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Behaviour Algebras

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Abstract. The paper is concerned with algebras whose elements can be used to represent runs of a system, called processes. These algebras, called behaviour algebras, are categories with respect to a partial binary operation called sequential composition, and they are partial monoids with respect to a partial binary operation called parallel composition. They are characterized by axioms such that their elements and operations can be represented by labelled posets and operations on such posets. The respective representation is obtained without assuming a discrete nature of represented elements. In particular, it remains true for behaviour algebras with infinitely divisible elements, and thus also with elements which can represent continuous and partially continuous processes. An important consequence of the representation of elements of behaviour algebras by labelled posets is that elements of some subalgebras of behaviour algebras can be endowed in a consistent way with structures such as a graph structure etc.

Keywords: Processes, states, sequential composition, parallel composition, category, partial monoid, structure.

1. Introduction

In this paper we study algebras whose elements can be interpreted as bounded processes, where by a process we mean a run of a system as in the theory of Petri nets (cf. for example [3], [15], and [8]). (Note that this understanding of term process is different from that in CCS and other similar calculi (cf. for example [2] and [11]), where processes are understood as evolving objects.)

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In [19] it has been described how to assign to every Condition/Event Petri net a partial algebra $\mathcal{A} = (A, dom, cod, ;, +, 0)$, where A is the set of finite processes of this net, $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are unary operations assigning respectively the initial and the final state to each process α, $(\alpha_1, \alpha_2) \mapsto \alpha_1; \alpha_2$, where $\alpha_1; \alpha_2$ is written also as $\alpha_1\alpha_2$, is the partial operation of composing sequentially processes of which α_1 leads to a state from which α_2 starts, $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ is the partial operation of composing in parallel concurrent processes, and 0 is the empty process. It has been shown that the following axioms hold in such an algebra.

(A1) The reduct $(A, dom, cod, ;)$ of $\mathcal{A}$ is a (morphisms-only) category $cat(\mathcal{A})$, and it enjoys the following properties:

(A1.1) if $\sigma\alpha$ and $\sigma'\alpha$ are defined and $\sigma\alpha = \sigma'\alpha$ then $\sigma = \sigma'$,

(A1.2) if $\alpha\tau$ and $\alpha\tau'$ are defined and $\alpha\tau = \alpha\tau'$ then $\tau = \tau'$,

(A1.3) if $\sigma\tau$ is an identity then σ and τ are also identities,

(A1.4) if $\sigma\alpha\tau$ is defined and $\sigma\alpha\tau = \alpha$ then σ and τ are identities.

(A2) The reduct $(A, +, 0)$ of $\mathcal{A}$ is a partial monoid $pmon(\mathcal{A})$ with the partial operation $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ and its neutral element 0, and it enjoys the following properties:

(A2.1) $\alpha_1 + (\alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2) + \alpha_3$ whenever either (that is at least one) side is defined,

(A2.2) $\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1$ whenever either side is defined,

(A2.3) if $\alpha + \sigma$ and $\alpha + \sigma'$ are defined and $\alpha + \sigma = \alpha + \sigma'$ then $\sigma = \sigma'$,

(A2.4) $\alpha + \alpha$ is defined only for $\alpha = 0$,

(A2.5) given a family $(\alpha_i : i \in \{1, ..., n\})$, where $n \geq 2$, if $\alpha_i + \alpha_j$ are defined for all $i, j \in \{1, ..., n\}$ such that $i \neq j$ then $\alpha_1 + ... + \alpha_n$ is defined,

(A2.6) the following relation $\sqsubseteq$ is a partial order:
$\alpha_1 \sqsubseteq \alpha_2$ iff α_2 contains α_1 in the sense that $\alpha_2 = \alpha_1 + \rho$ for some ρ,

(A2.7) for all α_1 and α_2 there exists the greatest lower bound of α_1 and α_2 with respect to $\sqsubseteq$, written as $\alpha_1 \sqcap \alpha_2$,

(A2.8) if $\alpha_1 + \alpha_2$ is defined then $(\alpha_1 \sqcap \sigma) + (\alpha_2 \sqcap \sigma)$ is defined and $(\alpha_1 \sqcap \sigma) + (\alpha_2 \sqcap \sigma) = (\alpha_1 + \alpha_2) \sqcap \sigma$,

(A2.9) if $\alpha_1 \sqcap \alpha_2 = 0$ and $\alpha_1 \sqsubseteq \alpha$ and $\alpha_2 \sqsubseteq \alpha$ for some α then $\alpha_1 + \alpha_2$ is defined,

(A2.10) each $\alpha \neq 0$ contains some β that is a $(+)$-*atom* in the sense that $\beta \neq 0$ and $\beta = \alpha_1 + \alpha_2$ only if either $\alpha_1 = \beta$ and $\alpha_2 = 0$ or $\alpha_1 = 0$ and $\alpha_2 = \beta$; in particular, each identity of the category $cat(\mathcal{A})$ contains a $(+)$-*atom* and this $(+)$-*atom* is an identity of $cat(\mathcal{A})$, called an *atomic identity*.

(A2.11) each α is determined uniquely by the set $h(\alpha)$ of $(+)$-atoms it contains in the sense that $h(\alpha_1) = h(\alpha_2)$ implies $\alpha_1 = \alpha_2$; in particular, each identity u is determined uniquely by the set $h(u)$ of atomic identities it contains.

(A3) The reducts $cat(\mathcal{A})$ and $pmon(\mathcal{A})$ are related to each other so that:

(A3.1) $dom(\alpha_1 + \alpha_2) = dom(\alpha_1) + dom(\alpha_2)$ whenever $\alpha_1 + \alpha_2$ is defined, (A3.2) $cod(\alpha_1 + \alpha_2) = cod(\alpha_1) + cod(\alpha_2)$ whenever $\alpha_1 + \alpha_2$ is defined,

(A3.3) $dom(\alpha) = 0$ implies $\alpha = 0$ and $cod(\alpha) = 0$ implies $\alpha = 0$,

(A3.4) if $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22})$ is defined then $\alpha_{11} + \alpha_{21}$, $\alpha_{11} + \alpha_{22}$, $\alpha_{12} + \alpha_{21}$, $\alpha_{12} + \alpha_{22}$ are also defined and $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22}) = (\alpha_{11} + \alpha_{21})(\alpha_{12} + \alpha_{22})$,

(A3.5) if $\alpha_{11}\alpha_{12}$ and $\alpha_{21}\alpha_{22}$ are defined, and $\alpha_{11} + \alpha_{21}$ is defined, or $\alpha_{11} + \alpha_{22}$ is defined, or $\alpha_{12} + \alpha_{21}$ is defined, or $\alpha_{12} + \alpha_{22}$ is defined, then $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22})$ is defined,

(A3.6) $\alpha_1 + \alpha_2 = \beta_1\beta_2$ implies the existence of unique α_{11}, α_{12}, α_{21}, α_{22} such that $\alpha_1 = \alpha_{11}\alpha_{12}$, $\alpha_2 = \alpha_{21}\alpha_{22}$, $\beta_1 = \alpha_{11} + \alpha_{21}$, $\beta_2 = \alpha_{12} + \alpha_{22}$.

(A4) In $pmon(\mathcal{A})$ there exists the least congruence $\sim$ such that $\alpha \sim dom(\alpha) \sim cod(\alpha)$ for all α.

(A5) A diagram $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in $cat(\mathcal{A})$ if and only if there exist c, φ_1, φ_2 such that c is an identity, $c + \varphi_1 + \varphi_2$ is defined, $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$, $\alpha_2 = c + dom(\varphi_1) + \varphi_2$, $\alpha_1' = c + \varphi_1 + cod(\varphi_2)$, $\alpha_2' = c + cod(\varphi_1) + \varphi_2$.

(A6) For all ξ_1, ξ_2, η_1, η_2 such that $\xi_1\xi_2 = \eta_1\eta_2$ there exist unique σ_1, σ_2, and a unique bicartesian square $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$, such that $\xi_1 = \sigma_1\alpha_1$, $\xi_2 = \alpha_2'\sigma_2$, $\eta_1 = \sigma_1\alpha_2$, $\eta_2 = \alpha_1'\sigma_2$.

(A7) Given α such that $dom(\alpha)$ contains an atomic identity p and $cod(\alpha)$ contains an atomic identity q, if α cannot be represented as $(p + \alpha_1)(q + \alpha_2)$ then for every ξ and η such that $\alpha = \xi\eta$ the state $cod(\xi) = dom(\eta)$ contains an atomic identity m such that ξ cannot be represented as $(p + \xi_1)(m + \xi_2)$ and η cannot be represented as $(m + \eta_1)(q + \eta_2)$.

(A8) Every α that is not an identity can be represented in the form $\alpha = \alpha_1 ... \alpha_n$, where α_1,...,α_n are $(;)$-*atoms* in the sense that they are not results of composing sequentially elements that are not identities.

(A9) For every atomic identity p there exists exactly one atomic identity p' which is different from p and such that $p' \sim p$.

It has been shown that every algebra in which the above axioms hold is isomorphic to the algebra of finite processes of a Condition/Event Petri net.

Algebras in which (A1) - (A9) hold are members of a larger class of algebras in which only (A1) -(A7) hold, called in [19] process systems, and of the subclass of this class in which also (A8) holds, called discrete process systems.

In the present paper we study algebras in which only (A1) - (A6) hold, called after [10] and [16] behaviour algebras, prove that some of such algebras can be represented as algebras of isomorphism classes of labelled partially ordered sets, and show how they can be used to represent processes of systems of various types, including continuous and hybrid systems, and systems with rich structures of states and processes.

2. Processes in a universe of objects

In this section we describe a subclass of concrete behaviour algebras that plays a particular role. Each member of this subclass is a behaviour algebra whose processes represent activities in a universe of objects, each object with a set of possible internal states and instances corresponding to these states, each activity changing states of some objects. Processes of such a behaviour algebra are defined following to some extent the definition in [6] of the topological structure of the physical world of Einstein's general theory of relativity.

In order to introduce suitable notions we start with some preliminaries.

Given a partial order $\leq$ on a set X, we call $\mathbf{X} = (X, \leq)$ a *partially ordered set*, or briefly a *poset*, by the *strict partial order* corresponding to $\leq$ we mean $<$, where $x < y$ iff $x \leq y \text{ and } x \neq y$. By a *strong cross-section* of $(X, \leq)$ we mean a maximal antichain Z of $(X, \leq)$ that has an element in every maximal chain of $(X, \leq)$. By a *weak cross-section*, or briefly a *cross-section*, of $(X, \leq)$ we mean a maximal antichain Z of $(X, \leq)$ such that, for every $x, y \in X$ for which $x \leq y$ and $x \leq z'$ and $z'' \leq y$ with some $z', z'' \in Z$, there exists $z \in Z$ such that $x \leq z \leq y$. We say that $\leq$ (and $(X, \leq)$) is K-*dense* (resp. *weakly* K-*dense*) iff every maximal antichain of $(X, \leq)$ is a strong cross-section (resp. (weak) cross-section) of $(X, \leq)$ (cf. [13] and [14]). For every cross-section Z of $(X, \leq)$, we define $X^-(Z) = \{x \in X : x \leq z \text{ for some } z \in Z\}$ and $X^+(Z) = \{x \in X : z \leq x \text{ for some } z \in Z\}$, and we say that a cross-section Z' *precedes* a cross-section Z'' and write $Z' \preceq Z''$ iff $X^-(Z') \subseteq X^-(Z'')$. The relation $\preceq$ thus defined is a partial order on the set of cross-sections of $(X, \leq)$. For cross-sections Z' and Z'' such that $Z' \preceq Z''$ we define a *segment* of $\mathbf{X}$ from Z' to Z'' as the restriction of $\mathbf{X}$ to the set $[Z', Z''] = X^+(Z') \cap X^-(Z'')$, written as $\mathbf{X}|[Z', Z'']$. A segment $\mathbf{X}|[Y', Y'']$ such that $Z' \preceq Y' \preceq Y'' \preceq Z''$ is called a *subsegment* of $\mathbf{X}|[Z', Z'']$. If $Z' \neq Y'$ or $Y'' \neq Z''$ (resp. if $Z' = Y'$, or if $Y'' = Z''$) then we call it a *proper* (resp. an *initial*, or a *final*) subsegment of $\mathbf{X}|[Z', Z'']$.

Given a partial order $\leq$ on a set X and a function $l : X \to W$ that assigns to every $x \in X$ a label $l(x)$ from a set W, we call $\mathcal{X} = (X, \leq, l)$ a *labelled partially ordered set*, or briefly an *lposet*, by a *chain* (resp. an *antichain*, a *cross-section*) of $\mathcal{X}$ we mean a chain (resp. an antichain, a cross-section) of $(X, \leq)$, by a *segment* of $\mathcal{X}$ we mean the restriction of $\mathcal{X}$ to a segment of $(X, \leq)$, and we say that $\mathcal{X}$ is K-*dense* (resp. *weakly* K-*dense*) iff $\leq$ is K-dense (resp. weakly K-dense).

By $LPOSETS$ we denote the category of lposets and their morphisms, where a *morphism* from an lposet $\mathcal{X} = (X, \leq, l)$ to an lposet $\mathcal{X}' = (X', \leq', l')$ is defined as an injection $b : X \to X'$ such that, for all x and y, $x \leq y$ iff $b(x) \leq' b(y)$, and, for all x, $l(x) = l'(b(x))$. In the category $LPOSETS$ a morphism from $\mathcal{X}$ to $\mathcal{X}'$ is an *isomorphism* iff it is bijective, and it is an *automorphism* iff it is bijective and $\mathcal{X} = \mathcal{X}'$. If there exists an isomorphism from an lposet $\mathcal{X}$ to an lposet $\mathcal{X}'$ then we say that $\mathcal{X}$ and $\mathcal{X}'$are *isomorphic*. A *partially ordered multiset*, or briefly a *pomset*, is defined as an isomorphism class ξ of lposets. Each lposet that belongs to such a class ξ is called an *instance* of ξ. The pomset corresponding to an lposet $\mathcal{X}$ is written as $[\mathcal{X}]$.

A universe of objects and processes in such a universe can be defined as follows.

2.1. Definition. By a *universe of objects* we mean a structure $U = (W, V, ob)$, where V is a set of *objects*, W is a set of *instances* of objects from V (a set of *object instances*), and $ob : W \to V$ is a mapping that assigns the respective object to each of its instances. $\sharp$

2.2. Example. Consider a producer p that produces some material for a distributor d and a car c that transports portions of material from p to d. Define an instance of p to be a pair (p, q), where $q \geq 0$ is amount of material at disposal of p. Define an instance of d to be a pair (d, r), where $r \geq 0$ is amount of material at disposal of d. Define an instance of c to be a triple (c, m, s), where $m \geq 0$ is amount of material in the car and s is position of the car that may change from 0 (when the car is at p) to 1 (when the car is at d). Define $V = \{p, d, c\}$, $W = W_p \cup W_d \cup W_c$, where $W_p = \{(p, q) : q \geq 0\}$, $W_d = \{(d, r) : r \geq 0\}$, $W_c = \{(c, m, s) : m \geq 0, 0 \leq s \leq 1\}$. Define $ob(w) = p$ for $w = (p, q) \in W_p$, $ob(w) = d$ for $w = (d, r) \in W_d$, and $ob(w) = c$ for $w = (c, m, s) \in W_c$. Then $U = (W, V, ob)$ is a universe of objects. $\sharp$

2.3. Definition. By a *concrete process* in a universe $U = (W, V, ob)$ of objects we mean a labelled partially ordered set $P = (X, \leq, ins)$, where

(1) X is a set (of *occurrences* of objects from V, called *object occurrences*),

(2) $ins : X \to W$ is a mapping (a *labelling* that assigns an object instance to each occurrence of the respective object),

(3) the partial order $\leq$ of P (the *flow order* of P) is such that

  (3.1) for every object $v \in V$, the set $X|v = \{x \in X : ob(ins(x)) = v\}$ is either empty or it is a maximal chain and has an element in every cross-secton,

  (3.2) every element of X belongs to a cross-section,

  (3.3) no segment of P is isomorphic to its proper subsegment. $\sharp$

Condition (3.1) means that P contains all information on the behaviour within P of every object which has in P an occurrence, and that every potential global state of P contains an element of this information. Condition (3.2) guarantees that each occurrence of an object in P belongs to a potential global state of P, since it excludes posets with elements which do not belong to any cross-section (as $\{a, b, c, d, e, f\}$ with $a < d$, $b < d$, $b < e$, $c < d$, $c < e$, $c < f$, $f < d$, where the only cross-sections are $\{a, b, c\}$ and $\{d, e\}$ and they do not contain f). Condition (3.3) allows one to distinguish every segment of P even if P is considered up to isomorphism. Note that it holds if for an object v with nonempty $X|v$ there is no flow order and labelling preserving bijection from an interval of $X|v$ to its proper subinterval.

2.4. Example. Let $U = (W, V, ob)$ be the universe described in 2.2.

Undisturbed production of material by the producer p in an interval $[t', t'']$ of global time is a concrete process that can be defined as $I = (X_I, \leq_I, ins_I)$, where

X_I is the set of numbers equal to variations $var(quantity; t', t)$ in $[t', t] \subseteq [t', t'']$ of the real valued function $quantity$ that specifies the amount of material at disposal of p at every moment of $[t', t'']$,

$\leq_I$ is the restriction of the usual order of numbers to X_I,

$ins_I(x) = (p, quantity(t))$ for $x = var(quantity; t', t)$.

(We recall that the variation of a real-valued function f on an interval $[a, b]$, written as $var(f; a, b)$, is the least upper bound of the set of numbers $|f(a_1) - f(a_0)| + ... + |f(a_n) - f(a_{n-1})|$ corresponding to subdivisions $a = a_0 < a_1 < ... < a_n = b$ of $[a, b]$.). Defining X_I as above instead of defining X_I as $[t', t'']$ is necessary in order to ensure the property (3.3) of 2.3.

Undisturbed distribution of material by the distributor d in an interval $[t', t'']$ of global time is a concrete process that can be defined as $J = (X_J, \leq_J, ins_J)$, where

X_J is the set of numbers equal to variations $var(reserve; t', t)$ in $[t', t] \subseteq [t', t'']$ of the real valued function $reserve$ that specifies the amount of material at disposal of d at every moment of $[t', t'']$,

$\leq_J$ is the restriction of the usual order of numbers to X_J,

$ins_J(x) = (d, reserve(t))$ for $x = var(reserve; t', t)$.

Undisturbed ride of the car c with a load m in an interval $[t', t'']$ of global time is a concrete process that can be defined as $P_K = (X_K, \leq_K, ins_K)$, where

X_K is the set of numbers equal to variations $var(position; t', t)$ in $[t', t] \subseteq [t', t'']$ of the real valued function $position$ that specifies the position of c at every moment of $[t', t'']$,

$\leq_K$ is the restriction of the usual order of numbers to X_K,

$ins_K(x) = (c, m, position(t))$ for $x = var(position; t', t)$.

Loading the empty car c by the producer p with an amount m of material and sending it to the distributor d is a concrete process that can be defined as $L = (X_L, \leq_L, ins_L)$, where

$X_L = \{x_1, x_2, x_3, x_4\}$,

$x_1 <_L x_3, \quad x_1 <_L x_4, \quad x_2 <_L x_3, \quad x_2 <_L x_4$,

$ins_L(x_1) = (p, q)$, $ins_L(x_2) = (c, 0, 0)$, $ins_L(x_3) = (p, q - m)$,
$ins_L(x_4) = (c, m, 0)$.

Delivery of an amount m of material that is the load of the car c to the distributor d is a concrete process that can be defined as $M = (X_M, \leq_M, ins_M)$, where

$X_M = \{y_1, y_2, y_3, y_4\}$,

$y_1 <_M y_3, \quad y_1 <_M y_4, \quad y_2 <_M y_3, \quad y_2 <_M y_4$,

$ins_M(y_1) = (d, r)$, $ins_M(y_2) = (c, m, 1)$, $ins_M(y_3) = (d, r + m)$,
$ins_M(y_1) = (c, 0, 1)$.

Undisturbed production of material by the producer p followed by loading the car with a load m and resuming production, ride of this car to the distributor d which in the meantime distributes the material, and delivery of the load to d, is a process $P = (X_P, \leq_P, ins_P)$, where

$X_P = X_{I'} \cup X_{I''} \cup X_{J'} \cup X_{K'} \cup X_{L'} \cup X_{M'}$,

$\leq_P$ is the transitive closure of $\leq_{I'} \cup \leq_{I''} \cup \leq_{J'} \cup \leq_{K'} \cup \leq_{L'} \cup \leq_{M'}$,

$ins_P = ins_{I'} \cup ins_{I''} \cup ins_{J'} \cup ins_{K'} \cup ins_{L'} \cup ins_{M'}$,

for variants I' and I'' of I, a variant J' of J, a variant K' of K, a variant L' of L, and a variant M' of M, such that the maximal element of $X_{I'}$ coincides with the respective minimal element of $X_{L'}$, the minimal element of $X_{I''}$ coincides with the respective maximal element of $X_{L'}$, the minimal element of $X_{K'}$ coincides with the respective maximal element of $X_{L'}$, the maximal element of $X_{K'}$ coincides with the respective minimal element of $X_{M'}$, the maximal element of $X_{J'}$ coincides with the respective element of $X_{M'}$, and these are the only common elements of pairs of sets from among $X_{I'}$, $X_{I''}$, $X_{J'}$, $X_{K'}$, $X_{L'}$, $X_{M'}$

Isomorphism classes of lposets corresponding to processes I, J, K, L, M, and P, are represented graphically in Figure 1 below. $\sharp$

Let $P = (X, \leq, ins)$ be a concrete process in $U = (W, V, ob)$.

Every cross-section of P contains an occurrence of each object v with nonempty $X|v$.
By $csections(P)$ we denote the set of cross-sections of P. This set is partially ordered by the relation $\preceq$ and $Z' \preceq Z''$ iff for every $z' \in Z'$ there exists $z'' \in Z''$ such that $z' \leq z''$. From (3) of 2.3 it follows that the set of objects occurring in a cross-section is the same for all cross-sections of P. We call it the *range* of P and write it as $objects(P)$. We say that P is *global* if $objects(P) = V$. We say that P is *bounded* if the set of elements of P that are minimal with respect to $\leq$ and the set of elements of P that are maximal with respect to $\leq$ are cross-sections; the respective cross-sections are then called the *origin* and the *end* of P, and they are written as $origin(P)$ and $end(P)$.

2.5. Proposition. The partially ordered set $(csections(P), \preceq)$ is a lattice. $\sharp$

Proof. We have to prove that for every two cross-sections Z' and Z'' of P there exist the greatest lower bound $Z' \bigtriangleup Z''$ and the least upper bound $Z' \bigtriangledown Z''$ of Z' and Z'' with respect to $\preceq$. To this end it suffices to define $Z' \bigtriangleup Z''$ as the set of those $z \in Z' \cup Z''$ for which $z \leq z'$ for some $z' \in Z'$ and $z \leq z''$ for some $z'' \in Z''$, and to define $Z' \bigtriangledown Z''$ as the set of those $z \in Z' \cup Z''$ for which $z' \leq z$ for some $z' \in Z'$ and $z'' \leq z$ for some $z'' \in Z''$.

Indeed, in order to see that $Z' \bigtriangleup Z''$ is an antichain suppose that $x < y$ for x and y in this set. If $x \in Z'$ then $y \in Z''$ and there exists $z' \in Z'$ such that $y \leq z'$. However, this is impossible since Z' is an antichain. Similarly for $x \in Z''$.

In order to see that $Z' \bigtriangleup Z''$ is a maximal antichain suppose that there exists x that is incomparable with all the elements of this set. Then there must be $x \leq z'$ and $x \leq z''$ for some $z' \in Z'$ and $z'' \in Z''$ that are not in $Z' \bigtriangleup Z''$. Consequently, there exists z, say in Z'', such that $x \leq z \leq z'$. Moreover, $z \in Z' \bigtriangleup Z''$ since otherwise there would be $t \in Z'$ such that $t \leq z \leq z'$, what is impossible.

In order to see that $Z' \bigtriangleup Z''$ is a cross-section consider $x \leq y$ such that $x \leq t$ and $u \leq y$ for some $t \in Z' \bigtriangleup Z''$ and $u \in Z' \bigtriangleup Z''$, where $t \in Z'$ and $u \in Z''$.

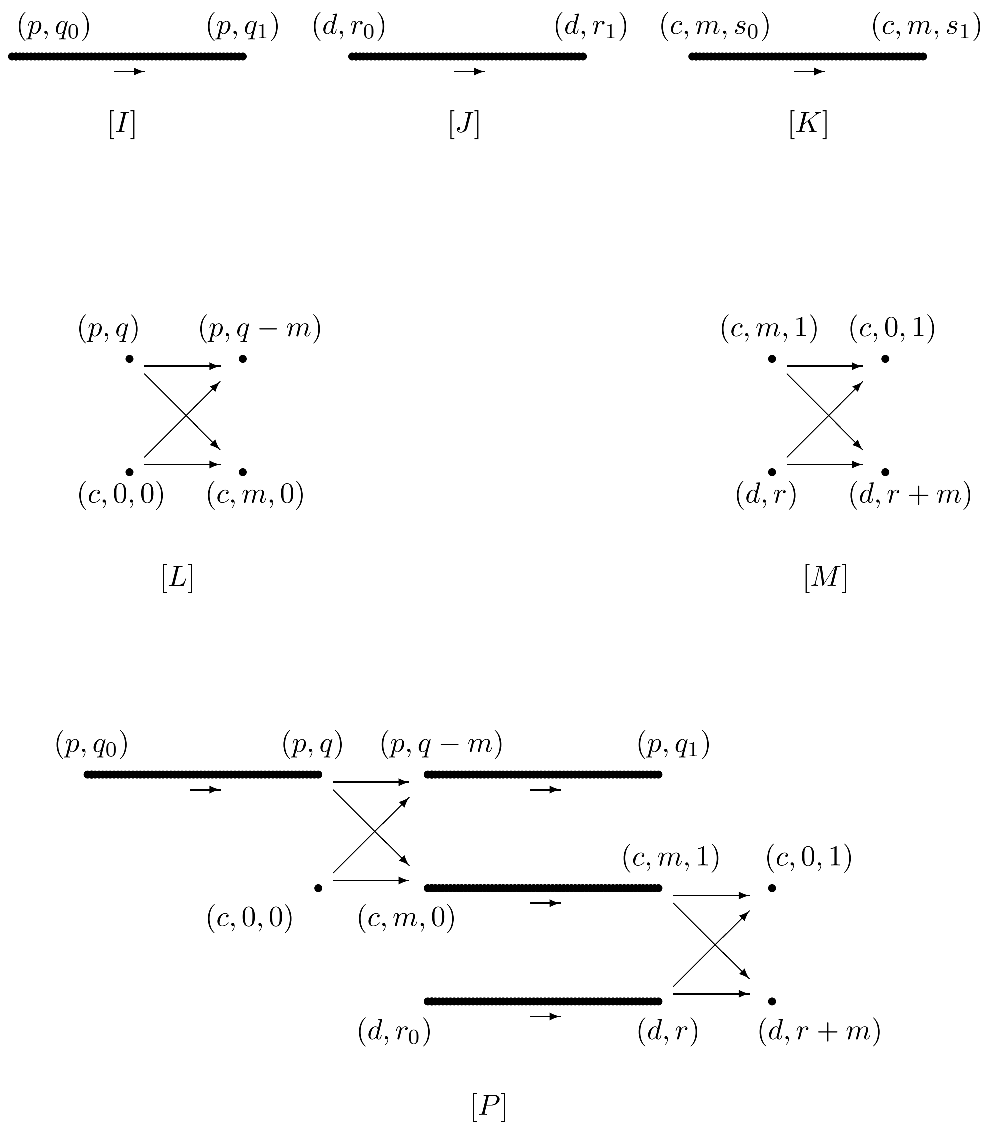

Figure 1: Processes

Without a loss of generality we can assume that $y \leq y'$ for some $y' \in Z'$ since otherwise we could replace y by an element of Z'. Consequently, there exists $z \in Z''$ such that $x \leq z \leq y$. On the other hand, $z \in Z' \bigtriangleup Z''$ since otherwise there would be $z' \in Z'$ such that $z' \leq z \leq y$, what is impossible. In a similar manner we can find $z \in Z' \bigtriangleup Z''$ for the other cases of t and u.

The fact that $Z' \bigtriangleup Z''$ is the greatest lower bound of Z' and Z'' follows from the definition. Similarly, we can see that $Z' \bigtriangledown Z''$ is a cross-section and the least upper bound of Z' and Z''. $\sharp$

2.6. Proposition. For every segment Q of P, every isomorphism between initial or final subsegments of Q is an identity. $\sharp$

Proof. Let R and S be two initial subsegments of Q.

Suppose that $f : R \to S$ is an isomorphism that it is not an identity. Then there exists an initial subsegment T of R such that the image of T under f, say T', is different from T. By (3.3) of 2.3 neither T' is a subsegment of T nor T is a subsegment of T'. Define T'' to be the least segment containing both T and T', and consider $f' : T \to T''$, where $f'(x) = f(x)$ for $x \leq f(x)$ and $f'(x) = x$ for $f(x) < x$. In order to derive a contradiction, and thus to prove that f is an identity, it suffices to verify, that f' is an isomorphism. It can be done as follows.

For injectivity suppose that $f'(x) = f'(y)$. If $x \leq f(x)$ and $y \leq f(y)$ then $f(x) = f'(x) = f'(y) = f(y)$ and thus $x = y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) = f'(y) = y$. The case $x \leq f(x)$ and $f(y) < y$ is excluded by $f'(x) = f'(y)$ since $x \leq f(x) = f'(x) = f'(y) = y$ and, on the other hand, $f(y) < y = f(x)$ implies $y < x$. Similarly, the case $f(x) < x$ and $y \leq f(y)$ is excluded. Consequently, f' is injective.

For surjectivity suppose that y is in T''. If $y \leq f(y)$ then $y = f(t)$ for some $t \leq y$ and thus $y = f'(t)$ since $t \leq y = f(t)$ and thus $f'(t) = f(t)$. If $f(y) < y$ then $y = f'(y)$. Consequently, f' is surjective.

For monotonicity suppose that $x \leq y$. If $x \leq f(x)$ and $y \leq f(y)$ then $f'(x) = f(x) \leq f(y) = f'(y)$. If $f(x) < x$ and $f(y) < y$ then $f'(x) = x \leq y = f'(y)$. If $x \leq f(x)$ and $f(y) < y$ then $f'(x) = f(x) \leq f(y) < y = f'(y)$. If $f(x) < x$ and $y \leq f(y)$ then $f'(x) = x \leq y \leq f(y) = f'(y)$. Consequently, f' is monotonic.

For monotonicity of the inverse suppose that $f'(x) < f'(y)$. If $x \leq f(x)$ and $y \leq f(y)$ then $f(x) = f'(x) < f'(y) = f(y)$ and thus $x < y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) < f'(y) = y$. If $x \leq f(x)$ and $f(y) < y$ then $x \leq f(x) = f'(x) < f'(y) = y$. If $f(x) < x$ and $y \leq f(y)$ then $f(x) < x = f'(x) < f'(y) = f(y)$ and thus $x < y$. Consequently, the inverse of f' is monotonic.

Verification for final subsegments is similar. $\sharp$

2.7. Corollary. For every segment Q of P, every isomorphism between initial or final subsegments of Q has an extension to an automorphism of the whole segment Q. $\sharp$

2.8. Definition. An *abstract process* is an isomorphism class π of concrete processes. $\sharp$

For every concrete process P' such that P and P' are isomorphic we have $objects(P') = objects(P)$. Consequently, for the abstract process $[P]$ that corresponds to a concrete process P we define $objects([P]) = objects(P)$. We say that an abstract process π is *global* (resp.: *bounded*, *K-dense*, *weakly K-dense*) if the instances of π are global (resp.: bounded, K-dense, weakly K-dense).

By $Proc(U)$ we denote the set of all bounded processes in U.

In $Proc(U)$ there exists a process with the empty set of object instances, called the *empty process* and denoted by 0.

Processes from $Proc(U)$ with flow orders reducing to identities, called *process identities*, or *identities*, or *states*, can be identified with the sets of instances of occurring objects.

For each process π from $Proc(U)$ there exists a unique process identity, called the *source*, or the *domain*, or the *initial state* of π, and written as $dom(\pi)$, (resp.: a unique process identity, called the *target*, or the *codomain*, or the *final state* of π, and written as $cod(\pi)$), whose instance can be obtained from an instance P of π by restricting P to the set $origin(P)$ of minimal elements (resp.: to the set $end(P)$ of maximal elements).

Thus we have two unary operations on processes: the operation $\pi \mapsto dom(\pi)$ of taking the source (the domain), and the operation $\pi \mapsto cod(\pi)$ of taking the target (the codomain).

We have also a sequential composition and a parallel composition.

The sequential composition allows one to combine two processes whenever one of them is a continuation of the other. It can be defined as follows.

2.9. Proposition. For each cross-section c of a concrete process $P = (X, \leq, ins)$, the restrictions of P to the subsets $X^-(c) = \{x \in X : x \leq z \text{ for some } z \in c\}$ and $X^+(c) = \{x \in X : z \leq x \text{ for some } z \in c\}$ are concrete processes, called respectively the ***head*** and the ***tail*** of P with respect to c, and written respectively as $head(P, c)$ and $tail(P, c)$. $\sharp$

A proof is straightforward.

2.10. Definition. A process π is said to *consist* of a process π_1 *followed* by a process π_2 if its instance P has a cross-section c such that $head(P, c)$ is an instance of π_1 and $tail(P, c)$ is an instance of π_2. $\sharp$

2.11. Proposition. For every two processes π_1 and π_2 such that $cod(\pi_1) = dom(\pi_2)$ there exists a unique process, written as $\pi_1; \pi_2$, or as $\pi_1\pi_2$, that consists of π_1 followed by π_2. $\sharp$

Proof. Take $P_1 = (X_1, \leq_1, ins_1) \in \pi_1$ and $P_2 = (X_2, \leq_2, ins_2) \in \pi_2$ with $X_1 \cap X_2 = end(P_1) = origin(P_2)$ and with the restriction of P_1 to $end(P_1)$ identical with the restriction of P_2 to $origin(P_2)$, and equip $X_1 \cup X_2$ with the least common extension of the flow orders and labellings of P_1 and P_2.

Let P be the lposet thus obtained. In order to prove that P is a process it suffices to show that P does not contain a segment with isomorphic proper subsegment. To this end suppose the contrary.

Suppose that $f : Q \to R$ is an isomorphism from a segment Q of P to a proper subsegment R of Q, where Q consists of a part Q_1 contained in P_1 and a part Q_2 contained in P_2. By applying twice the method described in the proof of 2.6 we can modify f to an isomorphism $f' : Q \to R$ such that the image of Q_1 under f', say R_1, is contained in Q_1, and the image of Q_2 under f', say R_2, is contained in Q_2. As R is a proper subsegment of Q, one of these images, say R_1, is a proper part of the respective Q_i. By taking the greatest lower bounds and the least upper bounds of appropriate cross-sections we can extend Q_1 and R_1 to segments Q_1' and R_1' of P_1 such that R_1' is a proper subsegment of Q_1' and there exists an isomorphism from Q_1' to R_1'. This is in a contradiction with the fact that P_1 is a process and implies that P is a process. $\sharp$

2.12. Definition. The operation $(\pi_1, \pi_2) \mapsto \pi_1\pi_2$ is called the *sequential composition*. $\sharp$

The parallel composition allows one to combine processes on disjoint sets of involved objects. It can be defined as follows.

2.13. Definition. Given a concrete process $P = (X, \leq, ins)$, by a *splitting* of P we mean an ordered pair $s = (X^F, X^S)$ of two disjoint subsets X^F and X^S of X such that $X^F \cup X^S = X$ and $x' \leq x''$ only if x' and x'' are both in one of these subsets. $\sharp$

2.14. Proposition. For each splitting $s = (X^F, X^S)$ of a concrete process $P = (X, \leq, ins)$, the restrictions of P to the subsets X^F and X^S are concrete processes, called respectively the *first part* and the *second part* of P with respect to s, and written respectively as $\mathit{first}(P, s)$ and $\mathit{second}(P, s)$. $\sharp$

A proof is straightforward.

2.15. Definition. A process π is said to *consist* of two *parallel* processes π_1 and π_2 if its instance P has a splitting s such that $\mathit{first}(P, s)$ is an instance of π_1 and $\mathit{second}(P, s)$ is an instance of π_2. $\sharp$

2.16. Proposition. If for two processes π_1 and π_2 there exists a process π with an instance P that has a splitting s such that $\mathit{first}(P, s)$ is an instance of π_1 and $\mathit{second}(P, s)$ is an instance of π_2 then such a process is unique. If such a process π exists then we write it as $\pi_1 + \pi_2$ and say that the processes π_1 and π_2 are *parallel*. $\sharp$

For a proof it suffices to take $P_1 = (X_1, \leq_1, ins_1) \in \pi_1$ and $P_2 = (X_2, \leq_2, ins_2) \in \pi_2$ with $X_1 \cap X_2 = \emptyset$, and to equip $X_1 \cup X_2$ with the least common extension of the flow orders and labellings of P_1 and P_2.

2.17. Definition. The operation $(\pi_1, \pi_2) \mapsto \pi_1 + \pi_2$ is called the *parallel composition*. $\sharp$

The sequential and the parallel composition of processes are operations which allow one to represent complex processes in terms of their components. For example, in the case of processes in 2.4 we can represent $[P]$ as $([I'] + (c, 0, 0) + (d, r_0))([L'] + (d, r_0))([I''] + [J'] + [K'])((p, q_1) + [M'])$.

From 2.7 we obtain that $\sigma\pi = \sigma'\pi$ implies $\sigma = \sigma'$. Indeed, if i is an isomorphism from an instance Q of $\sigma\pi$ to an instance Q' of $\sigma'\pi$, where $S = \mathit{head}(Q, c)$ is an instance of σ, $P = \mathit{tail}(Q, c)$ is an instance of π, $S' = \mathit{head}(Q', c')$ is an instance of σ', $P' = \mathit{tail}(Q', c')$ is an instance of π, and j is an isomorphism from P' to P, then P' is isomorphic to the image of P under i and, consequently, the composite $j \circ (i|P)$ has an extension to an automorphism k of Q'. Hence S' is isomorphic to the image of S under i and thus to S, too, and this implies $\sigma = \sigma'$.

Similarly, $\pi\tau = \pi\tau'$ implies $\tau = \tau'$.

From (3.3) of 2.3 we obtain also that if $\sigma\pi\tau$ is defined and $\sigma\pi\tau = \pi$ then σ and τ are identities.

Taking this into account and following [18] and [19] it is straightforward to prove the following result.

2.18. Theorem. The set $Proc(U)$ equipped with the operations

$\pi \mapsto dom(\pi)$, $\pi \mapsto cod(\pi)$, $(\pi_1, \pi_2) \mapsto \pi_1\pi_2$, $(\pi_1, \pi_2) \mapsto \pi_1 + \pi_2$,

and with the constant 0, is a behaviour algebra, called the *algebra of bounded processes in the universe* U, and written as $PROC(U)$. The set of K-dense processes from $Proc(U)$ forms a subalgebra $KPROC(U)$ of this algebra. $\sharp$

3. Behaviour algebras

Arbitrary behaviour algebras are defined formally as follows.

3.1. Definition. A *behaviour algebra* is a partial algebra $\mathcal{A} = (A, dom, cod, ;, +, 0)$, where A is a set, $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are unary operations in A, $(\alpha_1, \alpha_2) \mapsto \alpha_1; \alpha_2$ is a partial operation in A, $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ is a partial operation in A, and 0 is a constant, such that the axioms (A1) - (A6) hold. $\sharp$

The composite $\alpha_1; \alpha_2$ is written as $\alpha_1\alpha_2$. The category $cat(\mathcal{A}) = (A, dom, cod, ;)$ is called the *underlying category* of $\mathcal{A}$. The partial monoid $pmon(\mathcal{A}) = (A, +, 0)$ is called the *underlying partial monoid* of $\mathcal{A}$.

An element $\alpha \neq 0$ of A is said to be a $(+)$*-atom* of $\mathcal{A}$ provided that for every $\alpha_1 \in A$ and $\alpha_2 \in A$ the equality $\alpha = \alpha_1 + \alpha_2$ implies that either $\alpha_1 = 0$ and $\alpha_2 = \alpha$ or $\alpha_1 = \alpha$ and $\alpha_2 = 0$. An identity of $cat(\mathcal{A}) = (A, dom, cod, ;)$ that is a $(+)$-atom is said to be an *atomic identity*.

An element α of A is said to be a $(;)$*-atom* of $\mathcal{A}$ provided that it is not an identity of $cat(\mathcal{A})$ and for every $\alpha_1 \in A$ and $\alpha_2 \in A$ the equality $\alpha = \alpha_1\alpha_2$ implies that either α_1 is an identity and $\alpha_2 = \alpha$ or $\alpha_1 = \alpha$ and α_2 is an identity. An element α of A which is both a $(+)$-atom and $(;)$-atom is said to be a $(+, ;)$*-atom*. In particular, atomic identities are $(+, ;)$-atoms.

3.2. Definition. Given $\alpha \in A$, by a *cut* of α we mean a pair (α_1, α_2) such that $\alpha_1\alpha_2 = \alpha$. $\sharp$

Cuts of every $\alpha \in A$ are partially ordered by the relation $\preceq_\alpha$, where $x \preceq_\alpha y$ with $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ means that $\eta_1 = \xi_1\delta$ with some δ. From (A1) it follows that $\preceq_\alpha$ is a partial order, and that for $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \preceq_\alpha y$ there exists a unique δ such that $\eta_1 = \xi_1\delta$, written as $x \to y$. From (A6) it follows that this partial order makes the set of cuts of α a lattice L_α. Given two cuts x and y, by $x \bigtriangledown_\alpha y$ and $x \bigtriangleup_\alpha y$ we denote respectively the least upper bound and the greatest lower bound of x and y. From (A6) it follows that $(x \leftarrow x \bigtriangleup_\alpha y \to y, x \to x \bigtriangledown_\alpha y \leftarrow y)$ is a bicartesian square.

3.3. Definition. Given $\alpha \in A$ and its cuts $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \preceq_\alpha y$, by a *segment* of α from x to y we mean $\beta \in A$ such that $\xi_2 = \beta\eta_2$ and $\eta_1 = \xi_1\beta$, written as $\alpha|[x, y]$. A segment $\alpha|[x', y']$ of α such that $x \preceq_\alpha x' \preceq_\alpha y' \preceq_\alpha y$ is called a *subsegment* of $\alpha|[x, y]$. If $x = x'$ (resp. if $y = y'$) then we call it an *initial* (resp. a *final*) subsegment of $\alpha|[x, y]$. $\sharp$

In the sequel elements of A are called *processes* of $\mathcal{A}$. Processes of $\mathcal{A}$ which are identities of the underlying category $cat(\mathcal{A})$ are called *states* of $\mathcal{A}$. Processes which are atomic identities are called *atomic states*. For every process α, the states $u = dom(\alpha)$ and $v = cod(\alpha)$ are called respectively the *initial state* and the *final state* of α and we write α as $u \stackrel{\alpha}{\to} v$. The operations $(\alpha_1, \alpha_2) \mapsto \alpha_1\alpha_2$ and $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ are called respectively the *sequential composition* and the *parallel composition*.

3.4. Definition. If processes α_1 and α_1 are such that $\alpha_1 + \alpha_2$ is defined then we say that they are *concurrent* and write $\alpha_1 \ co \ \alpha_2$. The relation co thus defined is called the *concurrency relation* of $\mathcal{A}$. $\sharp$

For example, processes $[J']$ and $[K']$ in 2.4 are concurrent.

With the aid of concurrency relation we can generalize the introduced in [17] notions of parallel and sequential independence of processes of Condition/Event Petri nets (cf. also [9]).

3.5. Definition. Processes α_1 and α_2 such that $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$ and $\alpha_2 = c + dom(\varphi_1) + \varphi_2$ for a state c and processes φ_1 and φ_2 such that $c + \varphi_1 + \varphi_2$ is defined are said to be *parallel independent*. $\sharp$

In particular, processes $\alpha_1 = \varphi_1 + dom(\varphi_2)$ and $\alpha_2 = dom(\varphi_1) + \varphi_2$, where φ_1 and φ_2 are concurrent, are parallel independent.

3.6. Definition. Processes α_1 and α_2 such that $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$ and $\alpha_2 = c + cod(\varphi_1) + \varphi_2$ for a state c and processes φ_1 and φ_2 such that $c + \varphi_1 + \varphi_2$ is defined are said to be *sequential independent*. $\sharp$

In particular, processes $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$ and $\alpha_2 = c + cod(\varphi_1) + \varphi_2$, where φ_1 and φ_2 are concurrent, are sequential independent.

From (A5) we obtain the following characterization of the parallel and the sequential independence of processes.

3.7. Theorem. Processes of the pair $v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w$ (= $(v \stackrel{\alpha_1}{\leftarrow} u, u \stackrel{\alpha_2}{\to} w)$) are parallel independent iff there exists a unique pair $v \stackrel{\alpha_2'}{\to} u' \stackrel{\alpha_1'}{\leftarrow} w$ such that $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w, v \stackrel{\alpha_2'}{\to} u' \stackrel{\alpha_1'}{\leftarrow} w)$ is a bicartesian square. $\sharp$

3.8. Theorem. Processes of the pair $u \stackrel{\alpha_1}{\to} v \stackrel{\alpha_2'}{\to} u'$ are sequential independent iff there exists a unique pair $u \stackrel{\alpha_2}{\to} w \stackrel{\alpha_1'}{\to} u'$ such that $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w, v \stackrel{\alpha_2'}{\to} u' \stackrel{\alpha_1'}{\leftarrow} w)$ is a bicartesian square. $\sharp$

4. Underlying partial monoids

Let $\mathcal{A} = (A, dom, cod, ;, +, 0)$ be a behaviour algebra with the underlying category $cat(\mathcal{A})$, with the underlying partial monoid $pmon(\mathcal{A})$, with the operation $\sqcap$ of taking the greatest lower bound with respect to the partial order $\sqsubseteq$, where $\alpha_1 \sqsubseteq \alpha_2$ iff $\alpha_2 = \alpha_1 + \rho$ for some ρ, and with the function $\alpha \mapsto h(\alpha)$ that assigns to each α the set of $(+)$-atoms less than or equal α with respect to the partial order $\sqsubseteq$.

Let A_+ denote the set of $(+)$-atoms of $\mathcal{A}$. Let A_0 denote the set of identities of the underlying category $cat(\mathcal{A})$, and $A_{+0} = A_+ \cap A_0$ the subset of atomic identities.

4.1. Lemma. If $\alpha_1 + \alpha_2$ is defined then $\alpha_1 \sqcap \alpha_2 = 0$. $\sharp$

Proof. Let $\alpha_1 = (\alpha_1 \sqcap \alpha_2) + \xi$ and $\alpha_2 = (\alpha_1 \sqcap \alpha_2) + \eta$. Since $\alpha_1 + \alpha_2$ is defined, we have $\alpha_1 + \alpha_2 = (\alpha_1 \sqcap \alpha_2) + (\alpha_1 \sqcap \alpha_2) + \xi + \eta$. Thus $(\alpha_1 \sqcap \alpha_2) + (\alpha_1 \sqcap \alpha_2)$ is defined and, by (A2.4), $\alpha_1 \sqcap \alpha_2 = 0$. $\sharp$

4.2. Lemma. If $\alpha_1 + \alpha_2$ is defined then there exists the least upper bound of α_1 and α_2, written as $\alpha_1 \sqcup \alpha_2$, and $\alpha_1 \sqcup \alpha_2 = \alpha_1 + \alpha_2$. $\sharp$

Proof. $\alpha_1 + \alpha_2$ is an upper bound of α_1 and α_2. If ζ is another upper bound of α_1 and α_2 then for $\theta = \zeta \sqcap (\alpha_1 + \alpha_2)$ we have $\alpha_1 \sqsubseteq \theta$ and $\alpha_2 \sqsubseteq \theta$, $\theta + \gamma = \alpha_1 + \alpha_2$, $\alpha_2 + \delta = \theta$, and $\alpha_2 + \epsilon = \theta$. Hence $\alpha_1 + \delta + \gamma = \alpha_1 + \alpha_2$ and $\alpha_2 + \epsilon + \gamma = \alpha_1 + \alpha_2$. Thus $\delta + \gamma = \alpha_2$ and $\epsilon + \gamma = \alpha_1$. Hence $\gamma \sqsubseteq \alpha_1$ and $\gamma \sqsubseteq \alpha_2$, i.e., $\gamma = 0$ by 4.1. Consequently, $\theta = \zeta \sqcap (\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2$. Finally, $\alpha_1 + \alpha_2 \sqsubseteq \zeta$, i.e., $\alpha_1 + \alpha_2 = \alpha_1 \sqcup \alpha_2$. $\sharp$

4.3. Lemma. The correspondence $\alpha \mapsto h(\alpha)$ enjoys the following properties:

(1) if $\alpha_1 \neq \alpha_2$ then $h(\alpha_1) \neq h(\alpha_2)$,

(2) $h(\alpha_1 \sqcap \alpha_2) = h(\alpha_1) \cap h(\alpha_2)$,

(3) if $\alpha_1 + \alpha_2$ is defined then $h(\alpha_1) \cap h(\alpha_2) = \emptyset$,

(4) if $\alpha_1 + \alpha_2$ is defined then $h(\alpha_1 + \alpha_2) = h(\alpha_1) \cup h(\alpha_2)$. $\sharp$

Proof. For (1) refer to (A2.9). For (2) notice that $\xi \sqsubseteq \alpha_1 \sqcap \alpha_2$ iff $\xi \sqsubseteq \alpha_1$ and $\xi \sqsubseteq \alpha_2$. For (3) notice that if $\alpha_1 + \alpha_2$ is defined then by 4.1 we have $\alpha_1 \sqcap \alpha_2 = 0$. Consequently, $h(\alpha_1 \sqcap \alpha_2) = \emptyset$ and it suffices to apply (2). For (4) notice that if $\xi \in h(\alpha_1 + \alpha_2)$ then $\xi \sqsubseteq \alpha_1 + \alpha_2$ and thus $\xi \sqsubseteq \alpha_1$ or $\xi \sqsubseteq \alpha_2$ since ξ is a $(+)$-atom. Consequently, $\xi \in h(\alpha_1)$ or $\xi \in h(\alpha_2)$. Conversely, if $\xi \in h(\alpha_1)$ or $\xi \in h(\alpha_2)$ then $\xi \in \alpha_1$ or $\xi \in \alpha_2$, i.e., $\xi \in h(\alpha_1 + \alpha_2)$. $\sharp$

We recall that a tolerance relation in a set is a reflexive and symmetric binary relation in this set, that for such a relation a tolerance preclass is a set whose every two elements are in this relation, and that a tolerance class is a maximal tolerance preclass.

The relation $\overline{co}$, where $\alpha_1 \ \overline{co}\ \alpha_2$ iff α_1 and α_2 are concurrent or $\alpha_1 = \alpha_2$, is a tolerance relation. We call it *the tolerance relation of* $\mathcal{A}$ and say about processes α_1 and α_2 such that $\alpha_1 \ \overline{co}\ \alpha_2$ that they *tolerate* each other. By tol we denote the restriction of $\overline{co}$ to the set A_+ of $(+)$-atoms of $\mathcal{A}$.

The following fact is a consequence of (A2.9) and (A2.10).

4.4. Lemma. For each process α the set $h(\alpha)$ of $(+)$-atoms contained in α is a tolerance preclass of the relation tol. $\sharp$

The following fact is a consequence of (A2.5).

4.5. Lemma. For every finite tolerance preclass C of the relation tol there exists a process α such that $h(\alpha) = C$. $\sharp$

From (4.3) - (4.5) we obtain that elements of the partial monoid $pmon(\mathcal{A})$ can be representated as tolerance preclasses of the relation tol and combined with the aid of set theoretical operations. More precisely, we obtain the following theorem.

4.6. Theorem. The underlying partial monoid $pmon(\mathcal{A}) = (A, +, 0)$ of $\mathcal{A}$ is isomorphic to a partial monoid $\mathcal{M} = (A', +', 0')$ of tolerance preclasses of the tolerance relation tol, where

(1) A' is a set of tolerance preclasses of tol that contains all finite preclasses and is closed with respect to intersections and unions of families with an upper bound in A',

(2) the operation $+'$ is defined for pairs of disjoint preclasses from A' as the set theoretical union provided that its results belong to A',

(3) $0'$ is the empty set.

The isomorphism is given by the correspondence $\pi \mapsto h(\pi)$. $\sharp$

Let $\sim$ be the least congruence whose existence is guaranteed by (A4). Let nat be the natural homomorphism from $\mathcal{A}$ to the quotient algebra $\mathcal{A}/\sim$.

4.7. Definition. Given an atomic identity $p \in A_{+0}$, the image $nat(p)$ of p under the natural homomorphism nat is called an *object* corresponding to p, and p is called an *instance* of this object. $\sharp$

By $\mathcal{A}_{ob}$ we denote the set of objects corresponding to atomic identities of $\mathcal{A}$ and we call elements of $\mathcal{A}_{ob}$ *objects definable in* $\mathcal{A}$. We can show that the identities of $cat(\mathcal{A})$ can be represented by partial functions from $\mathcal{A}_{ob}$ to A_{+0} and combined in a natural way.

4.8. Theorem. The restriction of $pmon(\mathcal{A})$ to the subset A_0 of identities is isomorphic to a partial monoid $\mathcal{N} = (A'', +'', 0'')$ of partial functions, where A'' is a set of partial functions from $\mathcal{A}_{ob}$ to A_{+0}, $u +'' v$ denotes the set theoretical union of partial functions u and v provided that such functions have disjoint domains and their union belongs to A'', and $0''$ is the empty partial function. $\sharp$

Proof. Given an identity u, we define $H(u)$ as the set of pairs $(nat(p), p)$ with $p \in h(u)$. From the fact that $\sim$ is a congruence on $\mathcal{A}$ it follows that $nat(p_1) = nat(p_2)$ implies $p_1 = p_2$ since otherwise $p_1 + p_2$ would be defined and, consequently, $nat(p_1) + nat(p_2)$ would also be defined, and (A2.4) could not hold. Hence $H(u)$ is a partial function. The fact that H defines an isomorphism follows from (4.6). $\sharp$

Given an identity $u \in A_0$, each pair $(nat(p), p) \in H(u)$ can be interpreted as a representant of an instance p of the object $nat(p) \in \mathcal{A}_{ob}$. Consequently, $H(u)$ can be interpreted as the partial function defined on a set of objects definable in $\mathcal{A}$ that assigns an instance to each object from its domain. For example, conditions of a Condition/Event Petri net are objects definable in the algebra of finite processes of this net and a function that for each condition from a subset of conditions of the net assigns to this condition its logical value is a state of the net.

5. Towards a representation theorem

Let $\mathcal{A} = (A, dom, cod, ;, +, 0)$ be a behaviour algebra. With the characterization just described of identities of $cat(\mathcal{A})$ we can characterize arbitrary elements of $\mathcal{A}$.

We shall represent each such element α by a partially ordered labelled set $P(\alpha) = (X_\alpha, \leq_\alpha, l_\alpha)$. Each element $x \in X_\alpha$ will play the role of an occurrence of the instance $l_\alpha(x)$ of the object $nat(l_\alpha(x))$. The partial order $\leq_\alpha$ will reflect how occurrences of instances of objects arise from other instances.

This way of representing elements of A will allow us to extend the correspondence $u \mapsto H(u)$ by assigning to each $\alpha \in A$ the isomorphism class of partially ordered labelled sets that contains $P(\alpha)$.

The elements of X_α will be defined as packets of cuts of α, where a cut is a decomposition of α into two components the sequential composition of which yields α (see 3.2).

We start with some notions and observations.

Given a cut $x = (\xi_1, \xi_2)$ of α and an atomic identity p, we say that p *occurs* in x and call (x, p) an *occurrence* of p in x if p is contained in $cod(\xi_1) = dom(\xi_2)$.

Given an occurrence (x, p) of an atomic identity p in a cut $x = (\xi_1, \xi_2)$ of α and an occurrence (y, q) of an atomic identity q in a cut $y = (\eta_1, \eta_2)$ of α, we say that these occurrences are *adjoint* and write $(x, p) \sim_\alpha (y, q)$ if $p = q$ and $p \sqsubseteq (x \vartriangle_\alpha y \rightarrow x \triangledown_\alpha y)$, that is if $p = q$ and $(x \vartriangle_\alpha y \rightarrow x \triangledown_\alpha y) = c + \varphi_1 + \varphi_2$ with an identity c that contains p and with $(x \vartriangle_\alpha y \rightarrow x) = c + \varphi_1 + dom(\varphi_2)$, $(x \vartriangle_\alpha y \rightarrow y) = c + dom(\varphi_1) + \varphi_2$, $(y \rightarrow x \triangledown_\alpha y) = c + \varphi_1 + cod(\varphi_2)$, $(x \rightarrow x \triangledown_\alpha y) = c + cod(\varphi_1) + \varphi_2$.

Given a cut x of α, by $atomicid(x)$ we denote the set of atomic identities that occur in x. From (A3) we obtain that the cardinality of the set $atomicid(x)$ is the same for all cuts of α. We call it the *width* of α and write as $width(\alpha)$. Taking into account also (A4) we obtain that the set of objects definable in $\mathcal{A}$ and having instances in $atomicid(x)$ is also the same for all cuts of α. We call it the *range* of α and write as $range(\alpha)$.

5.1. Lemma. For each $\alpha \in A$ the relation $\sim_\alpha$ is an equivalence relation. $\sharp$

Proof. It suffices to prove that $\sim_\alpha$ is transitive. To this end suppose that $(x, p) \sim_\alpha (y, q)$ with $p = q$ and $p \sqsubseteq (x \vartriangle_\alpha y \rightarrow x \triangledown_\alpha y)$, and that $(y, q) \sim_\alpha (z, r)$ with $p = q = r$ and $p \sqsubseteq (y \vartriangle_\alpha z \rightarrow y \triangledown_\alpha z)$. Hence by (A3.6) we have $p \sqsubseteq \sigma$ for every σ that is a segment of $(x \vartriangle_\alpha y \rightarrow x \triangledown_\alpha y)$ or $(y \vartriangle_\alpha z \rightarrow y \triangledown_\alpha z)$. On the other hand, $(x \vartriangle_\alpha z \rightarrow x \triangledown_\alpha z)$ can be represented as the result of composing sequentially such segments. Consequently, $p \sqsubseteq (x \vartriangle_\alpha z \rightarrow x \triangledown_\alpha z)$. Hence $(x, p) \sim_\alpha (z, r)$. Thus $\sim_\alpha$ is transitive. $\sharp$

5.2. Definition. Given $\alpha \in A$ and an atomic identity p, by an *occurrence* of p in α we mean an equivalence class of occurrences of p in cuts of α. $\sharp$

5.3. Definition. Given $\alpha \in A$, the set of occurrences of atomic identities in α, written as X_α, is called the *canonical underlying set* of α. $\sharp$

5.4. Definition. Given $\alpha \in A$, the correspondence $[(x, p)] \mapsto p$ between occurrences of atomic identities in α and the atomic identities themselves, written as l_α, is called the *canonical labelling* of (occurrences of atomic identities in) α. $\sharp$

The partial order $\leq_\alpha$ on X_α can be defined as follows.

Given an occurrence (x, p) of an atomic identity p in a cut $x = (\xi_1, \xi_2)$ of α and an occurrence (y, q) of an atomic identity q in a cut $y = (\eta_1, \eta_2)$ of α, we say that (x, p) *precedes* (y, q) and write $(x, p) <_\alpha (y, q)$ if $x \preceq_\alpha y$, p occurs in x, q occurs in y, and there is no cut v of $x \to y$ such that $(x, p) \sim_\alpha (v, p)$ and $(y, q) \sim_\alpha (v, q)$.

5.5. Lemma. For each $\alpha \in A$ the relation $<_\alpha$ is irreflexive and transitive. $\sharp$

Proof. The irreflexivity of $<_\alpha$ follows directly from the definition. For the transitivity suppose that $(x, p) <_\alpha (y, q)$ and $(y, q) <_\alpha (z, r)$. Then from $x \preceq_\alpha y$ and $y \preceq_\alpha z$ we obtain $x \preceq_\alpha z$. On the other hand, p occurs in x and q occurs in z. So, it remains to prove that there is no cut v of $x \to y$ such that $(x, p) \sim_\alpha (v, p)$ and $(z, r) \sim_\alpha (v, r)$. To this end suppose the contrary and consider $y \triangle_\alpha v \to y \triangle_\alpha v = c + \varphi_1 + \varphi_2$, where c is an identity. From the fact that $(x, p) <_\alpha (y, q)$ excludes $(x, p) \sim_\alpha (y, q)$ we obtain that q does not occur in v. On the other hand, q cannot be contained in $cod(\varphi_1)$ since then there would be $(y \triangle_\alpha v, p) \sim_\alpha (x, p)$ and $(y \triangle_\alpha v, q) \sim_\alpha (y, q)$. Similarly, q cannot be contained in $dom(\varphi_2)$ since then there would be $(y \bigtriangledown_\alpha v, q) \sim_\alpha (y, q)$ and $(y \bigtriangledown_\alpha v, r) \sim_\alpha (z, r)$. Consequently, q could not occur in y as it follows from $(x, p) <_\alpha (y, q)$ and $(y, q) <_\alpha (z, r)$. $\sharp$

5.6. Lemma. For each $\alpha \in A$ the relation $\leq_\alpha$ on X_α, where $u \leq_\alpha v$ iff $u \sim_\alpha v$ or $(x, p) <_\alpha (y, q)$ for some $(x, p) \in u$ and $(y, q) \in v$, is a partial order. $\sharp$

Proof. It suffices to prove that $(x, p) <_\alpha (y, q)$ excludes $(y, q) <_\alpha (x, p)$. To this end it suffices to notice that otherwise the identity $x \to x$ would be the result of composing sequentially $x \to y$ and $y \to x$, what is impossible according to (A1.3). $\sharp$

5.7. Definition. Given $\alpha \in A$, the partial order $\leq_\alpha$ is called the *canonical partial order* of (occurrences of atomic identities in) α. $\sharp$

5.8. Lemma. Given an $\alpha \in A$, if $nat(l_\alpha(u)) = nat(l_\alpha(v))$ for some $u, v \in X_\alpha$ then $u \leq_\alpha v$ or $v \leq_\alpha u$. $\sharp$

Proof. It suffices to consider the case $u \neq v$. From $nat(l_\alpha(u)) = nat(l_\alpha(v))$ it follows that in this case $p = l_\alpha(u)$ and $q = l_\alpha(v)$ cannot occur in the same cut. Consequently, $(x, p) \in u$ and $(y, q) \in v$ for some cuts x and y such that $x \neq y$. Moreover, x and y can be chosen such that $x \preceq_\alpha y$ or $y \preceq_\alpha x$ and then we obtain respectively $(x, p) \leq_\alpha (y, q)$ or $(y, q) \leq_\alpha (x, p)$. $\sharp$

5.9. Lemma. For each $\alpha \in A$ and each object $s \in \mathcal{A}_{ob}$ the set $Z_\alpha(s)$ of $u \in X_\alpha$ such that $l_\alpha(u) = p$ for an instance p of s is a maximal chain with respect to the partial order $\leq_\alpha$ or it is empty. $\sharp$

Proof. Let $Z_\alpha(s) = \{u \in X_\alpha : l_\alpha(u) = p \text{ for some } p \text{ with } nat(p) = s\}$. Suppose that $u_1 <_\alpha u <_\alpha u_2$ for some $u_1, u_2 \in Z_\alpha(s)$ and u with $l_\alpha(u)$ not being an instance of s. Then there exists $(x, q) \in u$ with q being an instance of some $s' \in \mathcal{A}_{ob}$ that is different from s and has an occurrence in a cut that does not contain an occurrence of s. But this is impossible since every cut of α contains an occurrence of s. $\sharp$

5.10. Lemma. For each $\alpha \in A$ of finite width a subset $Y \subseteq X_\alpha$ is a maximal antichain of the partially ordered set $(X_\alpha, \leq_\alpha)$ iff it corresponds to the set of occurrences of atomic identities in a cut y of α. $\sharp$

Proof. Let y be a cut of α. From the definition of the partial order $\leq_\alpha$ we obtain that equivalence classes of occurrences of atomic identities in y are pairwise incomparable. Thus they form an antichain $Y = H'(y)$. According to (A4) for each $u \in X_\alpha$ that does not belong to Y there exists $v \in Y$ such that $nat(l_\alpha(u)) = nat(l_\alpha(v))$ and by 5.8 v is comparable with u. Consequently, Y is a maximal antichain.

Let Y be a maximal antichain of $(X_\alpha, \leq_\alpha)$. Then all different $u, v \in Y$ are incomparable with respect to $\leq_\alpha$ and it follows from the definition of $\leq_\alpha$ that there exists a cut x of α such that for some atomic identities p and q (x, p) is an instance of u and (x, q) is an instance of v. As α is of finite width, it is possible to construct step by step a cut y such that each element of Y has an instance in y. Namely, given a cut y_n such that (y_n, p_1),...,(y_n, p_n) are instances of elements u_1,...,u_n of Y, and an element u of Y that is incomparable with u_1,...,u_n and has instances (x_1, p_{n+1}),...,(x_n, p_{n+1}) such that $(x_1, p_1) \sim_\alpha (y_n, p_1)$,..., $(x_n, p_n) \sim_\alpha (y_n, p_n)$, we define y_{n+1} as $(x_1 \bigtriangledown_\alpha y_n) \bigtriangleup_\alpha ... \bigtriangleup_\alpha (x_n \bigtriangledown_\alpha y_n)$ if $(y_n, q) <_\alpha (x_1, p_{n+1})$ for some q, or as $(x_1 \bigtriangleup_\alpha y_n) \bigtriangledown_\alpha ... \bigtriangledown_\alpha (x_n \bigtriangleup_\alpha y_n)$ if $(x_1, p_{n+1}) <_\alpha (y_n, q)$ for some q. In the first case $(x_i \bigtriangleup_\alpha y_n \to x_i \bigtriangledown_\alpha y_n) = c_i + \varphi_{i1} + \varphi_{i2}$ with an identity c_i containing p_i and $cod(\varphi_{i2})$ containing p_{n+1}, and we obtain $(x_i \to x_i \bigtriangledown_\alpha y_n) = c_i + \varphi_{i1} + cod(\varphi_{i2})$ with p_{n+1} contained in $c_i + cod(\varphi_{i2})$ and $(y_n \to x_i \bigtriangledown_\alpha y_n) = c_i + cod(\varphi_{i1}) + \varphi_{i2}$ with p_i contained in $c_i + cod(\varphi_{i1})$. Hence $(x_i, p_i) \sim_\alpha (x_i \bigtriangledown_\alpha y_n, p_i)$ and $(x_i \bigtriangledown_\alpha y_n, p_{n+1}) \sim_\alpha (x_i, p_{n+1})$. From $(y_n \to x_i \bigtriangledown_\alpha y_n) = c_i + cod(\varphi_{i1}) + \varphi_{i2}$ and $y_n \to y_{n+1} \to x_i \bigtriangledown_\alpha y_n$ we obtain by (A2.6) $(y_n \to y_{n+1}) = c_i + cod(\varphi_{i1}) + \gamma_i$ and $(y_{n+1} \to x_i \bigtriangledown_\alpha y_n) = c_i + cod(\varphi_{i1}) + \delta_i$. Hence $(x_i, p_i) \sim_\alpha (y_{n+1}, p_i)$. From $(x_i \bigtriangledown_\alpha y_n, p_{n+1}) \sim_\alpha (x_i, p_{n+1})$ and $(x_1, p_{n+1}) \sim_\alpha ... \sim_\alpha (x_n, p_{n+1})$ we obtain $(x_i \bigtriangledown_\alpha y_n, p_{n+1}) \sim_\alpha (x_1, p_{n+1})$ for all $i \in \{1, ..., n\}$. Hence $(x_1 \bigtriangleup (x_i \bigtriangledown_\alpha y_n) \to x_1 \bigtriangledown (x_i \bigtriangledown_\alpha y_n)) = d_i + \psi_{i1} + \psi_{i2}$ with identities d_i containing p_{n+1} for all $i \in \{1, ..., n\}$ and, finally, $(x_1 \bigtriangleup y_{n+1} \to x_1 \bigtriangledown y_{n+1}) = d + \psi_1 + \psi_2$ with an identity d containing p_{n+1}. Thus $(y_{n+1}, p_1) \sim_\alpha (y_n, p_1)$,..., $(y_{n+1}, p_n) \sim_\alpha (y_n, p_n)$, $(y_{n+1}, p_{n+1}) \sim_\alpha (x_1, p_{n+1})$. Similarly, in the second case $(y_{n+1}, p_1) \sim_\alpha (y_n, p_1)$,..., $(y_{n+1}, p_n) \sim_\alpha (y_n, p_n)$, $(y_{n+1}, p_{n+1}) \sim_\alpha (x_1, p_{n+1})$. $\sharp$

5.11. Corollary. If the set $\mathcal{A}_{ob}$ of objects definable in $\mathcal{A}$ is finite then for every $\alpha \in A$ a subset $Y \subseteq X_\alpha$ is a maximal antichain of the partially ordered set $(X_\alpha, \leq_\alpha)$ iff it corresponds to the set of occurrences of atomic identities in a cut y of α. $\sharp$

5.12. Lemma. If $\mathcal{A}$ is a behaviour algebra in which (A7) holds and $\alpha \in A$ is of finite width then the canonical partial order $\leq_\alpha$ is K-dense. $\sharp$

Proof. Suppose that Y is a maximal antichain of $(X_\alpha, \leq_\alpha)$ that consists of the equivalence classes of occurrences of atomic identities in a cut y of α. Suppose that Z is a maximal chain of $(X_\alpha, \leq_\alpha)$. If all elements of Z are not above Y then for each $z \in Z$ the set $f(z, Y)$ of successors of z in Y is non-empty and it can at most decrease with the increase of z. As α is of finite width and thus $f(z, Y)$ is finite, there exists at least one element of Z that belongs to Y. Similarly when all elements of Z are not below Y. Finally, if Z has elements both below and above Y, then the set $g(z_1, z_2, Y)$ of elements of Y that are between an element z_1 of Z that is below Y and an element z_2 of Z that is above Y is non-empty due to (A7) and it can at most decrease when z_1 and z_2 approach Y. As α is of finite width and thus such a set is finite, Z has an element in Y. $\sharp$

It is straighforward that if $\mathcal{A}$ is such that (A7) holds and the set $\mathcal{A}_{ob}$ of objects definable in $\mathcal{A}$ is finite then the correspondence $P : \alpha \mapsto (X_\alpha, \leq_\alpha, l_\alpha)$ just described between elements of $\mathcal{A}$ and lposets enjoys the following properties.

5.13. Lemma. Let $\mathcal{A}$ be such that (A7) holds and the set $\mathcal{A}_{ob}$ of objects definable in $\mathcal{A}$ is finite. If $\gamma = \alpha + \beta$ then $P(\gamma)$ is a coproduct object in $LPOSETS$ of $P(\alpha)$ and $P(\beta)$ with the canonical morphisms given by the correspondences

$$i_{\alpha,\alpha+\beta} : [((\xi_1, \xi_2), p)] \mapsto [((\xi_1 + dom(\beta), \xi_2 + \beta), p)]$$

$$i_{\beta,\alpha+\beta} : [((\eta_1, \eta_2), p)] \mapsto [((dom(\alpha) + \eta_1, \alpha + \eta_2), p)] \quad \sharp$$

5.14. Lemma. Let $\mathcal{A}$ be such that (A7) holds and the set $\mathcal{A}_{ob}$ of objects definable in $\mathcal{A}$ is finite. If $\gamma = \alpha\beta$ with $cod(\alpha) = dom(\beta) = c$ then $P(\gamma)$ is the pushout object in $LPOSETS$ of the injections of $P(c)$ in $P(\alpha)$ and in $P(\beta)$ given by

$$k_{c,\alpha} : [((c, c), p] \mapsto ((\alpha, c), p)]$$

$$k_{c,\beta} : [((c, c), p] \mapsto [((c, \beta), p]$$

with the canonical morphisms given by the correspondences

$$j_{\alpha,\alpha\beta} : [((\xi_1, \xi_2), p)] \mapsto [((\xi_1, \xi_2\beta), p)]$$

$$j_{\beta,\alpha\beta} : [((\eta_1, \eta_2), p)] \mapsto [((\alpha\eta_1, \eta_2), p)] \quad \sharp$$

In the case of a behaviour algebra $\mathcal{A}$ in which (A7) and (A8) hold and $\mathcal{A}_{ob}$ is finite all the lposets $P(\alpha)$ are finite and thus they do not contain segments with isomorphic proper subsegments. Consequently, all $H(\alpha)$ are K-dense processes in the universe $U(\mathcal{A}) = (A_{+0}, \mathcal{A}_{ob}, nat|A_{+0})$ and they can be composed as it is described in section 2. Thus we come to the following representation of behaviour algebras.

5.15. Theorem. If $\mathcal{A}$ is is a behaviour algebra such that (A7) and (A8) hold and the set $\mathcal{A}_{ob}$ of objects definable in $\mathcal{A}$ is finite then the correspondence $\alpha \mapsto H(\alpha)$ is an isomorphism from $\mathcal{A}$ to a subalgebra of the algebra $KPROC(U(\mathcal{A}))$ of K-dense processes in the universe $U(\mathcal{A})$ of objects definable in $\mathcal{A}$. $\sharp$

In the case of a behaviour algebra $\mathcal{A}$ in which (A7) holds and $\mathcal{A}_{ob}$ is finite but (A8) does not hold the lposets $P(\alpha)$ need not be processes since they need not satisfy (3.3) of 2.3. However, in order to guarantee that also in this case the lposets $P(\alpha)$ are processes, it suffices to replace (A1.4) by the following axiom that holds in every algebra of processes over a universe of objects.

(A1.4') If $\sigma\alpha\tau$ is defined and the lattice $L_{\sigma\alpha\tau}$ of cuts of $\sigma\alpha\tau$ is isomorphic to the lattice L_α of cuts of α then σ and τ are identities.

Thus we come to the following result.

5.16. Theorem. If $\mathcal{A}$ is is a behaviour algebra such that (A1.4') and (A7) hold and the set $\mathcal{A}_{ob}$ of objects definable in $\mathcal{A}$ is finite then the correspondence $\alpha \mapsto H(\alpha)$ is a homomorphism from $\mathcal{A}$ to the algebra $KPROC(U(\mathcal{A}))$ of K-dense processes in the universe $U(\mathcal{A})$ of objects definable in $\mathcal{A}$. $\sharp$

6. Endowing processes with structures

We have shown that every element of a behaviour algebra defines a unique set (the canonical underlying set) and a unique structure on this set (the structure that consists of the canonical partial order and the canonical labelling). Now we want to show how some elements of such an algebra and the sets they define (their canonical underlying sets) can be endowed with some additional structures.

By structures we mean slightly modified versions of structures in the sense of Bourbaki's Elements (cf [4]). We define them as follows.

Let Ens and $BijEns$ denote respectively the category of sets and mappings and the category of sets and bijective mappings. Let $\mathcal{P} : Ens \to Ens$ be the powerset functor, i.e. the fuctor such that $\mathcal{P}(X)$ is the set of subsets of X and $(\mathcal{P}(f))(Z) = f(Z)$ for every mapping $f : X \to X'$ and every $Z \subseteq X$. Let $\times : Ens \times Ens \to Ens$ be the bifunctor of cartesian product, i.e. the functor such that $\times(X, Y)$ is the cartesian product $X \times Y$ of X and Y and $(\times(f, g))(x, y) = (f(x), g(y))$ for every mappings $f : X \to X'$, $g : Y \to Y'$ and every $(x, y) \in X \times Y$. For every set A let A denotes the constant functor from Ens to Ens, i.e., the functor that assigns the set A to every set X and the identity of A to every mapping $f : X \to X'$.

6.1. Definition. By a *structure form* we mean a functor $F : Ens \to Ens$ that can be built from the identity functor and constant functors using the powerset functor $\mathcal{P} : Ens \to Ens$ and the bifunctor $\times : Ens \times Ens \to Ens$ of cartesian product. $\sharp$

6.2. Definition. Given a structure form F, by a *structure* of the form F on a set X we mean an element S of the set $F(X)$. $\sharp$

For example, a binary relation ρ on a set X is a structure of the form $BREL : X \mapsto \mathcal{P}(X \times X)$, a graph with a set V of vertices, a set E of edges such that $E \cap V = \emptyset$, a source function $s : E \to V$, and a target function $t : E \to V$, is a structure $G = (V, E, s, t)$ of the form $\mathcal{G} : X \mapsto \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X \times X) \times \mathcal{P}(X \times X)$ on $X = V \cup E$, a topology τ on a set X is a structure of the form $\mathcal{T} : X \mapsto \mathcal{P}(\mathcal{P}(x))$ on X, etc.

6.3. Definition. Given a structure form F, by a *morphism* from a structure $S \in F(X)$ of the form F on X to a structure $S' \in F(X)$ of the same form F on X' we mean an injection $f : X \to X'$ such that S' is the image of S under the mapping $F(f)$. $\sharp$

By $STR(F)$ we denote the category of structures of a form F and their morphisms.

6.4. Definition. By a *structure type* we mean a pair $T = (F, D)$, where F is a structure form $F : Ens \rightarrow Ens$ and D is a functor $D : BijEns \rightarrow BijEns$ such that $D(b) = F(b)$ for every bijection $b : X \rightarrow X'$ and $D(X) \subseteq F(X)$ for every set X (cf. [5]). $\sharp$

For example, the type of partial orders can be defined as the pair $PO = (BREL, Po)$, where $Po : BijEns \rightarrow BijEns$ with $Po(X)$ being the set of partial orders on X, the type of graphs can be defined as the pair $GRAPHS = (Graphs, \mathcal{G})$, where $Graphs : BijEns \rightarrow BijEns$ with $Graphs(X)$ being the set of quadruples $G = (V, E, s, t)$ of the form $\mathcal{G} : X \mapsto \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X \times X) \times \mathcal{P}(X \times X)$ such that V and E are disjoint subsets of X, $X = V \cup E$, $s : E \rightarrow V$, and $t : E \rightarrow V$, etc.

By $STRUCT(T)$ we denote the category of structures of type T.

Given a behaviour algebra $\mathcal{A} = (A, dom, cod, ;, +, 0)$ and its subalgebra $\mathcal{A}'$ on $A' \subseteq A$, each $\alpha \in A'$ can be endowed with a structure str_α of type T on its canonical underlying set X_α, and such a structure can be transported from X_α to the underlying set of each instance of α. However, the choice of str_α for $\alpha \in A'$ cannot be arbitrary since elements of the subalgebra $\mathcal{A}'$ are related and thus the structures corresponding to such elements should also be related. We propose to formalize such a choice as follows.

6.5. Definition. Processes of a subalgebra with the carrier A' are said to be consistently endowed with structures of type T if there exists a correspondence $\alpha \mapsto str_\alpha$ such that, for every $\alpha \in A'$, str_α is a structure of type T on the canonical underlying set X_α of α and the following conditions are fulfilled:

(1) if $\alpha + \beta$ is defined then $str_{\alpha+\beta}$ is the coproduct object in $STRUCT(T)$ of str_α and str_β with the canonical injections $i_{\alpha,\alpha+\beta}$ and $i_{\beta,\alpha+\beta}$ as in 5.13,

(2) if $\alpha\beta$ is defined and $cod(\alpha) = dom(\beta) = c$ then $str_{\alpha\beta}$ is the pushout object in $STRUCT(T)$ of the injections $k_{c,\alpha}$ and $k_{c,\beta}$ of str_c in str_α and in str_β as in 5.14 with the canonical injections $j_{\alpha,\alpha\beta}$ and $j_{\beta,\alpha\beta}$ as in 5.14. $\sharp$

Examples that follow illustrate the idea.

Let LPO be the structure type of labelled partial orders. Let $\mathcal{A}$ be a behaviour algebra. To each element α of $\mathcal{A}$ we can assign the structure $lpo_\alpha = (\leq_\alpha, l_\alpha)$ on the canonical underlying set X_α. If the set $\mathcal{A}_{ob}$ of objects definable in $\mathcal{A}$ is finite then 5.13 and 5.14 imply that the correspondence $\alpha \mapsto lpo_\alpha$ fulfils the conditions (1) and (2) of 6.5 for the structure type LPO.

Let WPO be the structure type of weighted partial orders defined as pairs $wpo = (\leq, d)$, where $\leq$ is a partial order on a set X and $d : X \times X \rightarrow Real \cup \{-\infty, +\infty\}$ is a function such that

(a) $d(x, x) = 0$,

(b) $d(x, y) = -\infty$ if x and y are incomparable with respect to $\leq$,

(c) $d(x, y) = sup\{d(x, z) + d(z, y) : z \neq x, z \neq y, x \leq z \leq y\}$ if there exists z such that $z \neq x$, $z \neq y$, $x \leq z \leq y$.

Let $\mathcal{A}$ be a behaviour algebra and $\mathcal{A}'$ a subalgebra of $\mathcal{A}$ generated by a set of $(+, ;)$-atoms. If the set $\mathcal{A}_{ob}$ of objects definable in $\mathcal{A}$ is finite then to each element α of the subalgebra $\mathcal{A}'$ we can assign structure $wpo_\alpha = (\leq_\alpha, d_\alpha)$ To this end it suffices to define d_α on $(+, ;)$-atoms generating $\mathcal{A}'$ and then extend it on entire $\mathcal{A}'$ such that the conditions (1) and (2) of 6.5 are fulfilled for the structure type WPO. Values of functions d_α can be interpreted as delays between elements of the canonical underlying set X_α of α. Together with data about occurrence times of minimal elements of X_α they determine occurrence times of all elements of X_α. For instance, in the case of a process α with a linear flow order the occurrence time of each $x \in X_\alpha$ is $t' + d_\alpha(x', x)$, where x' is the minimal element of X_α and t' is the occurrence time of x'.

Let $\mathcal{A}$ be a behaviour algebra with finite $\mathcal{A}_{ob}$. Suppose that B is a subset of $(;)$-atoms of $\mathcal{A}$ such that to each $\beta \in B$ there corresponds a structure gr_β of a graph on the canonical set X_β of β. Suppose that $\mathcal{A}'$ is the subalgebra of $\mathcal{A}$ generated by B. Then $gr_{dom(\beta)}$ and $gr_{cod(\beta)}$ must be graphs and the correspondence $\beta \mapsto gr_\beta$ has a unique extension on entire subalgebra $\mathcal{A}'$ and this extension fulfils (1) and (2) of 6.5 for the structure type $GRAPHS$. Notice that elements of $\mathcal{A}'$ thus endowed can be interpreted as derivations of graphs from graphs by applying graph grammar productions in the sense of the so called double pushout approach (cf. [7]).

Let $ABREL$ be the structure type of acyclic binary relations. Let $\mathcal{A}$ be a subalgebra of $PROC(U)$ generated by a set A_0 of not necessarily atomic processes, and let A be the underlying set of $\mathcal{A}$. Suppose that we can assign to each $\alpha \in A$ an acyclic binary relation cxt_α on X_α (a *context relation* in the sense of [18]) such that, for all elements of X_α, $(x, y) \in cxt_\alpha$ excludes both $x \leq_\alpha y$ and $y \leq_\alpha x$, and the reflexive and transitive closure of the following relation R, where cxt_α^+ denotes the transitive closure of cxt_α, is a partial order:

$$\begin{aligned}(x,y) \in R \ \text{ iff } \ & x \leq_\alpha y \text{ or } (x <_\alpha z \text{ and } (z,y) \in cxt_\alpha^+ \text{ for some } z) \\ & \text{or } (x <_\alpha t \text{ and } z <_\alpha y \text{ and } (z,t) \in cxt_\alpha \text{ for some } z \text{ and } t).\end{aligned}$$

Then we can extend the correspondence $\alpha \mapsto cxt_\alpha$ on instances of processes from A such that the conditions (1) and (2) of 6.5 are fulfilled for the structure type $ABREL$.

7. Relation to earlier works and conclusions

The present paper is a natural extension of [16], where algebras of processes of Condition/Event Petri nets with invariant sets of admitted markings have been characterized and called behaviour algebras. The novelty of this extension consists in a new system of axioms such that a subsystem of this system does not require the existence of indivisible processes and thus allows one to model also continuous processes. The new system has been formulated due to discovery of the relation between independence of processes and existence of bicartesian squares in categories of processes that has been described in [17]. It has been obtained from the characterization of algebras of processes of finite Condition/Event Petri nets that has been described in [19] by omitting the axioms on decomposability of processes into atoms and on two only instances of each condition.

We have presented a class of algebras of processes in universa of objects that contains also algebras of continuous and partially continuous processes. We have shown that such algebras are models of the new system of axioms and thus that they are behaviour algebras in the new sense. We have shown that there exists a correspondence between elements of behaviour algebras and lposets, and that in the case of a subclass of this class this correspondence results in a representation theorem. Finally, we have shown a way of extending the obtained results on algebras of processes with rich internal structures.

An early attempt of formulating an adequate system of axioms has been described in [18]. Its main line was to introduce a model of processes with context-dependent actions and rich internal structures and by defining and studying algebras of such processes in order to find out their characteristic properties.

Now, due to the results obtained for the new system of axioms, it seems that an adequate framework for modelling complex processes can be obtained with the aid of behaviour algebras and their subalgebras. For instance, processes with context-dependent actions as in [12] and [1] can be represented as elements of the subalgebra of an algebra of processes in a universe of objects that is generated by processes consisting of two concurrent components: one representing the proper action and the other representing the necessary context. Similarly, processes with rich internal structures as in [18] can be represented as elements of suitable subalgebras of behaviour algebras that are consistently endowed with the respective structures as it is described in section 5. For example, graph processes in the sense of [7] can be represented as processes consistently endowed with graph structures.

A problem that still remains open is how to come from the representation of processes of behaviour algebras with finite sets definable objects to a representation of processes of behaviour algebras with infinite sets of definable objects.

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