Lattices generated by information systems and their internal structure

Józef Winkowski

Institute of Computer Science of the Polish Academy of Sciences
ul. Jana Kazimierza 5, 01-248 Warszawa, Poland
e-mail: wink@ipipan.waw.pl

Abstract. The paper exploits the fact that every information system generates a family of equivalence relations in the set of considered objects, and the corresponding family of partitions of this set, and that this family is a lattice with certain properties. It describes the internal structure of any lattice with such properties and shows that such a lattice is generated by an information system.

1 Introduction

In this paper abstract lattices with the properties of lattices of equivalence relations generated by information systems are defined and studied.

The information systems considered in the sequel are supposed to be given in the form proposed by Pawlak (cf. [5] and [6]), i.e., in the form of a structure $S = (U, A)$, where U is a nonempty set of *objects* called the *universe*, and A is a nonempty, set of *primitive attributes*, where every primitive attribute $a \in A$ is a total function $a : U \rightarrow V_a$ from U to a set V_a of possible values of a, called the *domain* of a. Every such a system with a finite set of objects and a finite set of attributes can be represented as a table with rows corresponding to objects and columns corresponding to attributes.

1.1. Example (after [6]). The structure $S = (U, A)$ with $U = \{a, b, c, d, e\}$ and $A = \{t, x, y, z\}$, where t, x, y, z are functions from U to $\{0, 1, 2, ...\}$ such that

$t(a) = 0$, $t(b) = t(c) = t(e) = 1$, $t(d) = 2$,

$x(a) = x(d) = x(e) = 1$, $x(b) = 2$, $x(c) = 0$,

$y(a) = 2$, $y(b) = y(d) = y(e) = 0$, $y(c) = 1$,

$z(a) = z(c) = 0$, $z(b) = z(e) = 2$, $z(d) = 1$

is an information system. It can be represented by a table as in Figure 1.1. $\sharp$

	t	x	y	z
a	0	1	2	0
b	1	2	0	2
c	1	0	1	0
d	2	1	0	1
e	1	1	0	2

Figure 1.1: A table representing S.

With each attribute of S the equivalence relation is associated which says that some objects have the same values of this attribute. By taking into account the greatest lower bounds and the least upper bounds of such equivalence relations we obtain a lattice of equivalence relations or, equivalently, the corresponding lattice of partitions of the universe of objects into disjoint classes of equivalent elements. Thus we come to a lattice of partitions of the universe U with the operations corresponding to the operations in the lattice of all equivalences in U. So, finally we can represent the considered information system by this lattice, call such a lattice an *information lattice*, and write it as $\mathbf{C}_S = (C_S, \rightarrow_S)$, where C_S is the set of possible partitions of the universe of objects and $\rightarrow_S$ is the relation to be *coarser*. Objects can be defined as elements of one-element members of the finest partition $\perp$. Attributes can be defined as generators of C_S.

1.2. Example. For the information system S in Example 1.1, the corresponding information lattice $\mathbf{C}_S = (C_S, \rightarrow_S)$ is depicted in Figure 1.2, where

$$\perp = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}, \quad u = \{\{a\}, \{b\}, \{c\}, \{d, e\}\},$$

$$v = \{\{a\}, \{c\}, \{b, e\}, \{d\}\}, \quad t = \{\{a\}, \{b, c, e\}, \{e\}\},$$

$$x = \{\{a, d, e\}, \{b\}, \{c\}\}, \quad y = \{\{a\}, \{b, d, e\}, \{c\}\},$$

$$z = \{\{a, c\}, \{b, e\}, \{d\}\}, \quad p = \{\{a, b, d, e\}, \{c\}\},$$

$$q = \{\{a, c\}, \{b, d, e\}\}, \quad s = \{\{a, b, c, e\}, \{d\}\}, \quad \top = \{\{a, b, c, d, e\}\}. \ \sharp$$

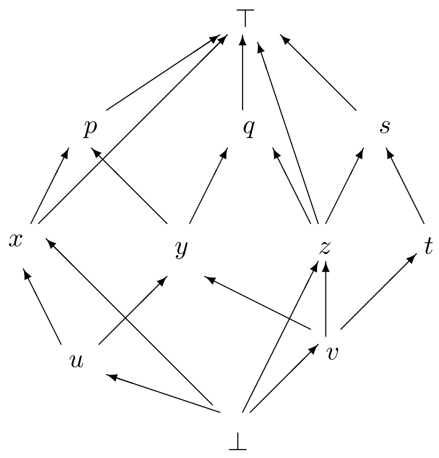

Figure 1.2

Attributes of an information system may depend on each other or they may be related in some ways. Consequently, it is natural to seek for the most influential attributes and to eliminate the irrelevant attributes.

In this paper we try to go further dealing not only with the given attributes but also trying to exploite the lattice structure to find some more natural attributes. In some cases it may allow us to considerably reduce the number of attributes that is essential for data representation, i.e., to reduce the dimension of representation. The problem of dimensionality reduction is important in many applications (cf. [1], [2] and related papers). In particular, it is important in applications with a relatively small number of objects and a large number of attributes.

2 Information lattices

As we have said, every information system generates an information lattice $\mathbf{C} = (C, \to)$. Elements of the underlying set C represent partitions of the considered universe of objects, every pair (c, c') of partitions such that $c \to c'$ represents a coarsening of c to c', and coarsenings $t \to x$ and $t \to y$ are regarded to be parallel independent if the diagram $(x \leftarrow t \to y, x \to z \leftarrow y)$ is a *diamond* in the sense that t is the greatest lower bound of x and y and z is the least upper bound of x and y. Moreover, $\mathbf{C}$ enjoys the following properties.

2.1. Proposition. In any lattice $\mathbf{C} = (C, \to)$ generated by an information system the following conditions are satisfied:

(A1) If $x \to z' \to z$ and $(x \leftarrow t \to y, x \to z \leftarrow y)$ is a diamond then there exists t' such that $t \to t' \to y$ and $(x \leftarrow t \to t', x \to z \leftarrow t')$ and $(z' \leftarrow t' \to y, z' \to z \leftarrow y)$ are diamonds.

(A2) If $t \to t' \to y$ and $(x \leftarrow t \to y, x \to z \leftarrow y)$ is a diamond then there exists z' such that $x \to z' \to z$ and $(x \leftarrow t \to t', x \to z \leftarrow t')$ and $(z' \leftarrow t' \to y, z' \to z \leftarrow y)$ are diamonds. $\sharp$

Prof outline. For any partitions x and y of the universe, the partition $x \sqcup y$ consists of maximal subsets of the universe with the property that their elements are equivalent with respect to the transitive closure of the union of the equivalence relations corresponding to x and y. The existence of such maximal subsets of the universe follows from the fact that every chain of subsets of the universe is contained in its own union. $\sharp$

In this paper we are interested not only in information lattices of information systems, but in arbitrary lattices that enjoy only the relatively weak properties (A1) and (A2).

What has been said can be reflected in the following definition.

2.2. Definition. An *abstract information lattice*, or briefly an *information lattice*, is a complete lattice $\mathbf{C} = (C, \to)$ that enjoys the properties (A1) and (A2) of Proposition 2.1. $\sharp$

2.3. Example. The structure $\mathbf{C} = (C, \to)$ with $C = C_S$ and $\to = \to_S$ as in Example 1.2 is an information lattice. $\sharp$

Let $\mathbf{C} = (C, \to)$ be an abstract information lattice. Elements of C are called partitions. If (c, c') is a pair of elements of $\mathbf{C}$ such that $c \to c'$ then it represents a *coarsening* of the partition c to a partition c'. Diagrams $(x \leftarrow t \to y, x \to z \leftarrow y)$ such that t is the greatest lower bound $x \sqcap y$ of x and y and z is the least upper bound $x \sqcup y$ of x and y are called *diamonds*. Given a diamond $D = (x \leftarrow t \to y, x \to z \leftarrow y)$, elements t, x, y, z are called *nodes* of D and coarsenings $t \to x$, $t \to y$, $x \to z$, $y \to z$ are called *sides* of D.

2.4. Example. The following diagrams in Figure 1.2 are diamonds of the information lattice in Example 1.2. The diagrams which consist of diamonds of this lattice with a common side are also its diamonds. $\sharp$

$(u \leftarrow \perp \to v, u \to y \leftarrow v)$, $(x \leftarrow u \to y, x \to p \leftarrow y)$,
$(y \leftarrow v \to z, y \to q \leftarrow z)$, $(z \leftarrow v \to t, z \to s \leftarrow t)$,
$(x \leftarrow \perp \to z, x \to \top \leftarrow z)$, $(p \leftarrow y \to q, p \to \top \leftarrow q)$,
$(q \leftarrow z \to s, q \to y \leftarrow s)$. $\sharp$

The following proposition follows from the definition of information lattice.

2.5. Proposition. For every $c \in C$ the restriction of $\mathbf{C}$ to the set $c^{\downarrow} = \{d \in C : d \to c\}$ is an information lattice (written also as $c^{\downarrow}$). $\sharp$

3 Independence and equivalence of coarsenings

The concept of a diamond can be used to define independence and equivalence of coarsenings of an information lattice.

The equivalence of coarsenings $t \to x$ and $t \to y$ is the basis of the present paper. It is represented by the fact that the diagram $(x \leftarrow t \to y, x \to z \leftarrow y)$ is a *diamond* in the sense that t is the greatest lower bound of x and y and z is the least upper bound of x and y.

3.1. Definition. If $(v \leftarrow u \to w, v \to u' \leftarrow w)$ is a diamond in an information lattice $\mathbf{C} = (C, \to)$ then the coarsenings $u \to v$ and $u \to w$ are said to be *parallel independent*, and the coarsenings $u \to v$ and $v \to u'$, as well as the coarsenings $u \to w$ and $w \to u'$, are said to be *sequential independent* (cf. [4]). $\sharp$

3.2. Example. For the information lattice $\mathbf{C}$ in example 2.3, the coarsenings $\perp \to u$ and $\perp \to v$ are parallel independent, and the coarsenings $\perp \to_1 u$ and $u \to_2 y$ are sequential independent. $\sharp$

3.3. Definition. By the *natural equivalence* of coarsenings in an information lattice $\mathbf{C} = (C, \to)$ we mean the least equivalence relation $\equiv$ between coarsenings such that $u \to v \ \equiv \ w \to u'$ whenever in this information lattice there exists a diamond $(v \leftarrow u \to w, v \to u' \leftarrow w)$. $\sharp$

3.4. Examples. Consider the information lattice $\mathbf{C}$ in example 2.3. In this lattice the coarsenings $\perp \to u$ and $v \to y$ are equivalent, and the coarsenings $u \to y$ and $\perp \to v$ are equivalent. $\sharp$

4 Regions of information lattices

The existence in information lattices of the natural equivalence of coarsenings makes it possible to adapt and exploit the concept of a region similar to that introduced in [3].

4.1. Definition. By a *region* of an information lattice $\mathbf{C} = (C, \to)$ we mean a nonempty subset r of the set of elements of $\mathbf{C}$ such that:

$u \in r$ and $v \notin r$ and $w \to u' \ \equiv \ u \to v$ implies $w \in r$ and $u' \notin r$,
$u \notin r$ and $v \in r$ and $w \to u' \ \equiv \ u \to v$ implies $w \notin r$ and $u' \in r$. $\sharp$

4.2. Example. In the information lattice $\mathbf{C}$ in example 2.3 the sets $\{\perp, v, z, t, s\}$, $\{u, x, y, p, q, \top\}$, are regions, and the sets $\{\perp, u, v, y, z, q\}$, $\{t, x, p, s, \top\}$ are regions. $\sharp$

From the definition of a region we obtain the following propositions.

4.3. Proposition. If $\mathbf{C} = (C, \to)$ is an information lattice, r is a region of $\mathbf{C}$, and $(v \leftarrow u \to w, v \to u' \leftarrow w)$ is a diamond in $\mathbf{C}$, then $v \in r$ implies that $u \in r$ or $u' \in r$. $\sharp$

4.4. Proposition. The set of all members of $\mathbf{C}$ is a region of $\mathbf{C}$. $\sharp$

4.5. Proposition. If p and q are disjoint regions of $\mathbf{C}$ then $p \cup q$ is a region of $\mathbf{C}$. $\sharp$

4.6. Proposition. If p and q are different regions of $\mathbf{C}$ such that $p \subseteq q$ then $q - p$ is a region of $\mathbf{C}$. $\sharp$

Let r be a region of $\mathbf{C}$ and let x be an element of r. Given a chain $(r_i : i \in I)$ of regions of $\mathbf{C}$ that are contained in r and contain and element x, for $r' = \bigcap(r_i : i \in I)$ and a transition $c \to d$ such that $c \in r'$ and $d \notin r'$, there exists $i_0 \in I$ such that $c \in r_i$ and $d \notin r_i$ for $i > i_0$. Consequently, for every coarsening $c' \to d'$ such that $c' \to d' \equiv c \to d$ we have $c' \in r_i$ and $d' \notin r_i$ for $i > i_0$, and thus $c' \in r'$ and $d' \notin r'$. Similarly, for $c \to d$ such that $c \notin r'$ and $d \in r'$ and for $c' \to d' \equiv c \to d$. So, r' is a region. Consequently, in the set of regions that are contained in r and contain x there exists a minimal region. Hence we obtain the following results.

4.7. Proposition. Every region of $\mathbf{C}$ contains a minimal region. $\sharp$

4.8. Proposition. Every element of $\mathbf{C}$ belongs to a minimal region. $\sharp$

4.9. Proposition. If a member s of $\mathbf{C}$ does not belong to a region r then there exists a minimal region r' such that $r \cap r' = \emptyset$ and $s \in r'$. $\sharp$

4.10. Proposition. Every region of $\mathbf{C}$ can be represented as a disjoint union of minimal regions. $\sharp$

Proof. Let m be the disjoint union of a family M of minimal regions of $\mathbf{C}$. Then m is a region of $\mathbf{C}$ and if it does not cover C then $C - m$ is a region of $\mathbf{C}$ and the family M can be extended by a minimal region of $\mathbf{C}$ that contains a given element of $C - m$ as in the text preceding Proposition 4.7. Consequently, a family of disjoint minimal regions of $\mathbf{C}$ can be defined such that its union covers C. $\sharp$

4.11. Example. In the information lattice $\mathbf{C}$ in example 2.3 we have the following decompositions of the set of members into disjoint union of minimal regions (see Figure 4.1): $A = \{\{\bot, v, z, t, s\}, \{u, x, y, p, q, \top\}\}$, $B = \{\{\bot, u, v, y, z, q\}, \{t, x, p, s, \top\}\}$, $G = \{\{\bot, u, x\}, \{v, y, z, t, p, q, s, \top\}\}$. $\sharp$

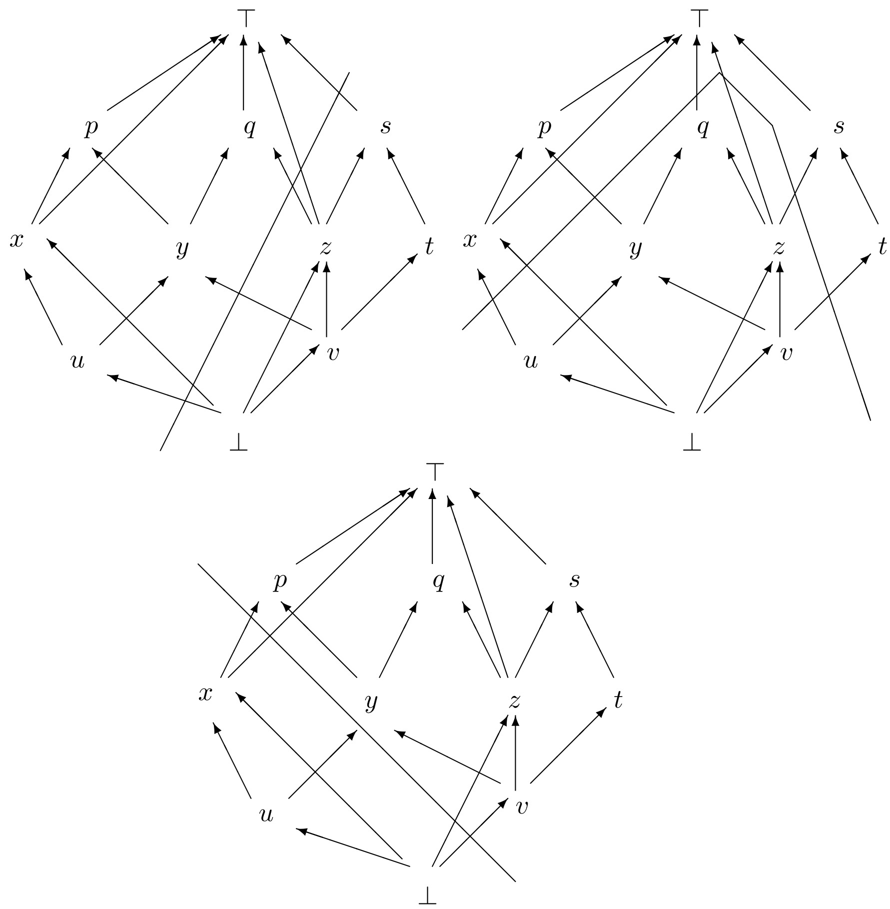

Figure 4.1

4.12. Proposition. For every element c of $\mathbf{C}$ and for every region r of $\mathbf{C}$ the subset $r|c = \{d \in r : d \to c\}$ of r is either empty or it is a region of $c^{\downarrow}$. $\sharp$

A proof follows from the fact that every diamond in $c^{\downarrow}$ is a diamond in $\mathbf{C}$.

5 Minimal regions of information lattices

We shall prove that every abstract information lattices are information lattices generated by an information system.

Given an information lattice $\mathbf{C} = (C, \to)$, we can assign to $\mathbf{C}$ a labelled partially ordered set $\mathbf{E} = (E, \leq, l)$ and an information system $\mathbf{S} = (U, A)$. This can be done as follows.

Let $R_{\mathbf{C}}$ denote the set of minimal regions of $\mathbf{C}$. Let $D_{\mathbf{C}}$ denote the set of decompositions of the set of elements of $\mathbf{C}$ into disjoint unions of minimal regions, every decomposition defined as a set d of mutually disjoint minimal regions from $R_{\mathbf{C}}$ such that $\bigcup d = C$.

The underlying set E of $\mathbf{E}$ is defined as the set $E_{\mathbf{C}}$ of pairs (d, r) consisting of a decomposition $d \in D_{\mathbf{C}}$ and of a minimal region $r \in d$.

The labelling l of E can be defined as $l : (d, r) \mapsto r$.

The universe U of S can be defined as the set of maximal antichains of $\mathbf{E}$.

The set A of attributes of S can be defined as the set of mappings a such that $a : m \mapsto r$ for every maximal antichain of $\mathbf{E}$ that contains (d, r).

We start from the partial order of $\mathbf{E}$.

The partial order $\leq$ can be introduced as follows.

Let $\preceq$ be the following relation in $R_{\mathbf{C}}$.

5.1. Definition. Given $x, y \in R_{\mathbf{C}}$, we write $x \preceq y$ iff for every $v \in y$ there exists $u \in x$ such that $u \to v$, for every $u \in x$ there exists $v \in y$ such that $u \to v$, and the following conditions are satisfied:

(1) $t \in x$ iff $w \in y$, for every diamond $(u \leftarrow t \to w, u \to v \leftarrow w)$ with $u \in x$ and $v \in y$,
(2) $t' \in x$ iff $w' \in y$, for every diamond $(t' \leftarrow u \to v, t' \to w' \leftarrow v)$ with $u \in x$ and $v \in y$. $\sharp$

5.2. Proposition. The relation $\preceq$ is a partial order on $R_{\mathbf{C}}$. $\sharp$

For a proof it suffices to notice that the relation $\preceq$ follows the partial order in $\mathbf{C}$.

The partial order $\leq$ can be defined as the least partial order $\leq_{\mathbf{C}}$ in E such that $(d, r) \leq_{\mathbf{C}} (d', r')$ if $r \preceq r'$ and $r \neq r'$ or if $d = d'$ and $r = r'$.

The properties of the partially ordered set $\mathbf{E}$ are consequences of the following observations.

First, the properties of information lattices imply an important property of minimal regions.

5.3. Proposition. Every minimal region r is *convex* in the sense that $w \in r$ for every w such that $u \to w \to v$ for some $u \in r$ and $v \in r$. $\sharp$

Proof. Suppose that r is a region of $\mathbf{C}$ and $a \sqsubseteq c \sqsubseteq b$ for $a, b \in r$ and $c \notin r$. Define r^- to be the set of $u \in r$ such that $u \sqsubseteq c$ or $u' \sqsubseteq c$ for some u' that can be connected with u by a side of a bicartesian square with the nodes of the opposite side not in r. Define r^+ to be the set of $u \in r$ such that $c \sqsubseteq u$ or $c \sqsubseteq u'$ for some u' that can be connected with u by a side of a bicartesian square with the nodes of the opposite side not in r. There is no bicartesian square with a side connecting some $u \in r$ and $v \in r$ such that $u \sqsubseteq c \sqsubseteq v$ and with the nodes of the opposite side not in r because by (A1) and (A2) it would imply $c \in r$. By (A1) and (A2) there are no bicartesian squares with sides connecting some u' with $u \in r$ and $v \in r$ such that $u \sqsubseteq c \sqsubseteq v$ and with the nodes of the opposite sides not in r. Consequently, the sets r^- and r^+ are disjoint. On the other hand, r is a minimal region of $\mathbf{C}$ and thus $r \subseteq r^- \cup r^+$. Moreover, there is no bicartesian square connecting an element of r^- with an element of r^+ and with the nodes of the opposite side not in r. Consequently, r cannot be a minimal region of $\mathbf{C}$ as supposed. $\sharp$

Second, minimal regions wich are not disjoint are incomparable with respect to the partial order $\preceq$.

5.4. Proposition. If minimal regions $x, y \in R_{\mathbf{C}}$ are not disjoint and different then neither $x \preceq y$ nor $y \preceq x$. $\sharp$

Proof. Suppose that x and y are different minimal regions of $R_{\mathbf{C}}$ such that $x \cap y \neq \emptyset$. Then $x - y$ and $y - x$ are nonempty and there exist $u \in x - y$, $v \in y - x$, and $w, z \in x \cap y$ such that u and w are adjacent nodes of a diamond U, z and v are adjacent nodes of a diamond V, and the nodes of the diamond $W = (w \leftarrow w \sqcap z \rightarrow z, w \rightarrow w \sqcup z \leftarrow z)$ are in $x \cap y$.

Consider the case in which $w = u \sqcup u'$ for some u' not in x and $z = v \sqcap v'$ for some v' not in y. Then $u' \in y$, $v' \in x$, and the condition (1) is not satisfied for $z \rightarrow v$ and the diamond $(v \leftarrow z \rightarrow v', v \rightarrow v \sqcup v' \leftarrow v')$. Consequently, $x \preceq y$ does not hold.

Similarly, in the other possible cases we come to the conclusion that neither $x \preceq y$ nor $y \preceq x$. $\sharp$

Third, in some configuration domains all disjoint minimal regions are comparable with respect to the partial order $\preceq$.

5.5. Proposition. If minimal regions $x, y \in R_{\mathbf{C}}$ are disjoint then either $x \preceq y$ or $y \preceq x$. $\sharp$

Proof. It is impossible that u and v are incomparable for all $u \in x$ and $v \in y$ since one of the regions x or y contains $u \sqcap_\alpha v$ or $u \sqcup_\alpha v$.

Suppose that $u \sqsubseteq v$ for $u \in x$ and $v \in y$. As x and y are disjoint and convex, it suffices to prove that every element of y has a predecessor in x. Consider $w \in y$. If $v \sqsubseteq w$ then $u \sqsubseteq w$. If $w \sqsubseteq v$ then $u' \sqsubseteq w$ for $u' = u \sqcap w$ and by considering the bicartesian square $(u \leftarrow u' \rightarrow w, u \rightarrow w' \leftarrow w)$ we obtain that $w' \in y$ because y

is convex. Hence $u' \in x$. If w and v are incomparable then either $v \sqcap w \in y$ and we may replace w by $v \sqcap w$ and proceed as in the previous case, or $v \sqcup w \in y$ and we may replace v by $v \sqcup w \in y$ and proceed as in the previous case. On the other hand, $u \sqsubseteq v$ for $u \in x$ and $v \in y$ excludes $v' \sqsubseteq u'$ for $u' \in x$ and $v' \in y$ since x and y are convex. Hence $x \preceq y$.

Similarly, in the case $v \sqsubseteq u$ we obtain $y \preceq x$. $\sharp$

One of the consequences of these observations is the following proposition.

5.6. Proposition. For every $d \in D_{\mathbf{C}}$ the subset $\{d\} \times \{r \in R_{\mathbf{C}} : r \in d\}$ of $E_{\mathbf{C}}$ is a maximal chain. $\sharp$

A proof follows from the fact that for every maximal $c \in C$ the restrictions $r|c$ of r from d form a decomposition of $R_{c\downarrow}$ into a disjoint union of minimal regions (see proposition 4.12).

5.7. Proposition. For every $c \in \mathbf{C}$, the subset $X_c = \{(d,r) :\in E_{\mathbf{C}} : c \in r\}$ of $E_{\mathbf{C}}$ is a maximal antichain of $\mathbf{E_C}$. $\sharp$

A proof follows from the fact that no minimal regions in X_c are disjoint and from the fact that c belongs to one minimal region of every decomposition $d \in D_{\mathbf{C}}$.

These results can be summarized in the following theorem.

5.8. Theorem. Given an information lattice $\mathbf{C} = (C, \rightarrow)$, the corresponding labelled partially ordered set is $\mathbf{E_C} = (E_{\mathbf{C}}, \leq_{\mathbf{C}}, l_{\mathbf{C}})$ with maximal antichains
$X_c = \{(d,r) :\in E_{\mathbf{C}} : c \in r\}$ of $\mathbf{E_C}$, and with maximal chains
$\{d\} \times \{r \in R_{\mathbf{C}} : r \in d\}$. $\sharp$

Each maximal antichain of $E_{\mathbf{C}}$ can be interpreted as an object. Each decomposition $d \in D_{\mathbf{C}}$ can be interpreted as an attribute. Each minimal region $r \in d$ can be interpreted as the value of the attribute d for the object (d,r). Thus we obtain the following result.

5.9. Theorem. The labelled partially ordered set $\mathbf{E_C}$ defines the information system $S_{\mathbf{C}}$, where $S_{\mathbf{C}} = (U_{\mathbf{C}}, A_{\mathbf{C}})$ with $U_{\mathbf{C}}$ is the set of maximal antichains of $\mathbf{E_C}$, and $A_{\mathbf{C}}$ is the set of functions a such that $a : m \mapsto r$ for every a maximal antichain of $\mathbf{E_C}$ that contains (d,r). $\sharp$

5.10. Example. For the information lattice $\mathbf{C}$ in example 2.3 the labelled partially ordered set $\mathbf{E_C}$ is $\mathbf{E_C} = (E_{\mathbf{C}}, \leq_{\mathbf{C}}, l_{\mathbf{C}})$ with
$E_{\mathbf{C}} = \{(A, \{\bot, v, z, t, s\}), (A, \{u, x, y, p, q, \top\}),$
$(B, \{\bot, u, v, y, z, q\}), (B, \{t, x, p, s, \top\}),,$
$(G, \{\bot, u, x\}), (G, \{v, y, z, t, p, q, s, \top\})\},$

$(A, \{\perp, v, z, t, s\}) \leq_{\mathbf{C}} (A, \{u, x, y, p, q, \top\})$,
$(B, \{\perp, u, v, y, z, q\}) \leq_{\mathbf{C}} (B, \{t, x, p, s, \top\})$,
$(G, \{\perp, u, x\}) \leq_{\mathbf{C}} (G, \{v, y, z, t, p, q, s, \top\})$,
$l_{\mathbf{C}}(A, r) = r$ for $r \in \{\{\perp, v, z, t, s\}, \{u, x, y, p, q, \top\}\}$,
$l_{\mathbf{C}}(B, r) = r$ for $r \in \{\{\perp, u, v, y, z, q\}, \{t, x, p, s, \top\}\}$,
$l_{\mathbf{C}}(G, r) = r$ for $r \in \{\{\perp, u, x\}, \{v, y, z, t, p, q, s, \top\}\}$,
$U_{\mathbf{C}}$ is the set of maximal antichains of $\mathbf{E_C}$,
$A_{\mathbf{C}}$ is the set of functions like m_0, where $m_0 : x' \mapsto \{\perp, v, z, t, s\}$ and $m_0 : x'' \mapsto \{\perp, v, u, y, z, q\}$ and $m_0 : x''' \mapsto \{\perp, u, x\}$ for the maximal antichain that consists of the minimal elements x', x'', x''' of $\mathbf{E_C}$, and so on (see Figure 5.1). $\sharp$

$$\begin{tikzcd}
(A, \{u, x, y, p, q, \top\}) & (B, \{t, x, p, s, \top\}) & (G, \{v, y, z, t, p, q, s, \top\}) \\
(A, \{\perp, v, z, t, s\}) \arrow[u, "\leq_{\mathbf{C}}"'] & (B, \{\perp, u, v, y, z, q\}) \arrow[u, "\leq_{\mathbf{C}}"'] & (G, \{\perp, u, x\}) \arrow[u, "\leq_{\mathbf{C}}"']
\end{tikzcd}$$

Figure 5.1

6 Concluding remarks

We have shown how an information system can be regarded as a set of generators of an algebraic system called an information lattice, and how elements of the latter can be regarded as maximal antichains of a labelled partially ordered set which represents a reduced information system.

The process of generating information lattices from information systems is relatively simple. It only requires computing the greatest lower bounds and the least upper bounds of partitions of the considered universe. Besides, intermadiate results can be exploited to execute it according to a strategy. For example, the greatest lower bound of partitions which are similar is similar to these partitions, and the greatest lower bound of partitions which are completely independent consists of intersections of the pairs of members of the argument partitions, and this can be used to control the choice of partitions to be combined in order to generate the information lattice corresponding to the given information system.

Moreover, the process of generating the information lattice can be combined with the process of finding the regions of this lattice and constructing a reduced information system and its information lattice.

Consequently, the paper contributes with a way of transforming an information system into another, usually less dimensional information system.

References

1. Charu C. Aggarwal (Editor): Data Streams Models and Algorithms, Springer, (2006)

2. Charu C. Aggarwal, You, P.: Finding generalized Projected Clusters in High Dimensional Data Spaces, ACM SIGMOD Conference, (2000)
3. Ehrenfeucht, A., Rozenberg, G., *Partial 2-structures*, Acta Informatica 27 (1990) 315-368
4. Ehrig, H., Kreowski, H. -J., *Parallelism of Manipulations in Multidimensional Information Structures*, in A. Mazurkiewicz (Ed.): Proc. of MFCS'76, Springer LNCS 45 (1976) 284-293
5. Pawlak Z., *Information systems - theoretical foundations*, Information Systems 6 (1981) 205-218
6. Pawlak Z., ROUGH SETS, Theoretical Aspects of Reasoning about Data. Kluwer Academic Publishers, (1991)