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A MATHEMATICAL MODEL OF ACTION [1]

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Abstract: In the paper a model of action is described that is universal in the sense that it may serve to represent actions of any kind: discrete, continuous, or partially discrete and partially continuous. The model is founded on the assumption that an action is executed in a universe of objects. It describes how the possible executions change the situation of involved objects. It exploits the fact that executions are represented such that their bounded segments admit only trivial automorphisms. Consequently, the model has an algebraic strucure and is a directed complete partial order.

Keywords: Action, object, object instance, occurrence of object instance, concrete execution, execution structure, history preserving equivalence of concrete execution structures, abstract execution, algebra of abstract executions, directed complete partial order, reduced execution structure.

PEWIEN MODEL AKCJI I JEGO WŁASNOŚCI

Streszczenie: Praca zawiera opis pewnego modelu akcji, który jest uniwersalny w tym sensie, że może służyć do reprezentowania akcji dowolnego rodzaju: dyskretnych, ciągłych, lub częściowo dyskretnych i częściowo ciągłych. Model ten opiera się na założeniu, że akcja jest wykonywana w pewnym środowisku obiektów. Opisuje jak możliwe wykonania akcji zmieniają sytuacje zaangażowanych obiektów. Wykorzystuje fakt, że wykonania akcji są reprezentowane tak, że ich ograniczone segmenty mają jedynie trywialne automorfizmy. Dzięki temu model ma pewną strukturę algebraiczną i częściowy porządek przy którym podzbiory skierowane mają kresy górne.

Słowa kluczowe: Akcja, obiekt, instancja obiektu, wystąpienie instancji obiektu, wykonanie konkretne, struktura wykonań, zachowująca historie równoważność struktur wykonań, wykonanie abstrakcyjne, algebra wykonań abstrakcyjnych, częściowy porządek z kresami górnymi podzbiorów skierowanych, zredukowana struktura wykonań.

1 Introduction

For discrete actions like behaviours of discrete concurrent systems there are models as labelled event structures which reflect well concurrency and indeterminism (cf. [GP 95], [NPW 81] and [WN 95]). Such models are based on local properties of modelled actions and they are compositional in the sense that the model of a compound action can be obtained by combining models of component actions (cf. [Wns 82]). However, usually they represent actions up to an equivalence (cf. [GG 01]).

Continuous actions like those considered in control theory are usually modelled by differential equations which reflect the relatons between variables representing the input of a system and variables representing the system state and output (cf. [KFA 69]).

Behaviours of systems working in a continuous way can also be represented as the behaviours of continuous Petri nets as those described in [D 91] and [DS 01]. In such nets transitions are regarded as continuous activities transforming some materials into some other materials and the behaviour consists of execution of such activities with intesities given by a control path.

Behaviours of hybrid systems like those including real-world components, which work in a continuous way, and controlling components, which operate in discrete steps, are modelled as a combination of continuous and discrete activities (cf. [H 96], [ML 07], and [NK 93]).

However, a typical way of representing in such models the possible continuity of work is to use a reference to a global time which is not a local property and may be artificial or uncertain.

An attempt of obtaining a free from a reference to a global time model of actions with continuous components has been described in [K 96]. The idea presented in [K 96] consists in a generalization of labelled event structures by dividing the set of possible events into a subset of discrete events and a family of subsets of remaining events, each subset of the family provided with a local time. However, this idea does not seem to be farther developed.

In this paper we propose a universal model of action, a model which uses only local properties of the modelled actions, is compositional, and is universal enough to represent in the same way actions of any kind: discrete, continuous, or partially discrete and partially continuous. In particular, we want to avoid a reference to a global time. So, instead of representing the continuity in actions with the aid of such a reference, we develop algebraic tools allowing to represent the possible continuity of action executions as infinite divisibility of such executions into segments.

A universal model of action of this type is needed for representing and relating behaviours of systems without a need of inventing a special model in every particular case.

Our model of action is founded on the assumption that an action is executed in a universe of objects, each object with a set of possible instances corresponding to its states and other temporary features, each execution transforming instances of objects. Formally it is a specific labelled partially ordered set (lposet), called execution structure. Each labelled element of such an lposet represents an occurrence during a possible execution of object instance given by the label. The partial order represents the causal dependency of occurrences of object instances, i.e., how occurrences of object instances arise from occurrences of object instances. Maximal downward closed subsets without incomparable occurrences of instances of the same object represent the possible executions.

Execution structures are similar to labelled event structures, but instead of occurrences of atomic actions they represent occurrences of object instances, and instead of conflict relation they represent the possible executions.

As in the case of labelled event structures, different execution structures may represent the same action, and formally it can be decided with the aid of the respective notion of equivalence. In particular, for execution structures we have a notion of history preserving equivalence which is similar to that for event structures. In the case of execution structures this notion is especially attractive since it leads to a very simple representation of equivalence classes and, consequently, to a very simple model of action. This model consists of isomorphism classes of executions and has an interesting algebraic structure.

The paper is organized as follows. In section 2 we define potential executions of actions and describe their properties. In section 3 we define execution structures of actions and describe their properties. In section 4 we define a history preserving equivalence of execution structures and characterize its equivalence classes as sets of abstract executions. In section 5 we define operations on abstract executions and the corresponding partial algebras. In section 6 we characterize sets of abstract executions of actions as specific subsets of such algebras. In section 7 describe actions whose abstract executions enjoy some particular properties.

2 Executions of actions

In order to define execution of actions we think of actions executed in a universe of objects (memory locations, messages, etc.), each object with a set of possible instances corresponding to its states and other temporary features (contents, positions, etc.), each execution transforming instances of objects (cf. [W 12]).

A universe of objects can be defined as follows.

2.1. Definition. A *universe of objects* is $\mathbf{U} = (W, V, ob)$, where V is a set of *objects*, W is a set of *instances* of objects from V (a set of *object instances*), and ob is a mapping $ob : W \to V$ that assigns the respective object to each of its instances. $\sharp$

2.2. Example. Suppose that V' is a set of tanks to keep a liquid. Define objects to be tanks $v \in V'$. Define instances of $v \in V'$ to be pairs $w = (v, s)$, where $s \geq 0$ is the amount of liquid in v. Define W' to be the set of possible instances of objects $v \in V'$ and $ob' : W' \to V'$ to be the mapping $(v, s) \mapsto v$. Then $\mathbf{U}' = (W', V', ob')$ is a universe of objects. $\sharp$

Potential executions of actions in a universe of objects can be defined without specifying what actions are executed. The notions used in the definition can be found in Appendix A (see Definitions A.1 and A.5).

2.3. Definition. A *concrete execution* in a universe $\mathbf{U} = (W, V, ob)$ of objects is a labelled partially ordered set (lposet) $E = (X, \leq, ins)$, where X is a set (of *occurrences* in E of (instances of) objects), $ins : X \to W$ is a mapping (a *labelling* that assigns the respective object instance to each occurrence of this object instance), and $\leq$ is a partial order (the *causal dependency relation* of E) such that

(1) for every object $v \in V$, the set $X|v = \{x \in X : ob(ins(x)) = v\}$ is either empty or it is a maximal chain and has an element in every cross-secton of $(X, \leq)$,

(2) every element of X belongs to a cross-section of $(X, \leq)$,

(3) no bounded segment of E is isomorphic to its proper bounded subsegment,

(4) the set of minimal elements of $(X, \leq)$ is a cross-section. $\sharp$

Put in another way, E is a partially ordered set of occurrences of object instances. Each object may have many instances and each of them may have in E many occurrences. Condition (1) says that the occurrences of instances of every object which takes part in E form a maximal chain, that E contains all information on such an object, and that every potential state of E contains the respective part of this information. Condition (2) says that every occurrence of an object in E belongs to a potential state of E. Condition (3) guarantees that the progress of E is a purely intrinsic property of E that is fully reflected by what happens to the involved objects. It implies that even in the case of continuous executions the lposets representing bounded segments of executions admit only trivial automorphisms, a property that will be crucial for simplifying a natural equivalence of execution structures and, consequently, models of actions. Condition (4) means that E has an initial state.

Representation of executions of actions in terms of object instances and their occurrences allows us to describe in the same way both discrete and continuous executions.

As concrete executions are lposets, their morphisms are defined as morphisms of lposets, that is as mappings that preserve the ordering and the labelling (see Appendix A).

2.4. Example. Suppose that $\mathbf{U}' = (W', V', ob')$ is the universe of object from Example 2.2.

What is going on in a tank $v \in V'$ in an interval $[t', t'']$ of time can be regarded as a concrete execution $E(v) = (X_{E(v)}, \leq_{E(v)}, ins_{E(v)})$ in $\mathbf{U}'$, where

$X_{E(v)}$ is the set of values of variations $var(t \mapsto s_v(t); t', t)$ in intervals $[t', t] \subseteq [t', t'']$ of the real valued function $t \mapsto s_v(t)$ that specifies the amount of liquid in v at every moment $t \in [t', t'']$,

$\leq_{E(v)}$ is the restriction of the usual order of numbers to $X_{E(v)}$,

$ins_{E(v)}(x) = (v, s_v(t))$ for $x = var(t \mapsto s_v(t); t', t)$.

One-element subsets of the set $X_{E(v)}$ are cross-sections.The set $X_{E(v)}$ with the order $\leq_{E(v)}$ represents the intrinsic time of $E(v)$ whatever is its nature (discrete, continuous, or partially discrete and partially continuous). In the case of a continuous function $t \mapsto s_v(t)$ it reduces to a closed interval. In the case $t'' = t'$ it reduces to the one-element set $\{0\}$ and $ins_{E(v)}(0) = s_v(t'') = s_v(t')$. If the function $t \mapsto s_v(t)$ is not too much pathological then the execution $E(v)$ can be identified with a partial function x such that $x(s) = x(inf(u \geq 0 : var(x; 0, u) \geq s))$ for all s for which $x(s)$ is defined. It suffices to define $x(s)$ as $s_v(inf(u \geq 0 : var(t \mapsto s_v(t); t', u) \geq s)$.

Pouring of an amount m of liquid from a tank $a \in V'$ to a tank $b \in V'$ can be regarded as a concrete execution $S = (X_S, \leq_S, ins_S)$ in $\mathbf{U}'$, where
$X_S = \{x_1, x_2, x_3, x_4\}$,
$x_1 <_S x_3, \quad x_1 <_S x_4, \quad x_2 <_S x_3, \quad x_2 <_S x_4$,
$ins_S(x_1) = (a, q)$, $ins_S(x_2) = (b, r)$,
$ins_S(x_3) = (a, q - m)$, $ins_S(x_4) = (b, r + m)$.

The subsets $\{x_1, x_2\}$ and $\{x_3, x_4\}$ of X_S are cross-sections. The set X_S with the order $\leq_S$ represents the intrinsic time of S.

What is going on in the tanks a and b if there is no pouring of liquid from a to b or from b to a can be regarded as a concrete execution $T = (X_T, \leq_T, ins_T)$ in $\mathbf{U}'$, where
$X_T = X_Q \cup X_R$,
$\leq_T = \leq_Q \cup \leq_R$,
$ins_T = ins_Q \cup ins_R$,
for a variant Q of $E(a)$ and a variant R of $E(b)$ such that $X_Q \cap X_R = \emptyset$.

Two-element subsets $\{x, y\}$ of X_T such that $x \in X_Q$ and $y \in X_R$ are cross-sections. The set X_T with the order $\leq_T$ represents the intrinsic time of T.

The isomorphism classes $[Q]$, $[R]$, $[S]$, $[T]$ of lposets representing the concrete executions Q, R, S, T are illustrated graphically in Figure 2.1.

A full run of the system consisting of the tanks a and b can be regarded as a concrete execution $P = (X_P, \leq_P, ins_P)$ in $\mathbf{U}'$ that consists of a countable sequence $T_1, S_1, T_2, S_2, ...$ of segments such that $T_1, T_2, ...$ are isomorphic to some variants of T, $S_1, S_2, ...$ are isomorphic to some variants of S, and, for every $i \in \{1, 2, ...\}$ and for every $x \in X_P$, x is a maximal element of X_{T_i} iff it is a minimal element of X_{S_i}, and x is a maximal element of X_{S_i} iff it is a minimal element of $X_{T_{i+1}}$. Such an execution is the inductive limit of the chain of its bounded initial segments P_i, where P_i consists of $T_1, S_1, ..., T, S_i$.

The isomorphism class $[P]$ of the lposet P is illustrated in Figure 2.2.
$\sharp$

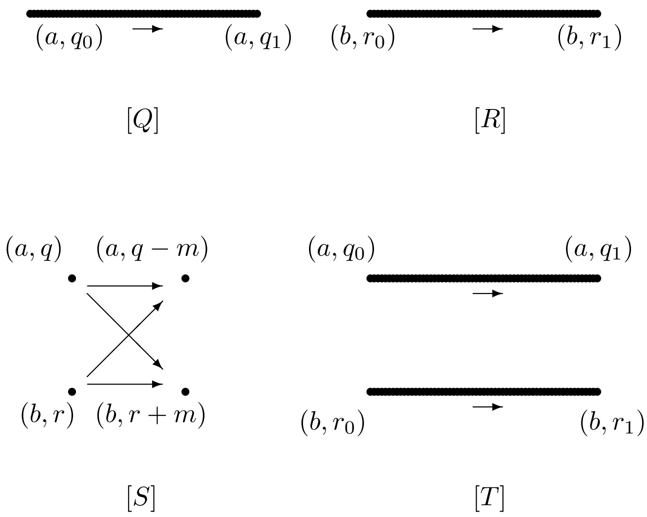

Figure 2.1: $[Q]$, $[R]$, $[S]$, $[T]$

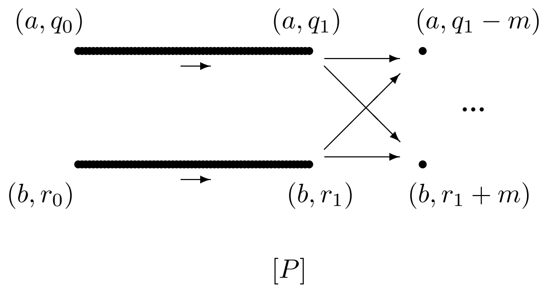

Figure 2.2: $[P]$

Let $\mathbf{U} = (W, V, ob)$ be a universe of objects.

Let $E = (X, \leq, ins)$ be a concrete process in $\mathbf{U}$.

Every cross-section of $(X, \leq)$ contains an occurrence of an instance of each object v with nonempty $X|v$, and it is called a *cross-section* of E. By $csections(E)$ we denote the set of cross-sections of E. This set is partially ordered by the relation $\preceq$ defined in Appendix A and, according to Proposition A.4, for every two cross-sections Z' and Z'' from $csections(E)$ there exist in $csections(E)$ the greatest lower bound $Z' \wedge Z''$ and the least upper bound $Z' \vee Z''$ of Z' and Z'' with respect to $\preceq$. It follows from (1) and (2) in Definition 2.3 that the set of objects with instances occurring in a cross-section is the same for all cross-sections of E. We call it the *range* of E and write as $objects(E)$. The set of elements of E that are minimal with respect to $\leq$ is a cross-section of E. We call it the *origin* of E and write as $origin(E)$. If the set of elements of E that are are maximal with respect to $\leq$ is also a cross-section then we call it the *end* of E and write as $end(E)$, and we say that E is *bounded*.

The following propositions are direct consequences of definition.

2.5. Proposition. Every segment of E is a concrete execution. $\sharp$

2.6. Proposition. For each cross-section c of E, the restrictions of E to the subsets $X^-(c) = \{x \in X : x \leq z \text{ for some } z \in c\}$ and $X^+(c) = \{x \in X : z \leq x \text{ for some } z \in c\}$ are concrete executions, called respectively the *head* and the *tail* of E with respect to c, and written respectively as $head(E, c)$ and $tail(E, c)$. $\sharp$

For example, for the concrete execution T in Example 2.4 and its cross-section c that consists of the maximal element x of X_Q and the minimal element y of X_R, $head(T, c)$ is the restriction of T to $X_Q \times \{y\}$, and $tail(T, c)$ is the restriction of T to $\{x\} \times X_R$.

2.7. Proposition. For every decomposition $s = (X^F, X^S)$ of the underlying set X of E into two disjoint subsets X^F and X^S such that $x' \leq x''$ only if x' and x'' are both in one of these subset, called a *splitting* of E, the restrictions of E to the subsets X^F and X^S are concrete executions, called respectively the *first part* and the *second part* of E with respect to s, and written respectively as $first(E, s)$ and $second(E, s)$. Each concrete execution E' such that $E' = first(E, s)$ or $E' = second(E, s)$ for some s is called an *independent component* of E. $\sharp$

For example, for the concrete execution T in Example 2.4 and its splitting s that consists of X_Q and X_R, $\mathit{first}(T, s)$ is the restriction of T to X_Q and $\mathit{first}(T, s) = Q$, and $\mathit{second}(T, s)$ is the restriction of T to X_R and $\mathit{second}(T, s) = R$. Moreover, Q and R are independent components of T.

The following proposition reflects an important property of concrete executions.

2.8. Proposition. For every cross-section c of E, every isomorphism between bounded initial segments of $\mathit{tail}(E, c)$ (resp.: between bounded final segments of $\mathit{head}(E, c)$) is an identity. $\sharp$

Proof. Let Q be the restriction of E to $X^+(c)$ and let R and S be two initial segments of Q. Suppose that $f : R \to S$ is an isomorphism that it is not an identity. Then there exists an initial subsegment T of R such that the image of T under f, say T', is different from T. By (3) of definition 2.3 neither T' is a subsegment of T nor T is a subsegment of T'. Define T'' to be the least segment containing both T and T', and consider $f' : T \to T''$, where $f'(x) = f(x)$ for $x \leq f(x)$ and $f'(x) = x$ for $f(x) < x$. In order to derive a contradiction, and thus to prove that f is an identity, it suffices to verify, that f' is an isomorphism. It can be done as follows.

For injectivity suppose that $f'(x) = f'(y)$. If $x \leq f(x)$ and $y \leq f(y)$ then $f(x) = f'(x) = f'(y) = f(y)$ and thus $x = y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) = f'(y) = y$. The case $x \leq f(x)$ and $f(y) < y$ is excluded by $f'(x) = f'(y)$ since $x \leq f(x) = f'(x) = f'(y) = y$ and, on the other hand, $f(y) < y = f(x)$ implies $y < x$. Similarly, the case $f(x) < x$ and $y \leq f(y)$ is excluded. Consequently, f' is injective.

For surjectivity suppose that y is in T''. If $y \leq f(y)$ then, by surjectivity of f and condition (1) of Definition 2.3, there exists $t \leq y$ such that $y = f(t)$ and thus $y = f(t) = f'(t)$ since $t \leq y = f(t)$. If $f(y) < y$ then $y = f'(y)$. Consequently, f' is surjective.

For monotonicity suppose that $x \leq y$. If $x \leq f(x)$ and $y \leq f(y)$ then $f'(x) = f(x) \leq f(y) = f'(y)$. If $f(x) < x$ and $f(y) < y$ then $f'(x) = x \leq y = f'(y)$. If $x \leq f(x)$ and $f(y) < y$ then $f'(x) = f(x) \leq f(y) < y = f'(y)$. If $f(x) < x$ and $y \leq f(y)$ then $f'(x) = x \leq y \leq f(y) = f'(y)$. Consequently, f' is monotonic.

For monotonicity of the inverse suppose that $f'(x) < f'(y)$. If $x \leq f(x)$ and $y \leq f(y)$ then $f(x) = f'(x) < f'(y) = f(y)$ and thus $x < y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) < f'(y) = y$. If $x \leq f(x)$ and $f(y) < y$

then $x \leq f(x) = f'(x) < f'(y) = y$. If $f(x) < x$ and $y \leq f(y)$ then $f(x) < x = f'(x) < f'(y) = f(y)$ and thus $x < y$. Consequently, the inverse of f' is monotonic.

A proof for final subsegments of E restricted $X^{-}(c)$ is similar. $\sharp$

2.9. Corollary. For every bounded segment Q of E, every automorphism of Q is an identity. $\sharp$

2.10. Corollary. For every bounded concrete execution E' there exists at most one isomorphism from E' to an initial segment of E. $\sharp$

2.11. Corollary. If E is bounded then for every bounded concrete execution E' there may be at most one isomorphism from E to E'. $\sharp$

Concrete executions with the same instances of objects and the same transformations of these instances are isomorphic lposets. Consequently, they are members of the same isomorphism class of lposets, that is members of the same pomset. This is reflected in the following definition.

2.12. Definition. An *abstract execution* in $\mathbf{U}$ is an isomorphism class ξ of concrete executions. Each member E of such a class ξ is called an instance of this class and ξ is written as $[E]$. $\sharp$

Collecting concrete executions into isomorphism classes, i.e. making abstract executions, is convenient because it allows one to define some natural operations on the latter (see section 5).

2.13. Example. The isomorphism classes $[Q]$, $[R]$, $[S]$, $[T]$ in Figure 2.1 of lposets corresponding to the concrete executions Q, R, S, T in Example 2.4 are abstract executions. $\sharp$

For every concrete execution E' such that E and E' are isomorphic we have $\mathit{objects}(E') = \mathit{objects}(E)$. Consequently, for the abstract execution $[E]$ that corresponds to a concrete execution E we define $\mathit{objects}([E]) = \mathit{objects}(E)$.

We say that an abstract execution is *bounded* if the instances of this execution are bounded.

By $\mathit{EXE}(\mathbf{U})$ we denote the set of executions in $\mathbf{U}$.

In the set $EXE(\mathbf{U})$ there exists the execution with the empty underlying set of its instance, called the *empty execution*, and written as 0. For each execution α from $EXE(\mathbf{U})$ with an instance $E \in \alpha$ and its cross-section $origin(E)$ there exists the unique execution $[origin(E)]$, called the *initial state* or the *source* or the *domain* of α and written as $dom(\alpha)$. For each bounded execution α from $EXE(\mathbf{U})$ with an instance $E \in \alpha$ and its cross-section $end(E)$ there exists the unique execution $[end(E)]$, called the *final state* or the *target* or the *codomain* of α and written as $cod(\alpha)$.

3 Execution structures

Actions in a universe of objects can be represented by tree-like structures called concrete execution structures. Branches of such structures represent concrete executions of represented actions.

Let $\mathbf{U} = (W, V, ob)$ be a universe of objects.

3.1. Definition. A *concrete execution structure* in a universe $\mathbf{U}$ is a labelled partially ordered set (lposet) $L = (X, \leq, ins)$, where X is a set (of *occurrences* of (instances of) objects), $ins : X \to W$ is a mapping (a *labelling* that assigns the respective object instance to each occurrence of this object instance), and $\leq$ is a partial order (the *causal dependency relation* of L), such that

(1) every maximal downward closed subset of X without incomparable occurrences of instances of an object is a concrete execution in $\mathbf{U}$, called a *full execution* of L (and of action represented by L),

(2) if a part d of a cross-section c of a full execution E of L and a part d' of a cross-section c' of a full execution E' of L are isomorphic then for every execution F with the origin d such that F is an independent component of an initial segment of $tail(E, c)$ there exists an isomorphic execution F' with the origin d' such that F' is an independent component of an initial segment of $tail(E', c')$, and vice-versa: for every execution F' with the origin d' such that F' is an independent component of an initial segment of $tail(E', c')$ there exists an isomorphic execution F with the origin d such that F is an independent component of an initial segment of $tail(E, c)$,

(3) every execution E in $\mathbf{U}$ such that every initial segment of E is isomorphic to an initial segment of a full execution of L is an initial segment of a full execution of L.

Full executions of L, their segments, and independent components of their segments are called executions of L (and of action represented by L). Full executions of L, their initial segments, and independent components of their initial segments are said to be *initial*. $\sharp$

Condition (2) means that the future of each initial execution depends only on presence of suitable object instances. Condition (3) characterizes those executions in $\mathbf{U}$ which are initial executions of L.

Concrete execution structures are similar to labelled trees used in [M 80] to represent behaviours of communicating systems. They more sophisticated because they reflect explicitly the concurrency existing in actions.

3.2. Example. The behaviour of a tank v as in Example 2.4 can be represented by the concrete execution structure $L(v) = (X_{L(v)}, \leq_{L(v)}, ins_{L(v)})$, where
$X_{E(v)}$ is the set of partial real valued functions x such that
$x(s) = x(inf(u \geq 0 : var(x; 0, u) \geq s))$
for all s for which $x(s)$ is defined (see Example 2.4),
$x \leq_{E(v)} y$ iff x is the rerstriction of y to an initial segment of its domain,
$ins_{E(v)}(x) = x(t)$ for x that is defined on a subset of $[0, t]$ that contains t.
The restrictions of $L(v)$ to maximal chains are full executions of $L(v)$. Each such a full execution is isomorphic to a concrete execution as $E(v)$ in Example 2.4 for the interval $[0, +\infty)$. $\sharp$

3.3. Example. Let K be the set of regular full runs $k = (X_k, \leq_k, ins_k)$ of the system consisting of tanks a and b as described in Example 2.4. Let $X_K = \bigcup(X_k : k \in K)$. For $k \in K$ and $x \in X_k$ define the history of x in k to be the restriction of k to the subset $\{y \in X_k : y \leq_k x\}$ (cf. [E 91]). Consider the lposet $L = (X, \leq, ins)$, where X is the set of pairs (x, h) such that $x \in X_K$ and h is a history of x in some $k \in K$ such that $x \in X_k$, $\leq$ is the partial order in X, where $(x, h) \leq (x', h')$ iff there exists $k \in K$ such that h' is the history of x' in k and $x \leq_k x'$ and h is a history of x in k, and $ins((x, h)) = ins_k(x)$ for every $(x, h) \in X$ such that h is a history of x in k. Then L is a concrete execution structure in $\mathbf{U}'$ and an execution E in $\mathbf{U}'$ is a full execution of L iff it is isomorphic to some $k \in K$. $\sharp$

From Definition 3.1 and from the properties of cross-sections of lposets described in Appendix A we obtain the following properties of concrete execution structures.

3.4. Proposition. The relation *prefix*, where E' *prefix* E'' iff E' is an independent component of an initial segment of E'', is a partial order on the set of initial executions of L, called the *prefix order*. If initial executions E' and E'' of L are such that E' *prefix* E'' then we say that E' is a *prefix* of E''. $\sharp$

3.5. Proposition. The set of initial executions of a concrete execution structure L in $\mathbf{U}$ with the prefix order is a directed complete partially ordered set (a DCPO as defined in Appendix B), written as $\mathit{initial}(L)$. $\sharp$

3.6. Proposition. Let $(E_1, E_2) \mapsto E_1; E_2$ be the partial operation defined for concrete executions E_1 and E_2 of L such that $\mathit{end}(E_1) = \mathit{origin}(E_2)$, where $E_1; E_2$ is defined as the unique execution E of L such that $\mathit{head}(E, c) = E_1$ and $\mathit{tail}(E, c) = E_2$ for a cross-section c of E. Let $(E_1, E_2) \mapsto E_1 + E_2$ be the partial operation defined for concrete executions E_1 and E_2 of L such that the sets $\mathit{objects}(E_1)$ and $\mathit{objects}(E_2)$ are disjoint, where $E_1 + E_2$ is defined as the unique execution E of L such that $\mathit{first}(E, s) = E_1$ and $\mathit{second}(E, s) = E_2$ for a splitting s of E. The set $\mathit{cexe}(L)$ of all concrete executions of L with these operations is a partial algebra $\mathbf{cexe}(L) = (\mathit{cexe}(L), ;, +)$. $\sharp$

4 Equivalence of execution structures

Different execution structures may be equivalent in the sense that they may be regarded as representing the same action. For execution structures we have a notion of history preserving equivalence which is similar to that for event structures (cf. [GG 01]).

4.1. Definition. A *history preserving bisimulation* between concrete execution structures $L = (X, \leq, \mathit{ins})$ and $L' = (X', \leq', \mathit{ins}')$ in a universe $\mathbf{U} = (W, V, \mathit{ob})$ of objects is a set R of triples (A, A', f) consisting of a bounded initial execution A of L, a bounded initial execution A' of L', and an isomorphism $f : A \to A'$, such that

(1) if $(A, A', f) \in R$ and A is a prefix of a bounded initial execution B of L then there exist a bounded initial execution B' of L' such that A' is a prefix of B' and an isomorphism $g : B \to B'$ such that $(B, B', g) \in R$ and g is an extention of f,

(2) if $(A, A', f) \in R$ and A' is a prefix of a bounded initial execution B' of

L' then there exist a bounded initial execution B of L such that A is a prefix of B and an isomorphism $g : B \to B'$ such that $(B, B', g) \in R$ and g is an extention of f.

If such a bisimulation exists then we say that L and L' are *history preserving equivalent* and write $L \approx L'$. $\sharp$

4.2. Example. Let L' be the restriction of the concrete execution structure $L(v) = (X_{L(v)}, \leq_{L(v)}, ins_{L(v)})$ from Example 3.2 to a subset $z \uparrow = \{x \in X_{L(v)} : z \leq_{L(v)} x\}$ with the obvious isomorphism $i : L(v) \to L'$. The set R of triples (A, A', f), where A is a bounded initial execution of $L(v)$, A' is the image of A under i, and f is the unique isomorphism from A to A', is a history preserving bisimulation. Consequently, $L(v)$ and L' are equivalent with respect to R, $L(v) \approx L'$. $\sharp$

4.3. Example. Let $L' = (X', \leq', ins')$ be the coproduct of two copies of an execution structure $L = (X, \leq, ins)$ with $X' = X \times \{1\} \cup X \times \{2\}$. The set R of triples (A, A', f), where A is a bounded initial execution of L and A' is the image of A under the isomorphism $f_1 : x \mapsto (x, 1)$ and $f = f_1$, or A' is the image of A under the isomorphism $f_2 : x \mapsto (x, 2)$ and $f = f_2$, is a history preserving bisimulation between L and L'. So, $L \approx L'$. $\sharp$

The history preserving equivalence of execution structures may be highly complex since execution structures may branch without restrictions and unfold in a continuous way. Nevertheless, it is still very interesting since it leads to a very simple representation of equivalence classes and, consequently, to a very simple model of action. More precisely, its equivalence classes can be regarded as some sets of abstract executions, and identity of such classes can be regarded as the identity of sets. This follows from the following theorem.

4.4. Theorem. Two concrete execution structures L and L' in $\mathbf{U}$ are history preserving equivalent if an only if they have the same set of bounded abstract initial executions. $\sharp$

For a proof it suffices to take into account Corollary 2.10 and Corollary 2.11 and consider the set R of triples (A, A', f) consisting of a bounded initial execution A of L, a bounded initial execution A' of L', and the unique isomorphism $f : A \to A'$, such that A and A' are isomorphic and f is the unique isomorphism from A to A'.

4.5. Example. In order to see that the execution structures $L(v)$ and L' in Example 4.2 are equivalent with respect to the history preserving equivalence it suffices to notice that they have the same bounded abstract executions. $\sharp$

5 Operations on abstract executions

The fact that concrete execution structures in a universe of objects are history preserving equivalent if an only if they have the same set of bounded abstract initial executions implies that each equivalence class of concrete execution structures is determined uniquely by the set of bounded abstract initial executions of its members. We shall exploit this observation and show that every such a set can be characterized as a specific subset of a partial algebra of abstract executions. In order to define the respective partial algebras we define natural partial operations on abstract executions in a universe of objects. Then we show that these operations can be used to define in every such an algebra a partial order.

In what follows, the word "execution" means "abstract execution".

Let $\mathbf{U} = (W, V, ob)$ be a universe of objects.

In the set $EXE(\mathbf{U})$ of executions in $\mathbf{U}$ there are two partial operations: a parallel composition and a sequential composition.

5.1. Definition. An execution α is said to *consist* of an execution α_1 *followed* by an execution α_2 iff an instance L of α has a cross-section c such that $head(L, c)$ is an instance of α_1 and $tail(L, c)$ is an instance of α_2. $\sharp$

5.2. Proposition. For every two executions α_1 and α_2 such that $cod(\alpha_1)$ is defined and $cod(\alpha_1) = dom(\alpha_2)$ there exists a unique execution, written as $\alpha_1;\alpha_2$, or as $\alpha_1\alpha_2$, that consists of α_1 followed by α_2. $\sharp$

Proof. Take $E_1 = (X_1, \leq_1, ins_1) \in \alpha_1$ and $E_2 = (X_2, \leq_2, ins_2) \in \alpha_2$ with $X_1 \cap X_2 = end(E_1) = origin(E_2)$ and with the restriction of E_1 to $end(E_1)$ identical with the restriction of E_2 to $origin(E_2)$, and provide $X = X_1 \cup X_2$ with the least common extension of the causal dependency relations and labellings of E_1 and E_2.

Let E be the lposet thus obtained. It suffices to prove that E is an execution and notice that $head(E, c) = E_1$ and $tail(E, c) = E_2$. In order to prove that E is an execution it suffices to show that E does not contain a segment with isomorphic proper subsegment. To this end suppose the contrary. Suppose that $f : Q \to R$ is an isomorphism from a segment Q of E to a proper subsegment R of Q, where Q consists of a part Q_1 contained in E_1 and a part Q_2 contained in E_2. By applying twice the method described in the proof of Proposition 2.8 we can modify f to an isomorphism $f' : Q \to R$ such that the image of Q_1 under f', say R_1, is contained in Q_1, and the image of Q_2 under f', say R_2, is contained in Q_2. As R is a proper subsegment of Q, one of these images, say R_1, is a proper part of the respective Q_i. By taking the greatest lower bounds and the least upper bounds of appropriate cross-sections we can extend Q_1 and R_1 to segments Q'_1 and R'_1 of P_1 such that R'_1 is a proper subsegment of Q'_1 and there exists an isomorphism from Q'_1 to R'_1. This is in a contradiction with the fact that E_1 is an execution. Consequently, E is an execution. $\sharp$

5.3. Definition. The operation $(\alpha_1, \alpha_2) \mapsto \alpha_1\alpha_2$ is called the *sequential composition* of executions. $\sharp$

Each execution which is a source or a target of an execution is an identity, i.e. an execution ι such that $\iota\phi = \phi$ whenever $\iota\phi$ is defined and $\psi\iota = \psi$ whenever $\psi\iota$ is defined. Moreover, $dom(\alpha)$ is the unique identity ι such that $\iota\alpha$ is defined, and if $cod(\alpha)$ is defined then it is the unique identity κ such that $\alpha\kappa$ is defined. Consequently, $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable partial operations on executions.

Identities are bounded executions with causal dependency relations reducing to identity relations. They are called *states*, or *identities*, and we can identify them with the sets of occurring instances of objects.

5.4. Definition. An execution α is said to *consist* of two *parallel* executions α_1 and α_2 iff an instance E of α has a splitting s such that $first(E, s)$ is an instance of α_1 and $second(E, s)$ is an instance of α_2. $\sharp$

5.5. Proposition. For every two executions α_1 and α_2 such that $objects(\alpha_1) \cap objects(\alpha_2) = \emptyset$ there exists an execution α with an instance E that has a splitting s such that $first(E, s)$ is an instance of α_1 and $second(E, s)$ is an instance of α_2. If such an execution α exists then it is unique, we write it as $\alpha_1 + \alpha_2$, and we say that the executions α_1 and α_2 are *parallel*. $\sharp$

For a proof it suffices to take $E_1 = (X_1, \leq_1, ins_1) \in \alpha_1$ and $E_2 = (X_2, \leq_2, ins_2) \in \alpha_2$ with $X_1 \cap X_2 = \emptyset$, and to provide $X_1 \cup X_2$ with the least common extension of the causal dependency relations and labellings of E_1 and E_2.

5.6. Definition. The operation $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ is called the *parallel composition* of executions. $\sharp$

The operations on executions allow one to represent complex executions in terms of their components.

5.7. Example. In the case of executions in Example 2.4 we can represent $[T]$ as $[Q]+[R]$, and an initial segment $[P_i]$ of $[P]$ with an instance consisting of $T_1, S_1, ..., T_i, S_i$ as $[T_1][S_1]...[T_i][S_i]$. $\sharp$

The operations of composing executions allow one to turn the set $EXE(\mathbf{U})$ into a partial algebra.

5.8. Definition. The partial algebra $\mathbf{EXE}(\mathbf{U}) = (EXE(\mathbf{U}), ;, +)$ is called the *algebra of executions* in $\mathbf{U}$. $\sharp$

The restriction of the algebra of executions to the subset of bounded executions is an arrows-only category (cf. [McL 71]). Other properties of the algebra of executions are described in the Appendix D.

The operations of the algebra of executions can be used to define in this algebra a partial order.

5.9. Proposition. The relation *pref*, where α *pref* β iff $\beta = (\alpha + \gamma)\delta$ for some γ and δ, is a partial order on $EXE(\mathbf{U})$. If α and β are such that α *pref* β then we say that α is a *prefix* of β. $\sharp$

Proof. For transitivity suppose that $\beta = (\alpha+\gamma)\delta$ and $\beta' = (\beta+\gamma')\delta'$. If $E_{\beta'}$ is an instance of β' then there exists c such that $head(E_{\beta'}, c)$ is an instance $E_{\beta+\gamma'}$ of $\beta+\gamma'$ and $head(first(E_{\beta+\gamma'}, s), c_1)$ is an instance E_β ofβ for some s and a part c_1 of c. Moreover, there exists d such that $head(E_\beta, d)$ is an instance $E_{\alpha+\gamma}$ of $\alpha+\gamma$ and $head(first(E_{\alpha+\gamma}, t), d_1)$ is an instance of E_α for some t and a part d_1 og d. Consequently, $head(E_{\beta'}, c')$ is an instance of $\alpha+\gamma+\gamma'$ for c' consisting of d and of the complement of c_1 to c, and $\beta' = (\alpha+\gamma+\gamma')\delta''$ for $\delta'' = tail(E_{\beta'}, c')$. For antisymmetry suppose that

$\beta = (\alpha + \gamma)\delta$ and $\alpha = (\beta + \gamma')\delta'$. As objects with instances occurring in α cannot occur in γ and objects with instances occurring in β cannot occur in γ', there must be $\gamma = \gamma' = 0$. Consequently, $\alpha = \alpha\delta\delta'$ and, by Corollary 2.10, δ and δ' must be identities. $\sharp$

5.10. Proposition. The extention $\sqsubseteq$ of the relation *pref*, where $\alpha \sqsubseteq \beta$ iff every prefix of α is a prefix of β, is a partial order on $EXE(\mathbf{U})$. The poset $(EXE(\mathbf{U}), \sqsubseteq)$ is a DCPO. Every element of $EXE(\mathbf{U})$ is the least upper bound of the directed set of its prefixes. $\sharp$

Proof. Given a directed subset D of the poset $(EXE(\mathbf{U}), \sqsubseteq)$, the pefixes of elements of D form a directed set D'. For every element of D' we choose a concrete instance, and we consider α and $\beta = (\alpha + \gamma)\delta$ such that E is the chosen instance of α, E_1 is the chosen instance of β, E_2 is the chosen instance of $\alpha + \gamma$ and $E_3 = head(E_1, c)$ is an instance of $\alpha + \gamma$. Then there exists a unique isomorphism f from E_2 to E_3 since otherwise there would be another isomorphism g and the correspondence $f(x) \mapsto g(x)$ would be different from identity isomorphism between two initial segments of E_1. On the other hand, f determines a unique isomorphism between E and $first(E_2, s)$ with a splitting s due to the fact that the first part of E_2 is determined uniquely by the set of objects which occur in it. Consequently, we can construct a direct system of instances of elements of D' such that the colimit of this system in the category **LPOSETS** described in Appendix A is an instance of the least upper bound of D' and of D.

The last part of the proposition is a simple consequence of the condition (2) of Definition 2.3. $\sharp$

5.11. Definition. The relation $\sqsubseteq$ on $EXE(\mathbf{U})$ is called the *prefix order*. The least upper bound of a directed subset D of the partially ordered set $(EXE(\mathbf{U}), \sqsubseteq)$ is called the *limit* of D. $\sharp$

Note that the least upper bounds of directed subsets of the poset $(EXE(\mathbf{U}), \sqsubseteq)$ are limits of the corresponding filters in $EXE(\mathbf{U})$ with the Scott topology induced by the partial order $\sqsubseteq$.

6 Reduced execution structures

The fact that concrete execution structures in a universe of objects are history preserving equivalent if an only if they have the same set of bounded abstract initial executions implies that each equivalence class of concrete execution structures is determined uniquely by the set of bounded abstract initial executions of its members. Now we are going to show that each such a set determines a specific partially ordered set (a poset) of abstract executions, a poset with an algebraic structure, called a reduced execution structure, and that every such a poset corresponds to a concrete execution structure. To this end we use algebras of abstract executions and their prefix order and define reduced execution structures as specific subsets of such algebras. The posets thus obtained inherit some algebraic structure from the algebras in which they are defined, and there is a natural concept of a morphism from one such a poset to another.

In what follows, the word "execution" means "abstract execution".

Let $\mathbf{U} = (W, V, ob)$ be a universe of objects.

The definition of a concrete execution structure implies the following property of the set of its abstract executions.

6.1. Proposition. The set of initial abstract executions of a concrete execution structure in $\mathbf{U}$ is a subset B of the partial algebra $\mathbf{EXE}(\mathbf{U})$ of abstract executions in $\mathbf{U}$ such that:

(1) B is downward closed with respect to $\sqsubseteq$,

(2) if α and β are initial segments of abstract executions in $\mathbf{U}$ that are maximal elements of B then $\alpha(\gamma + s) \in B$ iff $\beta(\gamma + t) \in B$ for every γ such that $dom(\gamma) + s = cod(\alpha)$ and $dom(\gamma) + t = cod(\beta)$,

(3) $\bigsqcup D \in B$ for every subset D of B such that $\bigsqcup D$ exists. $\sharp$

Due to Theorem 4.4 it is possible to represent actions by considering only their abstract executions. More precisely, every action considered up to the history preserving equivalence can be represented by a subset of the algebra of abstract executions that can be defined as follows.

6.2. Definition. A *reduced execution structure* in $\mathbf{U}$ is a subset B of the partial algebra $\mathbf{EXE}(\mathbf{U})$ of abstract executions in $\mathbf{U}$ that satisfies the conditions (1), (2), and (3) of Proposition 6.1. The abstract executions in $\mathbf{U}$ that are maximal elements of B are said to be *full* executions of B. By $seg(B)$ we denote the set of segments of full executions of B. By $\mathbf{seg}(B)$ we denote the restriction of the algebra $\mathbf{EXE}(\mathbf{U})$ to the set $seg(B)$. By $exe(B)$ we denote the set of elements of B, their segments, and independent components of their segments. By $\mathbf{exe}(B)$ we denote the restriction of the algebra $\mathbf{EXE}(\mathbf{U})$ to the set $exe(B)$. $\sharp$

Note that according to Proposition 5.10 every reduced execution structure is a DCPO.

6.3. Definition. A *morphism* from a reduced execution strucure B in $\mathbf{U}$ to a reduced execution structure B' in $\mathbf{U}'$ is a homomorphism $h : \mathbf{exe}(B) \to \mathbf{exe}(B')$. $\sharp$

Reduced execution structures play a role are similar to that of languages in the theory of automata and in the theory of Petri nets. However, they consist of pomsets rather than of strings, and their elements are combined with the aid of operations different from concatenation.

By considering arbitrarily chosen instances of abstract executions of a reduced execution structure and by repeating a construction as in Example 3.3 we can convert such a structure into a concrete execution structure. From this observation and from Propositions 3.4 - 3.6 and Proposition 5.10 we obtain the following property.

6.4. Theorem. A subset B the partial algebra $\mathbf{EXE}(\mathbf{U})$ of abstract executions in $\mathbf{U}$ is a reduced execution structure in $\mathbf{U}$ iff it is the image of the set of initial executions of a concrete execution structure L in $\mathbf{U}$ under the correspondence $L \mapsto [L]$. $\sharp$

Algebraic properties of $\mathbf{seg}(B)$ are related to those of $\mathbf{EXE}(\mathbf{U})$ due to the following theorem.

6.5. Theorem. For every reduced execution structure B in $\mathbf{U}$ the restriction $\mathbf{seg}(B)$ of $\mathbf{EXE}(\mathbf{U})$ to $seg(B)$ is a subalgebra of $\mathbf{EXE}(\mathbf{U})$. $\sharp$

Proof. As no segments of full executions can be composed in parallel, it suffices to prove that $\alpha\beta \in seg(B)$ whenever α and β are segments of full executions from B. To this end consider a concrete execution structure L such that B is the image of the set of initial executions of L under the correspondence $L \mapsto [L]$. Consider in L a full execution E such that $head(tail(E, c), d)$ is an instance of β, and a full execution E' such that $tail(head(E', c'), a)$ with c' isomorphic to c is an instance of α. According to condition (2) of Definition 3.1 there exists a full execution E'' with

$head(E'', c') = head(E', c')$ such that $tail(E'', c')$ is isomorphic to $tail(E, c)$. Consequently, there exists d' such that $head(tail(E'', c'), d')$ is an instance of β.On the other hand, $tail(head(E'', c'), a) = tail(head(E', c'), a)$ is an instance of α. Hence there exists d'' such that $tail(head(E'', d''), a)$ is an instance of $\alpha\beta$ and, consequently, $\alpha\beta \in seg(B)$. $\sharp$

This result cannot be extended on the set $\mathbf{exe}(B)$. The algebraic properties of this set are as follows.

6.6. Theorem. For every reduced execution structure B in $\mathbf{U}$ the set $\mathbf{exe}(B)$ is closed under the sequential composition. The resut $\alpha + \beta$ of the parallel composition of $\alpha \in \mathbf{exe}(B)$ and $\beta \in \mathbf{exe}(B)$ belongs to $\mathbf{exe}(B)$ iff $dom(\alpha) + dom(\beta)$ is defined and belongs to $\mathbf{exe}(B)$. $\sharp$

A proof is immediate.

Taking into account Theorem 6.5 and the results of section 5 we obtain the following result.

6.7. Theorem. Every reduced execution structure B in $\mathbf{U}$ is a set of abstract executions which can be obtained by combining abstract executions from the set $exe(B)$ with the aid of compositions and construction of limits. $\sharp$

This theorem has some consequences for applications of the model. Namely, it suggests how to construct an algorithm for symbolic generation of a reduced execution structure from a finite set of given abstract executions in a finite universe of objects for a give set of initial states. Such a reduced execution structure can be generated starting from the given abstract initial states and applyiung to each state which can be reached executions of the considered action in a way similar to that of generating unfoldings of Petri nets (cf. [Esp 94]). In some cases it can be used to investigate states which can be reached.

6.8. Example. Consider a tank a and a tank b as in Example 2.4. By combining the abstract executions corresponding to the possible variants of the concrete execution T with the aid of sequential composition and construction of limits, we obtain a set A_1 of abstract executions in the universe $\mathbf{U}'$. The set B_1 of executions from A_1 and their prefixes is a reduced execution structure in $\mathbf{U}'$. It represents an action that consists of independent actions of the tank a and the tank b. A scheme of B_1 is depicted in Figure 6.1.

By combining the abstract executions corresponding to the possible variants of the concrete executions S and T with the aid of sequential composition such that every two segments corresponding to components of type S are separated by a segment corresponding to a component of type T, and by construction of limits, we obtain a set A_2 of abstract executions in the universe $\mathbf{U}'$. The set B_2 of executions from A_2 and their prefixes is a reduced execution structure in $\mathbf{U}'$. It represents an action that consists of actions of the tank a and the tank b that are mainly independent, but from time to time are interrupted by the joint action of pouring of an amount of liquid from the tank a to the tank b.

Each of the reduced execution structures B_1 and B_2 is a DCPO. $\sharp$

$$
\begin{tikzcd}
 & \vdots & & \iddots \\
 & {[R]} \arrow[r] & \{(a,q)\}+[R] \arrow[r] & {[Q]+[R]} \\
 & \{(b,r)\} \arrow[u] \arrow[r] & \{(a,q),(b,r)\} \arrow[u] \arrow[r] & {[Q]+\{(b,r)\}} \arrow[u] \\
\cdots & 0 \arrow[u] \arrow[r] & \{(a,q)\} \arrow[u] \arrow[r] & {[Q]} \arrow[u] & \cdots \\
 & & \vdots & &
\end{tikzcd}
$$

Figure 6.1: A scheme of B_1

The algebraic properties of reduced execution structures follow from the properties of algebras of executions and from the properties described in Theorem 6.6 of the restrictions of operations of such algebras to reduced execution structures. They can be summarized as follows (cf. Proposition D.7).

6.9. Theorem. Given a reduced execution structure B in a universe $\mathbf{U}$ of objects, $\mathbf{exe}(B) = (exe(B), ;, +)$ is a partial algebra that enjoys the following properties:

(A) The reduct $(exe(B), ;)$ is a partial category with the properties (A1) - (A10).

(B) The reduct $(exe(B), +)$ is a partial commutative monoid with the properties (B1) - (B9).

(C) The reducts $(exe(B), ;)$ and $(exe(B), +)$ are related according to (C1) - (C4), (C5'), (C6), (C7'), and (C8), where

 (C5') If $\alpha_{11}\alpha_{12}$ and $\alpha_{21}\alpha_{22}$ are defined, and $\alpha_{11} + \alpha_{21}$ is defined then $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22})$ is defined.

 (C7') In $(exe(B), +)$ there exists the least congruence $\sim$ such that $\alpha \sim \beta$ for all α and β such that $\alpha = \gamma\beta\delta$ or $\alpha = \gamma\beta$ or $\alpha = \beta\delta$ for some γ and δ, and this congruence is strong in the sense that $\alpha_1 \sim \alpha'_1$ and $\alpha_2 \sim \alpha'_2$ implies that if $\alpha_1 + \alpha_2$ is defined and $\alpha_1 \sqsubseteq \alpha'_1$ and $\alpha_2 \sqsubseteq \alpha'_2$ then $\alpha'_1 + \alpha'_2$ is defined, and if $\alpha'_1 + \alpha'_2$ is defined and $\alpha'_1 \sqsubseteq \alpha_1$ and $\alpha'_2 \sqsubseteq \alpha_2$ then $\alpha_1 + \alpha_2$ is defined. $\sharp$

7 The case of locally complete executions

What has been said about the prefix order in **A** applies to every algebra of executions. Now we are going to describe subalgebras of algebras of executions that consist of executions which are locally complete in a sense and prove that the reduced execution structures contained in such subalgebras are continuous DCPOs. This will allow to simplify the corresponding reduced execution structures with probability measures.

Let $\mathbf{U} = (W, V, ob)$ be a universe of objects.

7.1. Definition. An execution in **U** is said to be *locally complete* if every every bounded segment of this execution is a complete lattice. $\sharp$

The following property of operations of composing executions implies that the subset of locally complete executions of the algebra of executions in **U** is a subalgebra of this algebra.

7.2. Proposition. The result of sequential or parallel composition of locally complete executions is locally complete. $\sharp$

Proof. In the case of parallel composition the proposition is obvious. In order to prove that $\alpha_1\alpha_2$ is locally complete if α_1 and α_2 are locally complete suppose that E is an instance of $\alpha_1\alpha_2$ with a cross-section c such that $head(E, c)$ is an instance of α_1 and $tail(E, c)$ is an instance of α_2. Given a segment Q of E and a subset S of cross-sections of E contained in Q, let c^- be the least upper bound of the set of cross-sections $s \wedge c$ with $s \in S$ and c^+ the least upper bound of cross-sections $s \vee c$ with $s \in S$. Then for every $v \in V$ define x_v as the greater of the two elements of $X|v$ in c^- and in c^+, and define d as the set of all x_v. As c^- and c^+ are cross-sections, d does not contain comparable elements and is an antichain. As all $v \in V$ have in d occurrences, d is a maximal antichain. It is also straightforward to verify that d is a cross-section and the least upper bound of S. In a similar way we can define a cross-section that is the greatest lower bound of S. $\sharp$

Subalgebras of locally complete executions of the algebra of executions in $\mathbf{U}$ enjoy the following property.

7.3. Proposition. If $\mathbf{A} = (A, ;, +)$ is the subalgebra of locally complete executions of the algebra of executions in $\mathbf{U}$ then $(A, \sqsubseteq)$ is a continuous DCPO. $\sharp$

Proof. Suppose that $\alpha \in B$ is a bounded execution with an instance E such that $E = head(E', c)$ for a concrete execution E' with $[E'] \in A$ and for c being the least upper bound of cross-sections c' of E' with the underlying sets of $head(E', c')$ containing occurrences $x_1, ..., x_n$ of instances of objects $v_1, ..., v_n$ from a finite subset of V. Then α is a compact element of A. Indeed, suppose that $\alpha \sqsubseteq \bigsqcup S$ for a directed subset S of A. Then all $s \in S$ and $\bigsqcup S$ have instances E_s and E_S that are initial segments of E' such that the underlying set of E_S is the union of the underlying sets of all E_s and it contains the underlying set of E. Consequently, for every $i \in \{1, ..., n\}$ there must be $s_i \in S$ such that the underlying set of E_{s_i} contains x_i. Consequently, $x_1, ..., x_n$ belong to the underlying set of E_s for an upper bound s of $s_1, ..., s_n$ that belongs to S. Consequently, c must be a cross-section of E_s and $\alpha \sqsubseteq s \in S$, as required.

In order to prove that A with the prefix order is algebraic domain, consider any $\alpha \in A$ and its instance E. As every execution is an inductive limit of a direct system of its bounded segments, it suffices to consider the case when α is bounded. Then for every finite set $f = \{x_1, ..., x_n\}$ of occurrences of instances of objects $v_1, ..., v_n$ in the underlying set of E

there exists the least cross-section c_f of E such that $x_1, ..., x_n$ belong to the underlyin set of $head(E, c_f)$. Then $s_f = [head(E, c_f)]$ is a compact element of A. On the other hand, executions s_f form a directed set S and $\alpha = \bigsqcup S$, as required. $\sharp$

The following theorem gives sufficient conditions of local completeness of executions in the algebra of executions in **U**.

7.4. Theorem. A concrete execution $E = (X, \leq, ins)$ in a universe $\mathbf{U} = (W, V, ob)$ of objects is locally complete if the following conditions are satisfied:

(1) For every object v that occurs in E the set $X|v$ of its occurrences in E is a locally complete chain.

(2) The relation of incomparability with respect to the flow order $\leq$ is a closed subset of the product $X \times X$ for X provided with the interval topology, i.e., the weakest topology in which all intervals $\{x \in X : a < x < b\}$ are open sets. $\sharp$

Proof. Let Z_1 and Z_2 be cross-sections of E such that $Z_1 \preceq Z_2$ and let S be the set of cross-sections of E such that $Z_1 \preceq s \preceq Z_2$. Due to (1) for every $v \in V$ that occurs in L there exists the least upper bound x_v of those elements of $X|v$ which belong to some $s \in S$. Due to (2) the set Z of all such elements is an antichain. This set is a maximal antichain of E and it is easy to verify that it is also a cross-section of E. $\sharp$

8 Concluding remarks

We have described a model of action that is based on local action properties and is universal in the sense that it allows to represent in the same way discrete, continuous and hybrid actions. The model is derived from execution structures that are similar to labelled event structures, but are not restricted to discrete actions only, and reflect in a more subtle way the possible action executions and their components. In particular, the structures representing bounded execution components admit only trivial automorphisms and unique isomorphisms. This leads to a simple characterization of history preserving equivalence of structures representing actions and to a simple characterization of its equivalence classes. More precisely, the equivalence classes of history preserving equivalence can be identified with reduced execution structures and there exists a bijective correspondence between equivalence classes of concrete execution structures and reduced execution structures. This allows us to represent actions by reduced execution structures rather than by concrete execution structures.

The representation of actions by reduced execution structures dramatically simplifies their studies.

As reduced execution structures are sets of abstract executions, they are partially ordered by inclusion. Consequently, a simulation of an action by another action can be defined as the usual inclusion of one reduced execution structure in another reduced execution structure. In particular, a bisimulation reduces to identity.

When ordered partially by inclusion the set of reduced execution structures representing actions in a universe of objects is a complete lattice.

Structures representing abstract executions can be combined with the aid of natural partial operations. This leads to partial algebras. In particular, the set of executions of each reduced execution structure can be regarded as a partial algebra of executions in a universe of objects, and a homomorphism from such a partial algebra to another such a partial algebra of executions can be used to represent a refinement of the represented action.

Every set of executions of a reduced execution structure with a partial prefix order is a directed complete partial order (a DCPO). Consequently, it can be provided with the Scott topology and the ideas described in AES 00], [JP 89], [LSV 07], and [VVW 04], can be applied to provide it with a probability measure.

The representation of actions by reduced execution structures leads to simple operations on actions and can be used to develop a calculus of actions playing a role similar to that of CCS (cf. [M 78], [M 80], and [WM 87]).

When ordered partially by inclusion the set of reduced execution structures representing actions in a universe of objects is a complete lattice. Consequently, we can speak of the *greatest lower bound* and the *least upper bound* of a family of reduced execution structures and the represented actions. The greatest lower bound of a nonempty family of actions is the action represented by the intersection of the reduced execution structures representing the members of the family. The least upper bound of a nonempty family of actions is the action represented by the reduced execution structure that consists of the executions of the members of the family and of the executions whose existence follows from the requirements of the definition of reduced execution structures.

These operations can be used to define compound actions as results of combining their component actions.

In order to illustrate this consider tanks as in Example 2.4 and actions represented by reduced execution structures described in Example 6.8.

According to Example 2.4, the behaviour of a tank v can be represented by the reduced execution structure $B(v)$ that consists of the possible abstract executions $[E(v)]$.

According to Example 6.8, the action that consists of independent actions of the tank a and the tank b can be represented by the reduced execution structure B_1. On the other hand, this action can be defined as the least upper bound of actions represented by $B(a)$ and $B(b)$ because B_1 is the least upper bound of $B(a)$ and $B(b)$.

According to Example 6.8, the action that consists of actions of the tank a and the tank b that are mainly independent but from time to time are interrupted by the joint action of pouring an amount of liquid from the tank a to the tank b can be represented by the reduced execution structure B_2. On the other hand, this action can be defined as the least upper bound of actions represented by B_1 and the least reduced execution structure containing the abstract executions correponding to the possible variants of the concrete execution S described in Example 2.4.

So, the proposed model of action is compositional in the sense that it allows to define complex actions as results of applying natural operations to models of simple component actions. In particular, it can be used to formulate finite definitions of actions whose components are infinite but can be described in a finite way in a logic.

The lattice theoretical operations on actions are not the only operations we can consider. In general, operations on actions can be defined like operations on data flows or operations of calculi like CCS. Such operations should be continuous in the sense that they should preserve the least upper bounds of chains of reduced execution structures. Then the list of operations on actions can be extended with the aid of fixed point equations, and a powerful calculus of actions can be developed.

Appendix A: Posets and their cross-sections

Given a partial order $\leq$ on a set X, i.e. a binary relation which is reflexive, anti-symmetric and transitive, we call $P = (X, \leq)$ a *partially ordered set*, or briefly a *poset*, by the *strict partial order* corresponding to $\leq$ we mean

$<$, where $x < y$ iff $x \leq y$ and $x \neq y$, by a *downward closed subset* we mean a subset $Y \subseteq X$ such that $x \leq y \in Y$ implies $x \in Y$, by an *upward closed subset* we mean a subset $Y \subseteq X$ such that $y \leq x$ for some $y \in Y$ implies $x \in Y$, by a *chain* we mean a subset $Y \subseteq X$ such that $x \leq y$ or $y \leq x$ for all $x, y \in Y$, and by an *antichain* we mean a subset $Z \subseteq X$ such that $x < y$ does not hold for any $x, y \in Z$.

A.1. Definition. Given a poset $P = (X, \leq)$, by a *strong cross-section* of P we mean a maximal antichain Z of P that has an element in every maximal chain of P. By a *weak cross-section*, or briefly a *cross-section*, of P we mean a maximal antichain Z of P such that, for every $x, y \in X$ for which $x \leq y$ and $x \leq z'$ and $z'' \leq y$ with some $z', z'' \in Z$, there exists $z \in Z$ such that $x \leq z \leq y$. $\sharp$

A.2. Definition. We say that a partial order $\leq$ on X (and the poset $P = (X, \leq)$) is *strongly K-dense* (resp.: *weakly K-dense*) iff every maximal antichain of P is a strong (resp.: a weak) cross-section of P (cf. [Pe 80] and [Plu 85], where K-density is defined as the strong K-density in our sense). $\sharp$

A.3. Definition. For every cross-section Z of a poset $P = (X, \leq)$, we define $X^-(Z) = \leq Z (= \{x \in X : x \leq z \text{ for some } z \in Z\})$ and $X^+(Z) = Z \leq = (\{x \in X : z \leq x \text{ for some } z \in Z\})$, and we say that a cross-section Z' *precedes* a cross-section Z'' and write $Z' \preceq Z''$ iff $X^-(Z') \subseteq X^-(Z'')$. $\sharp$

A.4. Proposition. The relation $\preceq$ is a partial order on the set of cross-sections of $P = (X, \leq)$. For every two cross-sections Z' and Z'' of P there exist the greatest lower bound $Z' \wedge Z''$ and the least upper bound $Z' \vee Z''$ of Z' and Z'' with respect to $\preceq$, where $Z' \wedge Z''$ is the set of those $z \in Z' \cup Z''$ for which $z \leq z'$ for some $z' \in Z'$ and $z \leq z''$ for some $z'' \in Z''$, and $Z' \vee Z''$ is the set of those $z \in Z' \cup Z''$ for which $z' \leq z$ for some $z' \in Z'$ and $z'' \leq z$ for some $z'' \in Z''$. Moreover, the set of cross-sections of P with the operations thus defined is a distributive lattice. $\sharp$

Proof. The set $Z' \wedge Z''$ is an antichain since otherwise there would be $x < y$ for some x and y in this set. If $x \in Z'$ then there would be $y \in Z''$ and there would exist $z' \in Z'$ such that $y \leq z'$. However, this is impossible since Z' is an antichain. Similarly for $x \in Z''$.

The set $Z' \wedge Z''$ is a maximal antichain since otherwise there would exist x that would be incomparable with all the elements of this set. Consequently, there would not exist $z' \in Z'$ and $z'' \in Z''$ such that $z' \leq x \leq z''$, or $z'' \leq x \leq z'$, or $z', z'' \leq x$, and thus there would be $x \leq z'$ and $x \leq z''$ for some $z' \in Z'$ and $z'' \in Z''$ that are not in $Z' \wedge Z''$. Consequently, there would exist z, say in Z'', such that $x \leq z \leq z'$. Moreover, $z \in Z' \wedge Z''$ since otherwise there would be $t \in Z'$ such that $t \leq z \leq z'$, what is impossible.

In order to see that $Z' \wedge Z''$ is a cross-section we consider $x \leq y$ such that $x \leq t$ and $u \leq y$ for some $t \in Z' \wedge Z''$ and $u \in Z' \wedge Z''$, where $t \in Z'$ and $u \in Z''$. Without a loss of generality we can assume that $y \leq y'$ for some $y' \in Z'$ since otherwise we could replace y by an element of Z'. Consequently, there exists $z \in Z''$ such that $x \leq z \leq y$. On the other hand, $z \in Z' \wedge Z''$ since otherwise there would be $z' \in Z'$ such that $z' \leq z \leq y$, what is impossible. In a similar manner we can find $z \in Z' \wedge Z''$ for the other cases of t and u.

In order to see that $Z' \wedge Z''$ is the greatest lower bound of Z' and Z'' consider a cross-section Y which precedes Z' and Z'' and observe that $y \leq z' \in Z'$ and $y \leq z'' \in Z''$ with z' and z'' not in $Z' \wedge Z''$ and $y \in Y$ implies the existence of $t \in Z'$ such that $y \leq t \leq z'$ or $u \in Z''$ such that $y \leq u \leq z''$.

Similarly, $Z' \vee Z''$ is the least upper bound of Z' and Z''.

The last part of the proposition follows from the fact that the correspondence $Z \mapsto X^-(Z)$ is an isomorphism from the lattice of cross-sections of P to a sublattice of the lattice of subsets of P. $\sharp$

A.5. Definition. For cross-sections Z' and Z'' of a poset $P = (X, \leq)$ such that $Z' \preceq Z''$ we write the restriction of P to the set $[Z', Z''] = X^+(Z') \cap X^-(Z'')$ as $P|[Z', Z'']$ and we call it a *segment* of P from Z' to Z''. We say that such a segment is *bounded*. The restrictions of P to the subsets $X^+(Z)$ and $X^-(Z)$ corresponding to a cross-section Z are also called segments of P even though they need not be bounded. A segment of $P|[Y', Y'']$ such that $Z' \preceq Y' \preceq Y'' \preceq Z''$ is called a *subsegment* of $P|[Z', Z'']$. If $Z' \neq Y'$ or $Y'' \neq Z''$ (resp.: if $Z' = Y'$, or if $Y'' = Z''$) then we call it a *proper* (resp.: an *initial*, or a *final*) subsegment of $P|[Z', Z'']$. In general, a segment of P that is contained in a segment Q of P is called a subsegment of Q. $\sharp$

The following proposition follows easily from definitions.

A.6. Proposition. For every strong or weak cross-section Z of a poset $P =$

$(X, \leq)$ the reflexive and transitive closure of the union of the restrictions of the partial order $\leq$ to $X^-(Z)$ and to $X^+(Z)$ is exactly the partial order $\leq$. $\sharp$

A.7. Proposition. A poset $P = (X, \leq)$ is said to be *locally complete* if every segment $P|[Z', Z'']$ of P is a complete lattice. $\sharp$

A.8. Definition. Given a partial order $\leq$ on a set X and a function $l : X \to W$ that assigns to every $x \in X$ a label $l(x)$ from a set W, we call $L = (X, \leq, l)$ a *labelled partially ordered set*, or briefly an *lposet*, by a *chain* (resp.: an *antichain*, a *cross-section*) of L we mean a chain (resp.: an antichain, a cross-section) of $P = (X, \leq)$, by a *segment* of L we mean each restriction of L to a segment of P, and we say that L is K*-dense* (resp.: *weakly* K*-dense*, *locally complete*) iff $\leq$ is K-dense (resp.: weakly K-dense, locally complete). $\sharp$

By **LPOSETS** we denote the category of lposets and their morphisms, where a *morphism* from an lposet $L = (X, \leq, l)$ to an lposet $L' = (X', \leq', l')$ is defined as a mapping $b : X \to X'$ such that, for all x and y, $x \leq y$ iff $b(x) \leq' b(y)$, and, for all x, $l(x) = l'(b(x))$. In the category **LPOSETS** a morphism from $L = (X, \leq, l)$ to $L' = (X', \leq', l')$ is an *isomorphism* iff it is bijective, and it is an *automorphism* iff it is bijective and $L = L'$. If there exists an isomorphism from an lposet L to an lposet L' then we say that L and $L'$are *isomorphic*. A *partially ordered multiset*, or briefly a *pomset*, is defined as an isomorphism class ξ of lposets. Each lposet that belongs to such a class ξ is called an *instance* of ξ. The pomset corresponding to an lposet L is written as $[L]$.

Appendix B: Directed complete posets

Let $(X, \sqsubseteq)$ be a partially ordered set (poset). A subset $Y \subseteq X$ is said to be *downward closed* (resp. : *upward closed*) if $Y = \sqsubseteq Y (= \{x \in X : x \sqsubseteq y \text{ for some } y \in Y\})$ (resp. : $Y = Y \sqsubseteq (= \{x \in X : y \sqsubseteq x \text{ for some } y \in Y\})$). A nonempty subset $Y \subseteq X$ is said to be em bounded complete if every bounded subset of Y has a least upper bound. A nonempty subset $Y \subseteq X$ is said to be *directed* if for all $x, y \in Y$ there exists $z \in Y$ such that $x, y \sqsubseteq z$. The *Scott topology* of $(X, \sqsubseteq)$ is the topology on X in which a subset $U \subseteq X$ is open iff it is upward closed and disjoint with every directed $Y \subseteq X$ which has the least upper bound $\sqcup Y$.

A poset is said to be *coherent* if every of its consistent subsets has a least upper bound. A poset is said to be *directed complete*, or a *directed complete partial order* (a DCPO), if every of its directed subsets has a least upper bound.

Let $(X, \sqsubseteq)$ be a DCPO. An element $x \in X$ is said to *approximate* an element $y \in X$, or that x is *way below* y, if in every directed set Z such that $y \sqsubseteq \sqcup Z$ there exists z such that $x \sqsubseteq z$. An element $x \in X$ is said to be a *compact* if it approximates itself. A subset $B \subseteq X$ is called a *basis* of $(X, \sqsubseteq)$ if for every $x \in X$ the set of those elements of B which approximate x is directed and has the least upper bound equal to x. The DCPO $(X, \sqsubseteq)$ is said to be *continuous* if it has a basis, and ω-*continuous* if it has a countable basis. The DCPO $(X, \sqsubseteq)$ is said to be an *algebraic domain* if every $y \in X$ is the directed least upper bound of all compact elements x such that $x \sqsubseteq y$ (see [AES 00], [JP 89], and [VVW 04]).

Appendix C: Partial categories

A partial category can be defined in exactly the same way as an arrows-only category, except that sources and targets may be not defined for some arrows that are not identities and then the respective compositions are not defined. Limits and colimits in partial categories can be defined as in usual categories (cf. McL 71]).

Let $\mathbf{A} = (A, ;)$ be a partial algebra with a binary partial operation $(\alpha, \beta) \mapsto \alpha; \beta$, where $\alpha; \beta$ is written also as $\alpha\beta$. An element $\iota \in A$ is called an *identity* if $\iota\phi = \phi$ whenever $\iota\phi$ is defined and $\psi\iota = \psi$ whenever $\psi\iota$ is defined. We call elements of A *arrows* or *morphisms* and say that $\mathbf{A}$ is a *partial category* if the following conditions are satisfied:

(1) For every α, β, and γ in A, if $\alpha\beta$ and $\beta\gamma$ are defined then $\alpha(\beta\gamma)$ and $(\alpha\beta)\gamma$ are defined and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$; if $\alpha(\beta\gamma)$ is defined then $\alpha\beta$ is defined; if $(\alpha\beta)\gamma$ is defined then $\beta\gamma$ is defined.

(2) For every identity $\iota \in A$, $\iota\iota$ is defined.

The conditions (1) and (2) imply the following properties.

(3) For every $\alpha \in A$, there exists at most one identity $\iota \in A$, called the *source* or the *domain* of α and written as $dom(\alpha)$, such that $\iota\alpha$ is defined, and at most one identity $\kappa \in A$, called the *target* or the *codomain* of α and written as $cod(\alpha)$, such that $\alpha\kappa$ is defined.

(4) For every α and β in A, $\alpha\beta$ is defined if and only if $cod(\alpha) = dom(\beta)$. If $\alpha\beta$ is defined then $dom(\alpha\beta) = dom(\alpha)$ and $cod(\alpha\beta) = cod(\beta)$.

For (3) suppose that ι_1 and ι_2 are identities such that $\iota_1\alpha$ and $\iota_2\alpha$ are defined. Then $\iota_2\alpha = \alpha$ and $\iota_1(\iota_2\alpha) = \iota_1\alpha$. Hence, by (1), $\iota_1\iota_2$ is defined and $\iota_1 = \iota_2$. Similarly for identities ι_1 and ι_2 such that $\alpha\iota_1$ and $\alpha\iota_2$ are defined.

For (4) suppose that $cod(\alpha) = dom(\beta) = \iota$. Then $\alpha\iota$ and $\iota\beta$ are defined and, by (1), $(\alpha\iota)\beta = \alpha\beta$ is defined. Conversely, if $\alpha\beta$ is defined then taking $\iota = cod(\alpha)$ we obtain that $\alpha\iota$ is defined and, consequently, $\alpha\beta = (\alpha\iota)\beta = \alpha(\iota\beta)$; the existence of $\iota\beta$ implies $dom(\beta) = \iota$. In a similar way we obtain $dom(\alpha\beta) = dom(\alpha)$ and $cod(\alpha\beta) = cod(\beta)$.

As usual, a morphism α with the source $dom(\alpha) = s$ and the target $cod(\alpha)$ is represented in the form $s \stackrel{\alpha}{\to} t$.

Note that $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable partial operations assigning to a morphism α respectively the source and the target of this morphism, if such a source or a target exists.

Dealing with arrows-only categories rather than with categories in the usual sense is sometimes more convenient since it allows us to avoid two sorted structures and more complicated denotations.

Given a partial category $\mathbf{A} = (A, ;)$, let A' be the set of quadruples $(\alpha, \sigma, \tau, \beta)$ where $\sigma\alpha\tau$ is defined and $\sigma\alpha\tau = \beta$, or $dom(\alpha)$ and σ are not defined and $\alpha\tau$ is defined and $\alpha\tau = \beta$, or $cod(\alpha)$ and τ are not defined and $\sigma\alpha$ is defined and $\sigma\alpha = \beta$, or $dom(\alpha)$ and $cod(\alpha)$ are not defined and $\alpha = \beta$. The set A' thus defined and the partial operation

$((\alpha, \sigma, \tau, \beta), (\beta, \sigma', \tau', \gamma)) \mapsto (\alpha, \sigma'\sigma, \tau\tau', \gamma)$

form a category $occ(\mathbf{A})$, called the *category of occurrences of morphisms in morphisms* in $\mathbf{A}$.

Given a partial category $\mathbf{A} = (A, ;)$ and its morphism α, let A'_α be the set of triples (ξ_1, δ, ξ_2) such that $\xi_1\delta\xi_2 = \alpha$.
The set A'_α thus defined and the partial operation

$((\eta_1, \delta, \varepsilon\eta_2), (\eta_1\delta, \varepsilon, \eta_2)) \mapsto (\eta_1, \delta\varepsilon, \eta_2)$

form a category dec_α, called the *category of decompositions* of α. In this category each triple (ξ_1, δ, ξ_2) in which δ is an identity, and thus $\delta = cod(\xi_1) = dom(\xi_2)$, is essentially a decomposition of α into a pair (ξ_1, ξ_2) such that $\xi_1\xi_2 = \alpha$ and it can be identified with this decomposition.

Given partial categories $\mathbf{A} = (A, ;)$ and $\mathbf{A}' = (A', ;')$, a mapping $f : A \to A'$ such that $f(\alpha);' f(\beta)$ is defined and $f(\alpha);' f(\beta) = f(\alpha\beta)$ for every α and β such that $\alpha\beta$ is defined, and $f(\iota)$ is an identity for every identity ι, is called a *morphism* or a *functor* from $\mathbf{A}$ to $\mathbf{A}'$. Note that such a morphism becomes a functor in the usual sense if $\mathbf{A}$ and $\mathbf{A}'$ are categories.

Diagrams, limits and colimits in partial categories can be defined as in usual categories.

A *direct system* is a diagram $(a_i \stackrel{\alpha_{ij}}{\to} a_j : i \leq j, i, j \in I)$, where $(I, \leq)$ is a directed poset, α_{ii} is identity for every $i \in I$, and $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ for all $i \leq j \leq k$. The *inductive limit* of such a system is its colimit, i.e. a family $(a_i \stackrel{\alpha_i}{\to} a : i, j \in I)$ such that $\alpha_i = \alpha_{ij}\alpha_j$ for all $i \in I$ and for every family $(a_i \stackrel{\beta_i}{\to} b : i, j \in I)$ such that $\beta_i = \alpha_{ij}\beta_j$ for all $i \in I$ there exists a unique $a \stackrel{\beta}{\to} b$ such that $\beta_i = \alpha_i\beta$ for all $i \in I$.

A *projective system* is a diagram $(a_i \stackrel{\alpha_{ij}}{\leftarrow} a_j : i \leq j, i, j \in I)$, where $(I, \leq)$ is a directed poset, α_{ii} is identity for every $i \in I$, and $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ for all $i \leq j \leq k$. The *projective limit* of such a system is its limit, i.e. a family $(a_i \stackrel{\alpha_i}{\leftarrow} a : i, j \in I)$ such that $\alpha_i = \alpha_j\alpha_{ij}$ for all $i \in I$ and for every family $(a_i \stackrel{\beta_i}{\leftarrow} b : i, j \in I)$ such that $\beta_i = \beta_j\alpha_{ij}$ for all $i \in I$ there exists a unique $a \stackrel{\beta}{\leftarrow} b$ such that $\beta_i = \beta\alpha_i$ for all $i \in I$.

A *bicartesian square* is a diagram $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w, v \stackrel{\alpha_2'}{\to} u' \stackrel{\alpha_1'}{\leftarrow} w)$ such that $v \stackrel{\alpha_2'}{\to} u' \stackrel{\alpha_1'}{\leftarrow} w$ is a pushout of $v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w$ and $v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w$ is a pullback of $v \stackrel{\alpha_2'}{\to} u' \stackrel{\alpha_1'}{\leftarrow} w$, i.e. such that for every $v \stackrel{\beta_1}{\to} u'' \stackrel{\beta_2}{\leftarrow} w$ such that $\alpha_1\beta_1 = \alpha_2\beta_2$ there exists a unique $u' \stackrel{\beta}{\to} u''$ such that $\beta_1 = \alpha_2'\beta$ and $\beta_2 = \alpha_1'\beta$, and for every $v \stackrel{\gamma_1}{\leftarrow} t \stackrel{\gamma_2}{\to} w$ such that $\gamma_1\alpha_2' = \gamma_2\alpha_1'$ there exists a unique $u \stackrel{\gamma}{\leftarrow} t$ such that $\gamma_1 = \gamma\alpha_1$ and $\gamma_2 = \gamma\alpha_2$.

The concept of a bicartesian square can be generalized to the concept of a bicartesian n-cube. This can be done as follows.

Given a partial graph G, by a *n-cube* in G we mean a subgraph G' of G whose nodes correspond to sequences $(a_1, ..., a_n)$ of binary coordinates $a_i =$ 0 or 1, and whose arrows lead from one node to another whenever one of the coordinates of the latter is obtained from the corresponding coordinate of the former by replacing 0 by 1. The arrow with all coordinates 0 and the arrows leading from this node to other nodes are termed *initial*. The node with all coordinates 1 and the arrows leading to this node from other nodes are termed *final*. Subgraphs of G' whose all nodes have some of the coordinates identical are m-cubes for the respective $m \leq n$, called *m-faces* of G'.

As partial categories are also partial graphs, all these notions apply to partial categories as well. In particular, one can define a *bicartesian n-cube* in a partial category C as an n-cube C' in $\mathbf{A}$ that commutes and is such that, for each face C'' of C', the family of initial arrows of C'' extends to a unique limiting cone for the remaining part of C'', and the family of final arrows of C'' extends to a unique colimiting cone for the remaining part of

C''. For example, each bicartesian square is a bicartesian 2-cube.

Appendix D: Properties of algebras of executions

Algebras of executions in a universe $\mathbf{U} = (W, V, ob)$ of objects enjoy a number of specific properties.

Taking into account the definitions of operations of composing executions we obtain the following proposition.

D.1. Proposition. $\mathbf{EXE}(\mathbf{U}) = (EXE(\mathbf{U}), ;, +)$ is a partial algebra that enjoys the following properties:

(1) The reduct $(EXE(\mathbf{U}), ;)$ is a partial category $\mathbf{pcatEXE}(\mathbf{U})$. For every $\alpha \in EXE(\mathbf{U})$, $dom(\alpha)$ is the source of α in this partial category, and if $cod(\alpha)$ is defined then it is the target of α in this partial category.

(2) The reduct $(EXE(\mathbf{U}), +)$ is a partial commutative monoid $\mathbf{pmonEXE}(\mathbf{U})$ with the empty execution 0 such that $\alpha + 0 = \alpha$ for every α.

For every concrete execution structure in $\mathbf{U}$, the correspondence $E \mapsto [E]$ between concrete executions of L and their isomorphism classes is a homomorphism from $\mathbf{c} - \mathbf{exe}(L)$ to $\mathbf{EXE}(\mathbf{U})$.
The restriction of $\mathbf{EXE}(\mathbf{U})$ to the subset $lcEXE(\mathbf{U})$ of locally complete executions in $\mathbf{U}$ is a subalgebra $\mathbf{lcEXE}(\mathbf{U}) = (lcEXE(\mathbf{U}), ;, +)$ that enjoys the following properties:

(3) The reduct $(lcEXE(\mathbf{U}), ;)$ is a partial category $\mathbf{pcatlcEXE}(\mathbf{U})$ that is a subalgebra of $\mathbf{pcatEXE}(\mathbf{U})$.

(4) The reduct $(EXE(\mathbf{U}), +)$ is a partial commutative monoid $\mathbf{pmonlcEXE}(\mathbf{U})$ that is a subalgebra of $\mathbf{pmonEXE}(\mathbf{U})$. $\sharp$

Due to the properties of concrete executions described in section 2 the partial category $\mathbf{pcatEXE}(\mathbf{U})$ enjoys properties which allow us to define in $EXE(\mathbf{U})$ a partial order.

An important property of the partial category $\mathbf{pcatEXE}(\mathbf{U})$ is that for its composition we have the following cancellation laws.

D.2. Proposition. If $\sigma\alpha$ and $\sigma'\alpha$ are defined, their targets are defined, and $\sigma\alpha = \sigma'\alpha$ then $\sigma = \sigma'$. If $\alpha\tau$ and $\alpha\tau'$ are defined and $\alpha\tau = \alpha\tau'$ then $\tau = \tau'$. $\sharp$

Proof. Suppose that $\sigma\alpha$ and $\sigma'\alpha$ are defined, their targets are defined, and $\sigma\alpha = \sigma'\alpha$. Suppose that E and E' are instances of $\sigma\alpha$ and $\sigma'\alpha$, that c and c' are cross-sections of E and E' such that $\sigma = [head(E, c)]$, $\sigma' = [head(E', c')]$, $\alpha = [tail(E, c)] = [tail(E', c')]$, and that f and f' are isomorphisms from E to E' such that $f(c) = c'$. Then $f|tail(E, c) = f'|tail(E, c)$ and $f'(c) = c'$ since otherwise $f \circ (f')^{-1}$ would be an automorphism from E to E whose restriction to $tail(E, c)$ would be different from identity isomorphism of final segments of E, and this would contradict to proposition 2.8. Thus f consists of two disjoint mappings $f|tail(E, c) : tail(E, c) \to tail(E', c')$ and $f|head(E, c) : head(E, c) \to head(E', c')$, Being disjoint restrictions of the isomorphism f both these mappings are isomorphisms. Consequently, $\sigma = [head(E, c)] = [head(E', c')] = \sigma'$.

The proof of the second law is similar. $\sharp$

Another important property of the partial category **pcatEXE(U)** is that bicartesian squares in this partial category can be characterized as follows.

D.3. Proposition. A diagram $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in **pcatEXE(U)** if and only if there exist c, φ_1, φ_2 such that c is an identity, there is no identity $d \neq 0$ such that $\varphi_1 = d + \varphi_1'$ for some φ_1' or $\varphi_2 = d + \varphi_2'$ for some φ_2', $c + \varphi_1 + \varphi_2$ is defined, $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$, $\alpha_2 = c + dom(\varphi_1) + \varphi_2$, $\alpha_1' = c + \varphi_1 + cod(\varphi_2)$, $\alpha_2' = c + cod(\varphi_1) + \varphi_2$. $\sharp$

Proof. Suppose that $D = (v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square, that E is an instance of $\alpha_1\alpha_2' = \alpha_2\alpha_1'$, and that Z_1, Z_2 are cross-sections of E such that $[head(E, Z_1)] = \alpha_1$, $[tail(E, Z_1)] = \alpha_2'$, $[head(E, Z_2)] = \alpha_2$, $[tail(E, Z_2)] = \alpha_1'$. Suppose that X' is the set of common elements of Z_1 and Z_2.

We have $Z_1 \vee Z_2 = end(E)$ since otherwise D could not be a pushout diagram, and $Z_1 \wedge Z_2 = origin(E)$ since otherwise D could not be a pullback diagram. Consequently, we can define c as the set of instances of objects of elements of X', φ_1 as $[E_1]$ for the restriction of E to the set

$X_1 = \{x \in X - X' : z_2 \leq x \leq z_1$ for some $z_1 \in Z_1$ and $z_2 \in Z_2\}$,

and φ_2 as $[E_2]$ for the restriction of E to the set

$X_2 = \{x \in X - X' : z_1 \leq x \leq z_2$ for some $z_1 \in Z_1$ and $z_2 \in Z_2\}$.

Conversely, suppose that there exist c, φ_1, φ_2 such that c is an identity, $c + \varphi_1 + \varphi_2$ is defined, $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$, $\alpha_2 = c + dom(\varphi_1) + \varphi_2$, $\alpha'_1 = c + \varphi_1 + cod(\varphi_2)$, $\alpha'_2 = c + cod(\varphi_1) + \varphi_2$, and consider the diagram $D = (v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha'_2}{\rightarrow} u' \overset{\alpha'_1}{\leftarrow} w)$.

Suppose that $\alpha_1\rho_2 = \alpha_2\rho_1 = \sigma$. Then in each instance E of σ there are cross-sections Z_1 and Z_2 such that $head(E, Z_1)$ is an instance of α_1 and $head(E, Z_2)$ is an instance of α_2. Consequently, $head(E, Z_1 \vee Z_2)$ is an instance of α and $tail(E, Z_1 \vee Z_2)$ is an instance of an execution ρ such that $\alpha\rho = \sigma$. By Proposition 5.9 such an execution is unique. Thus $v \overset{\alpha'_2}{\rightarrow} u' \overset{\alpha'_1}{\leftarrow} w$ is a pushout of $v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w$.

Suppose that $\xi_1\alpha'_2 = \xi_2\alpha'_1 = \tau$. Then in each instance T of τ there are cross-sections Y_1 and Y_2 such that $tail(T, Y_1)$ is an instance of α'_1 and $tail(T, Y_2)$ is an instance of α'_2. Consequently, $tail(T, Y_1 \wedge Y_2)$ is an instance of α and $head(T, Y_1 \wedge Y_2)$ is an instance of an execution ξ such that $\xi\alpha = \tau$. By Proposition 5.9 such an execution is unique. Thus $v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w$ is a pullback of $v \overset{\pi'_2}{\rightarrow} u' \overset{\pi'_1}{\leftarrow} w$.

Hence D is a bicartesian square. The uniqueness of α'_1 and α'_2 follows from the fact that in **pcatEXE**(**U**) only identity executions are isomorphisms. $\sharp$

D.4. Proposition. If $\mathbf{A} = (A, ;)$ is the partial category of executions in a universe of objects then it enjoys the following properties:

(A1) If $\sigma\alpha$ and $\sigma'\alpha$ are defined, their targets are defined, and $\sigma\alpha = \sigma'\alpha$ then $\sigma = \sigma'$.

(A2) If $\alpha\tau$ and $\alpha\tau'$ are defined and $\alpha\tau = \alpha\tau'$ then $\tau = \tau'$.

(A3) If $\sigma\tau$ is an identity then σ and τ are also identities.

(A4) If $\sigma\alpha\tau$ is defined, it has a source and a target, and the category $dec_{\sigma\alpha\tau}$ of decompositions of $\sigma\alpha\tau$ is isomorphic to the category dec_α of decompositions of α then σ and τ are identities.

(A5) For all ξ_1, ξ_2, η_1, η_2 such that $\xi_1\xi_2 = \eta_1\eta_2$ there exist unique σ_1, σ_2, and a unique bicartesian square $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha'_2}{\rightarrow} u' \overset{\alpha'_1}{\leftarrow} w)$, such that $\xi_1 = \sigma_1\alpha_1$, $\xi_2 = \alpha'_2\sigma_2$, $\eta_1 = \sigma_1\alpha_2$, $\eta_2 = \alpha'_1\sigma_2$.

(A6) If $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha'_2}{\rightarrow} u' \overset{\alpha'_1}{\leftarrow} w)$ is a bicartesian square then for every decomposition $u \overset{\alpha_1}{\to} v = u \overset{\alpha_{11}}{\to} v_1 \overset{\alpha_{12}}{\to} v$ (resp. $w \overset{\alpha'_1}{\to} u' = w \overset{\alpha'_{11}}{\to} w_1 \overset{\alpha'_{12}}{\to} u'$) there exist a unique decomposition $w \overset{\alpha'_1}{\to} u' = w \overset{\alpha'_{11}}{\to} w_1 \overset{\alpha'_{12}}{\to} u'$ (resp. $u \overset{\alpha_1}{\to} v = u \overset{\alpha_{11}}{\to} v_1 \overset{\alpha_{12}}{\to} v$), and a unique $v_1 \overset{\alpha''_2}{\to} w_1$, such that $(v_1 \overset{\alpha_{11}}{\leftarrow} u \overset{\alpha_2}{\to} w, v_1 \overset{\alpha''_2}{\to} w_1 \overset{\alpha'_{11}}{\leftarrow} w)$ and $(v \overset{\alpha_{12}}{\leftarrow} v_1 \overset{\alpha''_2}{\to} w_1, v \overset{\alpha'_2}{\to} u' \overset{\alpha'_{12}}{\leftarrow} w_1)$ are bicartesian squares.

(A7) Given a family $\alpha = (u \overset{\alpha_i}{\to} v_i : i \in \{1, ..., n\})$, $n \geq 2$, the existence for all $i, j \in \{1, ..., n\}$ such that $i \neq j$ of bicartesian squares of the form $(v_i \overset{\alpha_i}{\leftarrow} u \overset{\alpha_j}{\to} v_j, v_i \overset{\alpha'_j}{\to} u'_{ij} \overset{\alpha'_i}{\leftarrow} v_j)$ implies the existence in $\mathbf{A}$ of a unique bicartesian n-cube with α being the family of its initial morphisms.

(A8) Every decomposition of $\alpha \in A$ into a pair $c = (\xi_1, \xi_2)$ of $\xi_1 \in A$ and $\xi_2 \in A$ such that $\xi_1 \xi_2 = \alpha$ separates bicartesian squares in the category dec_α of decompositions of α in the sense that every two bicartesian squares in dec_α, one with $a = (\eta, \delta\xi_2)$ such that $\eta \neq \xi_1$ among the nodes, and another with $b = (\xi_1 \varepsilon, \zeta)$ such that $\zeta \neq \xi_2$ among the nodes, do not share a node.

(A9) Every direct system D in the category $occ(\mathbf{A})$ of occurrences of morphisms in morphisms in $\mathbf{A}$ such that elements of D are bounded in the sense that they possess sources and targets has an inductive limit (a colimit).

(A10) Every $\alpha \in A$ is the inductive limit of the direct system of its bounded segments, that is of bounded $\xi \in A$ such that $\alpha = \alpha_1 \xi \alpha_2$ for some α_1 and α_2. $\sharp$

Proof. The properties (A1) - (A2) have been proved as Proposition D.2.

(A3) is a direct consequence of definition.

For (A4) suppose that there exists an isomorphism b between the restriction of $\mathbf{A}$ to the set of components of α and the restriction of $\mathbf{A}$ to the set of components of $\sigma\alpha\tau$, and consider an instance E of α and an instance E' of $\sigma\alpha\tau$. The isomorphism b induces an isomorphism $\bar{b}$ between the lattice of cross-sections of E and the lattice of cross-sections of E'. As every object has a unique instance in every cross-section of E and a unique instance in every cross-section of E', by considering for every occurrence of an object in E the cross-sections containing this occurrence and by using the isomorphism $\bar{b}$ we can construct an isomorphism between E and E'. To this end it suffices to notice that an occurrence of an instance p of an object in a cross-section c_1 of E and an occurrence of p in a cross-section c_2 of E correspond to the same occurrence of p in E iff $[tail(head(E, c_1 \vee c_2), c_1 \wedge c_2)] = p + \delta$ for some δ, and that for E' we have a similar property.

Consequently, E cannot be a proper segment of E', and we obtain (A4).

For (A5) we refer to the characterization of bicartesian squares in the partial category $\mathbf{A} = \mathbf{pcatEXE}(\mathbf{U})$ as described in Proposition D.3. With this characterization a proof of (A5) can be carried out as follows. Consider an instance L of $\xi_1\xi_2 = \eta_1\eta_2$ and its cross-sections c_1 and c_2 such that $\xi_1 = [head(E, c_1)]$, $\xi_2 = [tail(E, c_1)]$, $\eta_1 = [head(E, c_2)]$, $\xi_1 = [tail(E, c_2)]$. Define $\sigma_1 = [head(E, c_1 \wedge c_2)]$, $\sigma_2 = [tail(E, c_1 \vee c_2)]$,
$\alpha_1 = [head(tail(E, c_1 \wedge c_2), c_1)]$, $\alpha_2 = [head(tail(E, c_1 \wedge c_2), c_2)]$,
$\alpha_1' = [head(tail(E, c_2), c_1 \vee c_2)]$, $\alpha_2' = [head(tail(E, c_1), c_1 \vee c_2)]$.
Follow the proof of Proposition D.3 to show that the diagram
$D = (v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square.

For (A6) it suffices to apply the characterization D.3 of bicartesian squares and notice that a decomposition of α_1 induces a decomposition of φ_1.

The properties (A7) and (A8) follow easily from Proposition D.3.

For (A9) it suffices to take into account corollary 2.10 and consider the respective colomits in the category **LPOSETS**.

The property (A10) follows from the condition (2) of Definition 2.3. $\sharp$

The following two proposition are direct consequences of definitions.

D.5. Proposition. If $\mathbf{A} = (A, +)$ is the partial monoid of executions in a universe of objects then it enjoys the following properties:

(B1) If $\alpha + \sigma$ and $\alpha + \sigma'$ are defined and $\alpha + \sigma = \alpha + \sigma'$ then $\sigma = \sigma'$.

(B2) $\alpha + \alpha$ is defined only for $\alpha = 0$.

(B3) The following relation $\triangleleft$ is a partial order:

$\alpha_1 \triangleleft \alpha_2$ iff α_2 contains α_1 in the sense that $\alpha_2 = \alpha_1 + \rho$ for some ρ.

(B4) Given a subset B of A, if $\alpha_1 + \alpha_2$ is defined for all $\alpha_1, \alpha_2 \in B$ such that $\alpha_1 \neq \alpha_2$ then in A there exists the least upper bound $\bigtriangledown B$ of B with respect to $\triangleleft$.

(B5) For all α_1 and α_2 there exists the greatest lower bound of α_1 and α_2 with respect to $\triangleleft$, written as $\alpha_1 \bigtriangleup \alpha_2$.

(B6) If $\alpha_1 + \alpha_2$ is defined then $(\alpha_1 \bigtriangleup \sigma) + (\alpha_2 \bigtriangleup \sigma)$ is defined and $(\alpha_1 \bigtriangleup \sigma) + (\alpha_2 \bigtriangleup \sigma) = (\alpha_1 + \alpha_2) \bigtriangleup \sigma$.

(B7) If $\alpha_1 \bigtriangleup \alpha_2 = 0$ and $\alpha_1 \triangleleft \alpha$ and $\alpha_2 \triangleleft \alpha$ for some α then $\alpha_1 + \alpha_2$ is defined.

(B8) Each $\alpha \neq 0$ contains some β that is a (+)-*atom* in the sense that $\beta \neq 0$ and $\beta = \alpha_1 + \alpha_2$ only if either $\alpha_1 = \beta$ and $\alpha_2 = 0$ or $\alpha_1 = 0$ and $\alpha_2 = \beta$.

(B9) Each α is determined uniquely by the set $h(\alpha)$ of (+)-atoms it contains in the sense that $h(\alpha_1) = h(\alpha_2)$ implies $\alpha_1 = \alpha_2$. $\sharp$

D.6. Proposition. The partial category **pcatEXE**(**U**) and the partial monoid **pmonEXE**(**U**) are related to each other as follows:

(C1) $dom(\alpha_1 + \alpha_2) = dom(\alpha_1) + dom(\alpha_2)$.

(C2) $cod(\alpha_1 + \alpha_2)$ and $cod(\alpha_1) + cod(\alpha_2)$ are defined and $cod(\alpha_1 + \alpha_2) = cod(\alpha_1) + cod(\alpha_2)$ whenever $\alpha_1 + \alpha_2$, $cod(\alpha_1)$, $cod(\alpha_2)$ are defined.

(C3) $dom(\alpha) = 0$ implies $\alpha = 0$ and $cod(\alpha) = 0$ implies $\alpha = 0$.

(C4) If $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22})$ is defined then $\alpha_{11} + \alpha_{21}$, $\alpha_{11} + \alpha_{22}$, $\alpha_{12} + \alpha_{21}$, $\alpha_{12} + \alpha_{22}$ are also defined and $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22}) = (\alpha_{11} + \alpha_{21})(\alpha_{12} + \alpha_{22})$.

(C5) If $\alpha_{11}\alpha_{12}$ and $\alpha_{21}\alpha_{22}$ are defined, and $\alpha_{11} + \alpha_{21}$ is defined, or $\alpha_{11} + \alpha_{22}$ is defined, or $\alpha_{12} + \alpha_{21}$ is defined, or $\alpha_{12} + \alpha_{22}$ is defined, then $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22})$ is defined.

(C6) $\alpha_1 + \alpha_2 = \beta_1\beta_2$ implies the existence of unique α_{11}, α_{12}, α_{21}, α_{22} such that $\alpha_1 = \alpha_{11}\alpha_{12}$, $\alpha_2 = \alpha_{21}\alpha_{22}$, $\beta_1 = \alpha_{11} + \alpha_{21}$, $\beta_2 = \alpha_{12} + \alpha_{22}$.

(C7) In **pmonPROC**(**U**) there exists the least congruence $\sim$ such that $\alpha \sim \beta$ for all α and β such that $\alpha = \gamma\beta\delta$ or $\alpha = \gamma\beta$ or $\alpha = \beta\delta$ for some γ and δ, and this congruence is strong, that is $\alpha_1 \sim \alpha_1'$ and $\alpha_2 \sim \alpha_2'$ implies that $\alpha_1 + \alpha_2$ is defined iff $\alpha_1' + \alpha_2'$ is defined.

(C8) A diagram $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in **pcatPROC**(**U**) if and only if there exist c, φ_1, φ_2 such that c is an identity, there is no identity $d \neq 0$ such that $d \triangleleft \varphi_1$ or $d \triangleleft \varphi_1$, $c + \varphi_1 + \varphi_2$ is defined, $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$, $\alpha_2 = c + dom(\varphi_1) + \varphi_2$, $\alpha_1' = c + \varphi_1 + cod(\varphi_2)$, $\alpha_2' = c + cod(\varphi_1) + \varphi_2$. $\sharp$

A proof of (C7) is straightforward assuming $\alpha \sim \beta$ whenever $objects(\alpha) = objects(\beta)$ and taking into account proposition D.3.

The obtained results can be summarized as follows.

D.7. Proposition. $\mathbf{EXE}(\mathbf{U}) = (EXE(\mathbf{U}), ;, +)$ is a partial algebra that enjoys the following properties:

(A) The reduct $(EXE(\mathbf{U}), ;)$ is a partial category $\mathbf{pcatEXE}(\mathbf{U})$ with the properties (A1) - (A10).

(B) The reduct $(EXE(\mathbf{U}), +)$ is a partial commutative monoid $\mathbf{pmonEXE}(\mathbf{U})$ with the properties (B1) - (B9).

(C) The reducts $(EXE(\mathbf{U}), ;)$ and $(EXE(\mathbf{U}), +)$ are related according to (C1) - (C8). $\sharp$

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