \mu_{IJ}(F|\xi')$ whenever $\mathit{cod}(\xi) = \mathit{cod}(\xi')$ and $\mu_{IJ}(F|\xi) = \mu_{IJ}(F'|\xi)$ whenever $\mathit{link}_{IJ}(F) = \mathit{link}_{IJ}(F')$. Then the conditional probabilities $\mu_{IJ}(F|\xi)$ can be regarded as values $P_{IJ}(G|x)$ of a function P_{IJ} for $G = \mathit{link}_{IJ}(F)$ and $x = \mathit{cod}(\xi)$, where

(*) $P_{IJ}(G|x) = \int_{G'} P_{KJ}(G''|u) dP_{IK}(u|x)$

for $G = G'G''$ with $G' \in \mathcal{F}_{IK}$ and $G'' \in \mathcal{F}_{KJ}$.

Consequently, knowing μ_I for some I and the functions P_{IJ} we can find μ_J using the formula

(**) $\mu_J(F) = \int_{\Gamma_I} P_{IJ}(link_{IJ}(F)|cod(\xi)) d\mu_I(\xi)$.

6.13. Example. For the sections

$I = \{a+c, a+d, b+c, b+d\}$,

$K = \{a+d, b+c, b+d, \alpha+c, \beta+c\}$,

$L = \{a+d, b+c, b+d, \alpha+\delta, \beta+c\}$

of the behaviour B_1 in example 5.3 we have

$I \ll K \ll L$,

$[I,K] = \{a+d, b+c, b+d, \alpha+c, \beta+c\}$,

$\pi_{IK}(\alpha+c) = a+c$,

$link_{IK}(\alpha+c) = \alpha+c$,

$[K,L] = \{a+d, b+c, b+d, \alpha+\delta, \beta+c\}$,

$\pi_{KL}(\alpha+\delta) = \alpha+c$,

$link_{KL}(\alpha+\delta) = a+\delta$.

Consequently, for

$\mu_I(\{a+c\}) = 1$,

$P_{IK}(\{\alpha+c\}|a+c) = P_{IK}(\{\beta+c\}|a+c) = 0.5$,

$P_{KL}(\{a+\delta\}|a+c) = P_{KL}(\{b+c\}|b+c) = 1$,

we obtain

$\mu_K(\{\alpha+c\}) = \int_{\Gamma_I} P_{IK}(\{\alpha+c\}|cod(\xi)) d\mu_I(\xi)$
$= P_{IK}(\{\alpha+c\}|a+c)\mu_I(\{a+c\}) = 0.5$,

$\mu_K(\{\beta+c\}) = \int_{\Gamma_I} P_{IK}(\{\beta+c\}|cod(\xi)) d\mu_I(\xi)$
$= P_{IK}(\{\beta+c\}|a+c)\mu_I(\{a+c\}) = 0.5$,

$\mu_L(\{\alpha+\delta\}) = \int_{\Gamma_K} P_{KL}(\{a+\delta\}|cod(\xi)) d\mu_K(\xi)$
$= P_{KL}(\{a+\delta\}|a+c)\mu_K(\{\xi \in K : cod(\xi) = a+c\})$
$= P_{KL}(\{b+c\}|a+c)\mu_K(\{\alpha+c\}) = 0.5$,

$\mu_L(\{\beta+c\}) = \int_{\Gamma_K} P_{KL}(\{\beta+c\}|cod(\xi)) d\mu_K(\xi)$
$= P_{KL}(\{b+c\}|b+c)\mu_K(\{\xi \in K : cod(\xi) = b+c\})$
$= P_{KL}(\{b+c\}|b+c)\mu_K(\{\beta+c\}) = 0.5$.

Similarly for other initial segments. $\sharp$

6.14. Example. Consider the behaviour B_2 in example 5.4.

Let Φ and Ψ be respectively the set of runs of the producer and the set of runs of the distributor.

Let Σ be the set of variants of the run $[S]$ of transfer of material from the producer to the distributor.

Let Π be the set of runs of the form $\varphi + \psi$, where $\varphi \in \Phi$ and $\psi \in \Psi$ are respectively the component of the producer and the component of the distributor.

Let $f_s : \Pi \to [0, +\infty)$ be the function with $f_s(\pi)$ defined for every run $\pi \in \Pi$ as the amount of material at disposal of the producer participating in π at the moment s of its local time.

Let $g_s : \Pi \to [0, +\infty)$ be the function with $g_t(\pi)$ defined for every run $\pi \in \Pi$ as the amount of material at disposal of the distributor participating in π at the moment t of its local time.

Given real $b \geq a \geq 0$, $q \geq 0$, and a Borel subset X of the interval $[0, +\infty)$, suppose that $P'_{ab}(X|q)$ is the probability that the producer, which has at the moment a of its local time the amount q of material and acts, gets at the moment b of its local time an amount x of material such that $x \in X$. Suppose that

$$P'_{ac}(X|q) = \int_{[0,+\infty)} P'_{bc}(X|\xi) dP'_{ab}(\xi|q)$$

for all $c \geq b \geq a \geq 0$ and $q \geq 0$.

Given real $b \geq a \geq 0$, $r \geq 0$, and a Borel subset Y of the interval $[0, +\infty)$, suppose that $P''_{ab}(Y|r)$ is the probability that the distributor, which has at the moment a of its local time the amount r of material and acts, gets at the moment b of its local time an amount y of material such that $y \in Y$. Suppose that

$$P''_{ac}(Y|r) = \int_{[0,+\infty)} P''_{bc}(Y|\eta) dP''_{ab}(\eta|r)$$

for all $c \geq b \geq a \geq 0$ and $r \geq 0$.

Given a section $I(s,t)$ of B_2, let $\mathcal{F}_{I(s,t)}$ be the least σ-algebra of subsets of $I(s,t)$ that contains all the inverse-images of Borel subsets of the product $[0, +\infty) \times [0, +\infty)$ under the mappings $h_{s',t'} : \pi \mapsto (f_{s'}(\pi), g_{t'}(\pi))$ with $0 \leq s' \leq s$ and $0 \leq t' \leq t$.

For $0 \leq s' \leq s''$ and $0 \leq t' \leq t''$ we have the σ-algebra $\mathcal{F}_{I(s',t')I(s'',t'')}$ of those $F \subseteq [I(s',t'), I(s'',t'')]$ for which $link^{-1}_{I(s',t')I(s'',t'')}(F) \in \mathcal{F}_{I(s'',t'')}$.

For $q \geq 0$, $r \geq 0$, and Borel subsets X and Y of the interval $[0, +\infty)$, we define

$$P_{I(s',t')I(s'',t'')}(link^{-1}_{I(s',t')I(s'',t'')}\,(f^{-1}_{s''}(X) \cap g^{-1}_{t''}(Y))|\{(p,q),(d,r)\}) =$$

$$= P'_{s's''}(X|q)P''_{t't''}(Y|r)$$

Then for every $q \geq 0$ and $r \geq 0$ the function thus defined extends to a unique probability measure $P_{I(s',t')I(s'',t'')}(.|\{(p,q),(d,r)\})$ on the σ-algebra $\mathcal{F}_{I(s',t')I(s'',t'')}$ of subsets of $[I(s',t'), I(s'',t'')]$ such that the rule (*) is satisfied. Consequently, given a probability measure $\mu_{I(0,0)}$ on the σ-algebra $\mathcal{F}_{I(0,0)}$ of subsets of $I(0,0)$, by applying the rule (**) it is possible to define the probability measures $\mu_{I(s,t)}$ on $\mathcal{F}_{I(s,t)}$ for all $s \geq 0$ and $t \geq 0$, and construct the respective projective system and its limit. As every section of B_2 is dominated by some $I(s,t)$, the result gives the required probability space.

Consider the behaviour B_3 in example 5.4.

Let Φ, Ψ, Π, f_s, g_t, P'_{ab}, P''_{ab}, $h_{s,t}$, $\mathcal{F}_{I(s',t')I(s'',t'')}$, $P_{I(s',t')I(s'',t'')}$, $\mu_{I(s,t)}$ be as before, and let Δ' and Δ'' be given positive real numbers.

Suppose that the producer and the distributor act in steps, the producer Δ' units of its local time in each step, the distributor Δ'' units of its local time in each step, and that each step ends with a transfer of an amount m of material from the producer to the distributor, where $m = \lambda(q', r')$ for the producer with an amount q' of material and the distributor with an amount r' of material.

Then the probability of the system consisting of the producer and the distributor to pass from a state $\xi = \{(p,q),(d,r)\}$ to a state in a Borel subset Z of the product $[0, +\infty) \times [0, +\infty)$ is

$$P_{I(0,0)I(\Delta',\Delta'')}(\Lambda^{-1}_{\Delta'\Delta''}(Z|\xi)$$

where $\Lambda_{\Delta'\Delta''} : \pi \mapsto (f_{\Delta'}(\pi) - \lambda(f_{\Delta'}(\pi), g_{\Delta''}(\pi)), g_{\Delta''}(\pi) - \lambda(f_{\Delta'}(\pi), g_{\Delta''}(\pi)))$.

On the other hand, $\mathcal{F}_{J(n)J(n+1)}$ is the σ-algebra of sets $G(F)$, where $F \in \mathcal{F}_{I(0,0)I(\Delta',\Delta'')}$ and $\gamma \in G(F)$ iff $\gamma = \pi\sigma_\pi$ with $\pi \in F$ and σ_π being the transfer of the amount $\lambda(q', r')$ of material for $\{(p,q'),(d,r')\}$ being the final state of π.

Consequently, for every $n = 1, 2, ...,$ every state $\xi = \{(p,q),(d,r)\}$, and every $G(F) \in \mathcal{F}_{J(n)J(n+1)}$ we can define

$$P_{J(n)J(n+1)}(G(F)|\xi) = P_{I(0,0)I(\Delta',\Delta'')}(F|\xi)$$

and then combine $P_{J(n)J(n+1)}$ to define $P_{J(n)J(m)}$ for arbitrary $1 \leq n \leq m$ such that the rule (*) is satisfied. Hence, given a probability measure $\mu_{I(0,0)}$, we can define $\mu_{J(0)} = \mu_{I(0.0)}$ and $\mu_{J(n)}$ for $n = 0, 1, \ldots$, and construct the respective projective system and its limit. As every section of B_3 is dominated by some $J(n)$, the result gives the required probability space. $\sharp$

7 Models related to Scott topology

The idea described in [14] can be applied to provide with probability measures behaviours which are continuous directed complete posets. Every such a behaviour B together with its Scott open subsets is a topological space with the Borel σ-algebra $\mathcal{B}$ of subsets generated by Scott open subsets. Every normalized continuous valuation ν of Scott open subsets of B extends uniquely to a probability measure ν' on $\mathcal{B}$. Then the probability measure ν' can be transported to the restriction of B to the subspace $\Omega(B)$ formed by the maximal elements of B. To this end, it suffices to define $\mathcal{B}' = \{f \cap \Omega(B) : F \in \mathcal{B}$ and to assign the value $\nu'(F)$ to every $F \cap \Omega(B)$ with $F \in \mathcal{B}$. Consequently, we obtain a probability space $(\Omega(B), \mathcal{B}', \mu)$, as required.

However, in the present paper we try to develop a basis as universal as possible for describing and studying random behaviours of concurrent systems, a basis that would allow us to describe in a uniform way behaviours of systems of various kinds, including behaviours that need not to be continuous directed complete posets. To this end, we shall describe again how the required measure μ on the σ-algebra $\mathcal{B}$' of subsets of the set $\Omega(B)$ of maximal elements of a behaviour B can be obtained from probability distributions on the sets of maximal elements of initial parts of B. The idea is similar to that for set theoretical models, but now it exploits the topological properties of behaviours.

First of all, we define a directed partially ordered set of subspaces of a behaviour B representing initial parts of B and a directed partially ordered set of subspaces of these subspaces consisting of their maximal elements. This can be done as follows.

7.1. Definition. Each subspace of a behaviour B that is downward closed and contains all the exising least upper bounds of its subsets and all the sources of initial segments of maximal elements of B is called an *initial fragment* of B. The subspace $I = \Omega(P)$ of an initial fragment P of B that consists of the maximal elements of P is called a *topological section* (or briefly a *section*) of B.

The set of subsets of $I = \Omega(P)$ of the form $F \cap I$, where F belongs to the Borel σ-algebra $\mathcal{B}$ of subsets of B, is a σ-algebra $\mathcal{B}_I$, called the *natural* σ-algebra of subsets of I. $\sharp$

According to this definition every initial fragment of a behaviour is Scott closed, that it is a directed complete poset, and every topological section consisting of bounded runs is a section in the sense of definition 6.4.

7.2. Example. Each downward closed subspace of the behaviour B_1 in example 5.3 that contains the existing least upper bounds of its subsets of B_1 and contains the subset $I = \{a+c, a+d, b+c, b+d\}$ of B_1 is an *initial fragment* of B_1. In particular, the following subsets I, E, E', E'', F, G of B_1 are initial fragments of B_1 and the following I, J, J', J'', K, L of B_1 are the corresponding sections of B_1:

$I = \{a+c, a+d, b+c, b+d\}$
$E = \{a+c, a+d, b+c, b+d, a+\delta\}$
$E' = \{a+c, a+d, b+c, b+d, \alpha+c, a+\delta\}$
$E'' = \{a+c, a+d, b+c, b+d, \beta+c, a+\delta\}$
$F = \{a+c, a+d, b+c, b+d, \alpha+c, \beta+c\}$
$G = \{a+c, a+d, b+c, b+d, \alpha+c, a+\delta, \alpha+\delta, \beta+c\}$

and the following subsets I, J, J', J'', K, L of B_1 are the corresponding sections of B_1 (see figure 5.1):

$I = \Omega(I) = \{a+c, a+d, b+c, b+d\}$
$J = \Omega(E) = \{a+d, b+c, b+d, a+\delta\}$
$J' = \Omega(E') = \{a+d, b+c, b+d, \alpha+c, a+\delta\}$
$J'' = \Omega(E'') = \{a+d, b+c, b+d, \beta+c, a+\delta\}$
$K = \Omega(F) = \{a+d, b+c, b+d, \alpha+c, \beta+c\}$
$L = \Omega(G) = \{a+d, b+c, b+d, \alpha+\delta, \beta+c\}$ $\sharp$

7.3. Example. Each set of elements of the behaviour B_2 in example 5.4 that are dominated with respect to the prefix order by elements of a section $I(s,t)$ of this behaviour as in example 5.6 is an initial fragment of B_2. Each section $I(s,t)$ as in example 6.6 is a topological section of B_2 in the sense of definition 7.1.

The σ-algebra $\mathcal{F}_{I(s,t)}$ of subsets of $I(s,t)$ that was defined in example 6.14 consists of intersections of $I(s,t)$ with members of the least σ-algebra cotaining sets $\{\pi \in B_2 : f_{s'}(\pi) \leq x\}$ with $0 \leq s' \leq s$ and sets $\{\pi \in B_2 : g_{t'}(\pi) \leq y\}$ with

$0 \le t' \le t$. On the other hand, such sets are Scott closed if processes of the producer and distributors consist of continuous segments. Consequently, the σ-algebra $\mathcal{F}_{I(s,t)}$ is then a subalgebra of the natural σ-algebra $\mathcal{B}_{I(s,t)}$.

Each set of elements of the behaviour B_3 in example 5.4 that are dominated by elements of a section $J(n)$ of this behaviour as in example 6.6 is an initial fragment of B_3 and $J(n)$ itself is a topological section of B_3. $\sharp$

A projective system consisting of a directed family of probability spaces characterizing initial parts of a behaviour can be constructed due to the existence of a directed set of topological sections of this behaviour and due to the existence of projections of topological sections on dominated topological sections.

7.4. Proposition. Let P and Q be two initial fragments of a behaviour B such that $P \subseteq Q$, and let $I = \Omega(P)$ and $J = \Omega(Q)$. For every $j \in J$ there exists a unique $i \in I$, written as $\rho_{IJ}(j)$, such that $i \sqsubseteq j$. $\sharp$

Proof. Let X_j be the set of $k \in P$ such that $k \sqsubseteq j$. The set X_j is nonempty since it contains $dom(j)$. It is directed since every two elements of X_j consist of prefixes of j and have the least upper bound that belongs to X_j. Consequently, there exists the least upper bound m of X_j and $m \sqsubseteq j$. As P is Scott closed, we have $m \in P$. As m is the least upper bound of X_j, it must belong to $I = \Omega(P)$, and we can define $\rho_{IJ}(j)$ as m. $\sharp$

From the fact that an initial fragment of a behaviour is downward closed and contains the existing least upper bounds of its subsets we obtain the following proposition.

7.5. Proposition. A subset X of an initial fragment P of a behaviour B is Scott closed iff it is Scott closed in the directed complete poset P. $\sharp$

It follows from proposition 7.4 that for every $U \cap I$ with Scott open U the set $U \cap J$ is the inverse image of $U \cap I$ under $\rho_{IJ}(j)$. Consequently, we obtain the following proposition.

7.6. Proposition. The correspondence $\rho_{IJ} : J \to I$ is a measurable mapping from J equipped with the σ-algebra $\mathcal{B}_J$ to I equipped with the σ-algebra $\mathcal{B}_I$. $\sharp$

The set of initial fragments of a behaviour B is ordered by inclusion. According to proposition 7.4 the set of topological sections of B can be defined as follows.

7.7. Definition. We say that a topological section I of B *precedes* another such a section J, and we write $I \ll J$, iff each element of J has a predecessor in I. $\sharp$

7.8. Proposition. The set of all topological sections of B with the partial order $\ll$ is a directed set $\mathcal{R}(B)$. $\sharp$

For a proof it suffices to consider two arbitrary sections of B, say I and J, and to notice that the set K of maximal elements of the union of the downward closures of I and J is a section of B.

Now we may use the directed set $\mathcal{R}(B)$ to construct the required probability space as a projective limit of a projective system of probability spaces.

A projective system consisting of a directed family of probability spaces characterizing initial fragments of a behaviour can be defined as follows.

For $I \in \mathcal{R}(B)$, let $\mathbf{\Xi}_I = (\Xi_I, \mathcal{X}_I, \mu_I)$ be probability spaces such that

(1) $\Xi_I = I$,

(2) $\mathcal{X}_I$ is the σ-algebra $\mathcal{B}_I$ of subsets of I.

For $I, J \in \mathcal{R}(B)$ such that $I \ll J$, let $\rho_{IJ} : \Xi_J \to \Xi_I$ be the mappings as in proposition 7.4.

The following facts follow easily from definitions.

7.9. Proposition. Every mapping $\rho_{IJ} : \Xi_J \to \Xi_I$ is measurable and the induced mapping $F \mapsto \rho_{IJ}^{-1}(F)$ maps $\mathcal{X}_I$ into $\mathcal{X}_J$. $\sharp$

7.10. Proposition. If $\mu_I(\rho_{IJ}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{X}_I$ then $\rho_{IJ} : \Xi_J \to \Xi_I$ is a morphism $\rho_{IJ} : \boldsymbol{\Xi}_J \to \boldsymbol{\Xi}_I$ in **PSPACES**. $\sharp$

7.11. Theorem. If $\mu_J(\rho_{IJ}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{X}_I$ then $(\boldsymbol{\Xi}_I \stackrel{\rho_{IJ}}{\leftarrow} \boldsymbol{\Xi}_J : I, J \in \mathcal{R}(B), I \ll J)$ is a projective system in **PSPACES**. $\sharp$

Let $\boldsymbol{\Xi} = (\Omega(B), \mathcal{F}, \mu)$ be a probability space such that $\mathcal{F}$ is the σ-algebra $\mathcal{B}_B$ of subsets of $\Omega(B)$.

7.12. Theorem. The probability space $\boldsymbol{\Xi} = (\Omega(B), \mathcal{F}, \mu)$ is the projective limit of the projective system $(\boldsymbol{\Xi}_I \stackrel{\rho_{IJ}}{\leftarrow} \boldsymbol{\Xi}_J : I, J \in \mathcal{R}(B), I \ll J)$, where each $\boldsymbol{\Xi}_I = (\Xi_I, \mathcal{X}_I, \mu_I)$ is the probability space such that

(1) $\Xi_I = I$,

(2) $\mathcal{X}_I$ is the σ-algebra $\mathcal{B}_I$,

(3) $\mu(\rho_{IB}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{X}_I$. $\sharp$

The fact that the probability space characterizing a random behaviour of a concurrent system is a projective limit of probability spaces characterizing initial fragments of this behaviour can be exploited in an effective way because referring only to initial fragments of this behaviour we are able to decide which subsets of topological sections belong to the respective σ-algebras. Consequently, we can try approximate the required probability space by simpler probability spaces.

Another approach can be to try to characterize the required probability distribution on the set $\Omega(B)$ with the aid of a probability space $(B, \mathcal{B}, \mu)$ and try to approximate the space $(B, \mathcal{B}, \mu)$ by simpler probability spaces. To this end, we can exploit simple theorems of measure theory.

Given an initial fragment P of a behaviour B, let $\mathcal{B}(P)$ be the σ-algebra of those Borel subsets of B whose inverse images under ρ_{PB} are Borel subsets of P.

7.13. Theorem. For every initial fragments P and Q of B such that $P \subseteq Q$ there exists a conditional probability distribution $\mu_{PQ} : \mathcal{B}(Q) \times \Omega_P \to [0,1]$ on $\mathcal{B}(Q)$ with respect to $\mathcal{B}(P)$ and we have

$\int_E \mu_{PQ}(F|x)d\mu_P(x) = \mu_Q(F \cap E)$

for all $F \in \mathcal{B}(Q)$ and $E \in \mathcal{B}(P)$. $\sharp$

A proof follows from the definition of the conditional probability.

7.14. Theorem. For every initial fragments P, Q, R of B such that $P \subseteq Q \subseteq R$, every $G \in \mathcal{B}(R)$, and every $x \in B$, it holds

$\mu_{PR}(G|x) = \int_B \mu_{QR}(G|y)d\mu_{PQ}(y|x)$ $\sharp$

For a proof it suffices to notice that

$\mu_R(E \cap G) = \int_E \mu_{QR}(G|y)d\mu_Q(y) = \int_E \int_B \mu_{QR}(G|y)d\mu_Q(y|x)d\mu_P(x)$
and
$\mu_R(E \cap G) = \int_E \mu_{PR}(G|x)d\mu_P(x)$

Once a probability space $(B, \mathcal{B}, \mu)$ as described is found, it is possible to use it to transport the required probability measure μ to the set $\Omega(B)$. It suffices to define $\mu'(F \cap \Omega(B))$ as $\mu(F)$ for every $F \cap \Omega(B)$ with $F \in \mathcal{B}$.

8 Concluding remark

The possibility of representing the probability space that characterizes the random behaviour of a concurrent system as a projective limit of a projective system of probability spaces characterizing initial parts of the behaviour gives a chance to approximate this space by spaces containing only partial runs. In particular, it may open a chance to work out methods for automatic checking of satisfiability of formulas of probabilistic temporal logics.

Appendix A: Posets and their cross-sections

Given a partial order $\leq$ on a set X, i.e. a binary relation which is reflexive, anti-symmetric and transitive, we call $P = (X, \leq)$ a *partially ordered set*, or briefly a *poset*, by the *strict partial order* corresponding to $\leq$ we mean $<$, where

$x < y$ iff $x \leq y$ and $x \neq y$, by a *chain* we mean a subset $Y \subseteq X$ such that $x \leq y$ or $y \leq x$ for all $x, y \in Y$, and by an *antichain* we mean a subset $Z \subseteq X$ such that $x < y$ does not hold for any $x, y \in Z$.

A.1. Definition. Given a poset $P = (X, \leq)$, by a *strong cross-section* of P we mean a maximal antichain Z of P that has an element in every maximal chain of P. By a *weak cross-section*, or briefly a *cross-section*, of P we mean a maximal antichain Z of P such that, for every $x, y \in X$ for which $x \leq y$ and $x \leq z'$ and $z'' \leq y$ with some $z', z'' \in Z$, there exists $z \in Z$ such that $x \leq z \leq y$. $\sharp$

A.2. Definition. We say that a partial order $\leq$ on X (and the poset $P = (X, \leq)$) is *strongly K-dense* (resp.: *weakly K-dense*) iff every maximal antichain of P is a strong (resp.: a weak) cross-section of P (cf. [11] and [12], where K-density is defined as our strong K-density). $\sharp$

A.3. Definition. For every cross-section Z of a poset $P = (X, \leq)$, we define $X^-(Z) = \leq Z (= \{x \in X : x \leq z \text{ for some } z \in Z\})$ and $X^+(Z) = Z \leq = (\{x \in X : z \leq x \text{ for some } z \in Z\})$, and we say that a cross-section Z' *precedes* a cross-section Z'' and write $Z' \preceq Z''$ iff $X^-(Z') \subseteq X^-(Z'')$. $\sharp$

A.4. Proposition. The relation $\preceq$ is a partial order on the set of cross-sections of $P = (X, \leq)$. For every two cross-sections Z' and Z'' of P there exist the greatest lower bound $Z' \wedge Z''$ and the least upper bound $Z' \vee Z''$ of Z' and Z'' with respect to $\preceq$, where $Z' \wedge Z''$ is the set of those $z \in Z' \cup Z''$ for which $z \leq z'$ for some $z' \in Z'$ and $z \leq z''$ for some $z'' \in Z''$, and $Z' \vee Z''$ is the set of those $z \in Z' \cup Z''$ for which $z' \leq z$ for some $z' \in Z'$ and $z'' \leq z$ for some $z'' \in Z''$. Moreover, the set of cross-sections of P with the operations thus defined is a distributive lattice. $\sharp$

Proof. The set $Z' \wedge Z''$ is an antichain since otherwise there would be $x < y$ for some x and y in this set. If $x \in Z'$ then there would be $y \in Z''$ and there would exist $z' \in Z'$ such that $y \leq z'$. However, this is impossible since Z' is an antichain. Similarly for $x \in Z''$.

The set $Z' \wedge Z''$ is a maximal antichain since otherwise there would exist x that would be incomparable with all the elements of this set. Consequently, there would not exist $z' \in Z'$ and $z'' \in Z''$ such that $z' \leq x \leq z''$, or $z'' \leq x \leq$

z', or $z', z'' \leq x$, and thus there would be $x \leq z'$ and $x \leq z''$ for some $z' \in Z'$ and $z'' \in Z''$ that are not in $Z' \wedge Z''$. Consequently, there would exist z, say in Z'', such that $x \leq z \leq z'$. Moreover, $z \in Z' \wedge Z''$ since otherwise there would be $t \in Z'$ such that $t \leq z \leq z'$, what is impossible.

In order to see that $Z' \wedge Z''$ is a cross-section we consider $x \leq y$ such that $x \leq t$ and $u \leq y$ for some $t \in Z' \wedge Z''$ and $u \in Z' \wedge Z''$, where $t \in Z'$ and $u \in Z''$. Without a loss of generality we can assume that $y \leq y'$ for some $y' \in Z'$ since otherwise we could replace y by an element of Z'. Consequently, there exists $z \in Z''$ such that $x \leq z \leq y$. On the other hand, $z \in Z' \wedge Z''$ since otherwise there would be $z' \in Z'$ such that $z' \leq z \leq y$, what is impossible. In a similar manner we can find $z \in Z' \wedge Z''$ for the other cases of t and u.

In order to see that $Z' \wedge Z''$ is the greatest lower bound of Z' and Z'' consider a cross-section Y which precedes Z' and Z'' and observe that $y \leq z' \in Z'$ and $y \leq z'' \in Z''$ with z' and z'' not in $Z' \wedge Z''$ and $y \in Y$ implies the existence of $t \in Z'$ such that $y \leq t \leq z'$ or $u \in Z''$ such that $y \leq u \leq z''$.

Similarly, $Z' \vee Z''$ is a cross-section and the least upper bound of Z' and Z''.

The last part of the proposition is a consequence of the easily verifiable inequality $Z \wedge (Z' \vee Z'') \preceq (Z \wedge Z') \vee (Z \wedge Z'')$ ♯

A.5. Definition. For cross-sections Z' and Z'' of a poset $P = (X, \leq)$ such that $Z' \preceq Z''$ we define a *segment* of P from Z' to Z'' as the restriction of P to the set $[Z', Z''] = X^+(Z') \cap X^-(Z'')$, written as $P|[Z', Z'']$. A segment $P|[Y', Y'']$ such that $Z' \preceq Y' \preceq Y'' \preceq Z''$ is called a *subsegment* of $P|[Z', Z'']$. If $Z' \neq Y'$ or $Y'' \neq Z''$ (resp.: if $Z' = Y'$, or if $Y'' = Z''$) then we call it a *proper* (resp.: an *initial*, or a *final*) subsegment of $P|[Z', Z'']$. ♯

The following proposition follows easily from definitions.

A.6. Proposition. For every strong or weak cross-section Z of a poset $P = (X, \leq)$ the reflexive and transitive closure of the union of the restrictions of the partial order $\leq$ to $X^-(Z)$ and to $X^+(Z)$ is exactly the partial order $\leq$. ♯

A.7. Proposition. A poset $P = (X, \leq)$ is said to be *locally complete* if every segment $P|[Z', Z'']$ of P is a complete lattice. ♯

A.8. Definition. Given a partial order $\leq$ on a set X and a function $l : X \to W$ that assigns to every $x \in X$ a label $l(x)$ from a set W, we call $L = (X, \leq, l)$ a *labelled partially ordered set*, or briefly an *lposet*, by a *chain* (resp.: an *antichain*, a *cross-section*) of L we mean a chain (resp.: an antichain, a cross-section) of $P = (X, \leq)$, by a *segment* of L we mean each restriction of L to a segment of P, and we say that L is *K-dense* (resp.: *weakly K-dense*, *locally complete*) iff $\leq$ is K-dense (resp.: weakly K-dense, locally complete). $\sharp$

By **LPOSETS** we denote the category of lposets and their morphisms, where a *morphism* from an lposet $L = (X, \leq, l)$ to an lposet $L' = (X', \leq', l')$ is defined as a mapping $b : X \to X'$ such that, for all x and y, $x \leq y$ iff $b(x) \leq' b(y)$, and, for all x, $l(x) = l'(b(x))$. In the category **LPOSETS** a morphism from $L = (X, \leq, l)$ to $L' = (X', \leq', l')$ is an *isomorphism* iff it is bijective, and it is an *automorphism* iff it is bijective and $L = L'$. If there exists an isomorphism from an lposet L to an lposet L' then we say that L and $L'$are *isomorphic*. A *partially ordered multiset*, or briefly a *pomset*, is defined as an isomorphism class ξ of lposets. Each lposet that belongs to such a class ξ is called an *instance* of ξ. The pomset corresponding to an lposet L is written as $[L]$.

Appendix B: Directed complete posets

Let $(X, \sqsubseteq)$ be a partially ordered set (poset). A subset $Y \subseteq X$ is said to be *downward closed* (resp. : *upward closed*) if $Y = \sqsubseteq Y (= \{x \in X : x \sqsubseteq y \text{ for some } y \in Y\})$ (resp. : $Y = Y \sqsubseteq (= \{x \in X : y \sqsubseteq x \text{ for some } y \in Y\})$). A nonempty subset $Y \subseteq X$ is said to be em bounded complete if every bounded subset of Y has a least upper bound. A nonempty subset $Y \subseteq X$ is said to be *directed* if for all $x, y \in Y$ there exists $z \in Y$ such that $x, y \sqsubseteq z$. The *Scott topology* of $(X, \sqsubseteq)$ is the topology on X in which a subset $U \subseteq X$ is open iff it is upward closed and disjoint with every directed $Y \subseteq X$ which has the least upper bound $\sqcup Y$. A poset is said to be *coherent* if every of its consistent subsets has a least upper bound. A poset is said to be a *directed complete partial order (DCPO)* if every of its directed subsets has a least upper bound.

Let $(X, \sqsubseteq)$ be a DCPO. An element $x \in X$ is said to *approximate* an element $y \in X$, or that x is *way below* y, if in every directed set Z such that $y \sqsubseteq \sqcup Z$ there exists z such that $x \sqsubseteq z$. An element $x \in X$ is said to be a *compact* if it approximates itself. A subset $B \subseteq X$ is called a *basis* of $(X, \sqsubseteq)$ if for every $x \in X$ the set of those elements of B which approximate x is directed and has the least upper bound equal to x. The DCPO $(X, \sqsubseteq)$ is said to be *continuous* if it has a basis, and *ω-continuous* if it has a countable basis. The DCPO $(X, \sqsubseteq)$ is said to be an *algebraic domain* if every $y \in X$ is the directed least upper bound of all compact elements x such that $x \sqsubseteq y$.

Appendix C: Probability spaces

Given a set X, by a *σ-algebra* of subsets of X we mean a set $\mathcal{F}$ of subsets of X such that $X \in \mathcal{F}$ and $\mathcal{F}$ is closed under complements and countable unions, and we call the pair $(X, \mathcal{F})$ a *measurable space*. If X is given with a topology τ then the least σ-algebra that contains τ is called the *Borel σ-algebra* of the topological space (X, τ).

Given measurable spaces $(X, \mathcal{F})$ and $(X', \mathcal{F}')$, a mapping $f : X \to X'$ is said to be *$\mathcal{F}$-measurable*, or a morphism from $(X, \mathcal{F})$ to $(X', \mathcal{F}')$, iff $f^{-1}(F') \in \mathcal{F}$ for every $F' \in \mathcal{F}'$.

By **MES** we denote the category of measurable spaces and their morphisms.

By a *probability space* we mean a triple $(\Omega, \mathcal{F}, \mu)$, where Ω is a set (the set of possible realizations of a random phenomenon), $\mathcal{F}$ is a σ-algebra of subsets of Ω, and μ is a real valued function on $\mathcal{F}$, called a *probability measure*, such that $0 \leq \mu(F) \leq 1$ for all $F \in \mathcal{F}$, $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, and $\mu(F_0 \cup F_1 \cup ...) = \mu(F_0) + \mu(F_1) + ...$ for mutually disjoint F_0, F_1,... from $\mathcal{F}$.

Given two probability spaces $\mathbf{\Omega} = (\Omega, \mathcal{F}, \mu)$ and $\mathbf{\Omega}' = (\Omega', \mathcal{F}', \mu')$ by a morphism from $\mathbf{\Omega}$ to $\mathbf{\Omega}'$ we mean a triple $f : \mathbf{\Omega} \to \mathbf{\Omega}'$, where f is a mapping from Ω to Ω' such that $f^{-1}(F') \in \mathcal{F}$ and $\mu(f^{-1}(F')) = \mu'(F')$ for every $F' \in \mathcal{F}'$.

By **PSPACES** we denote the category of probability spaces and their morphisms.

Given a probability space $\mathbf{\Omega} = (\Omega, \mathcal{F}, \mu)$ and a σ-algebra $\mathcal{E} \subseteq \mathcal{F}$, there exists a function $f : \mathcal{F} \times \Omega \to [0, 1]$ such that, for every $F \in \mathcal{F}$, the function $\omega \mapsto f(F|\omega)$ ($= f(F, \omega)$), is $\mathcal{E}$-measurable and for all $E \in \mathcal{E}$ it satisfies the equation

$$\int_E f(F|\omega) d\mu(\omega) = \mu(F \cap E).$$

Function f is called a *conditional probability distribution* in $(\Omega, \mathcal{F})$ with respect to $\mathcal{E}$. If f is such that $F \mapsto f(F|\omega)$ is a probability measure on $\mathcal{F}$ for every $\omega \in \Omega$ then it is called a *strict conditional probability distribution* in

$(\Omega, \mathcal{F})$ with respect to $\mathcal{E}$. Every function $\omega \mapsto f(F|\omega)$ is called a *variant* of *conditional probability* of F with respect to $\mathcal{E}$.

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