Multiplicative Transition Systems

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Abstract. The paper is concerned with algebras whose elements can be used to represent runs of a system. These algebras, called multiplicative transition systems, are partial categories with respect to a partial binary operation called multiplication. They can be characterized by axioms such that their elements and operations can be represented by partially ordered multisets of a certain type and operations on such multisets. The representation can be obtained without assuming a discrete nature of represented elements. In particular, it remains valid for systems with infinitely divisible elements, and thus also for systems with elements which can represent continuous and partially continuous runs.

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1 Introduction

This paper is an attempt to develop a universal framework for describing systems that may exhibit arbitrary combination of discrete and continuous behaviour.

There are reasons for which we need such a universal framework.

First, in order to describe and analyse systems including computer components, which operate in discrete steps, and real-world components, which operate in a continuous way, we need a framework including ideas from both computer science and control theory (cf. [4]). Consequently, we need a simple language to describe in the same way and to relate behaviours of systems of any nature, including discrete, continuous, and hybrid systems. Second, we need basic axioms valid for systems of any nature such that every particular subclass of systems could be characterized by only adding to the list of basic axioms the respective specific axioms. Third, we need a representation theorem resulting in a representation of system runs by well defined mathematical structures and in a representation of the composition of system runs by a composition of such structures. In particular, we need runs of discrete, continuous, and hybrid systems to be represented by structures of the same type. This will allow us to avoid inventing a special representation in every particular case.

Our idea of a universal framework for describing systems consists in a generalization of the concept of a transition system.

Transition systems are models of systems which operate in discrete indivisible steps called transitions (cf. [7], [8], [9],[10]). They specify system states and transitions. Consequently, they have means to represent implicitly partial and complete system runs viewed as sequences of successive transitions, including one-element sequences representing states. They can be provided in a natural way with a composition of runs of which one starts from the final state of the other, and this results in the structure of a partial category.

Models more precise than simple transition systems are needed to reflect that some system steps can be executed in parallel (parallel independent steps) or in arbitrary order (sequentially independent steps). Consequently, the corresponding simple transition systems must be provided with information reflecting the independence of transitions and the fact that some sequences of transitions may represent the same run.

Finally, together with runs of entire system also runs of subsystems can be considered and partially ordered by inclusion. Consequently, the corresponding transition systems can be provided with a partial order.

In the case of systems with continuous behaviour runs cannot be viewed as sequences of discrete indivisible steps. Nevertheless, the concept of a run still makes sense, and there is a natural composition of runs of which one starts from the resulting state of the other. Then the continuity can be expressed as infinite divisibility of runs with respect to such a composition, and the existing independence of transitions can be defined using the composition.

Moreover, together with runs of entire system also runs of subsystems can be considered and partially ordered by inclusion. Consequently, a partially ordered partial category of states and runs of the possible subsystems is obtained, called a transition structure.

Thus the partial category consisting of system runs and of the respective composition is a good candidate for a universal structure allowing one to represent both discrete and continuous behaviour. We call it a multiplicative transition system and call system runs represented in it transitions.

Note that the concept of a multiplicative transition system generalizes the standard concept of a transition system in the sense that every usual transition system can be regarded as the set of generators of the multiplicative transition system of the respective runs.

The paper is organized as follows. In section 2 we present formal tools exploited in the paper. In section 3 we introduce multiplicative transition systems. In section 4 we define an equivalence of transitions. In section 5 we define regions. In section 6 we represent transitions as labelled posets. In section 7 we represent multiplicative transition systems as a partial category of pomsets.

The paper is an essential extention of [13]. In the paper we exploit the concepts and properties of processes and operations on processes described in [11], [12], [14].

2 Preliminaries

In this section we represent the necessary tools related to partial categories and labelled partially ordered sets exploited in the paper.

A partial category can be defined in exactly the same way as an arrows-only category in the sense of [6], except that sources and targets may be not defined for some arrows that are not identities, and that it may restrict the composability of arrows.

Let $\mathbf{A} = (A, ;)$ be a partial algebra with a binary partial operation $(\alpha, \beta) \mapsto \alpha;\beta$ called *composition*, where $\alpha;\beta$ is written also as $\alpha\beta$. An element $\iota \in A$ is called an *identity* if $\iota\phi = \phi$ whenever $\iota\phi$ is defined and $\psi\iota = \psi$ whenever $\psi\iota$ is defined. We call elements of A *arrows* or *morphisms* and say that $\mathbf{A}$ is a *partial category* if the following conditions are satisfied:

(1) For every α, β, and γ in A, if $\alpha\beta$ and $\beta\gamma$ are defined then $\alpha(\beta\gamma)$ and $(\alpha\beta)\gamma$ are defined and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$; if $\alpha(\beta\gamma)$ is defined then $\alpha\beta$ is defined; if $(\alpha\beta)\gamma$ is defined then $\beta\gamma$ is defined.
(2) For every identity $\iota \in A$, $\iota\iota$ is defined.

The conditions (1) and (2) imply the following properties.

(3) For every $\alpha \in A$, there exists at most one identity $\iota \in A$, called the *source* or the *domain* of α and written as $dom(\alpha)$, such that $\iota\alpha$ is defined, and at most one identity $\kappa \in A$, called the *target* or the *codomain* of α and written as $cod(\alpha)$, such that $\alpha\kappa$ is defined.
(4) For every α and β in A, $\alpha\beta$ is defined if and only if $cod(\alpha) = dom(\beta)$. If $\alpha\beta$ is defined then $dom(\alpha\beta) = dom(\alpha)$ and $cod(\alpha\beta) = cod(\beta)$.

As usual, a morphism α with the source $dom(\alpha) = s$ and the target $cod(\alpha) = t$ is represented in the form $s \xrightarrow{\alpha} t$.

Note that $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable partial operations assigning to a morphism α respectively the source and the target of this morphism, if such a source or a target exists.

Dealing with arrows-only categories rather than with categories in the usual sense is sometimes more convenient since it allows us to avoid two sorted structures and more complicated denotations.

Given a morphism α, a morphism β such that $\alpha = \gamma\beta\varepsilon$ or $\alpha = \beta\varepsilon$ or $\alpha = \gamma\beta$ is called a *segment* of α. If $\alpha = \gamma\beta\varepsilon$ then β is said to be a *closed* segment of α. A segment of a segment β of α is said to be a *subsegment* of α.

Given a partial category $\mathbf{A} = (A, ;)$, let A' be the set of quadruples $(\alpha, \sigma, \tau, \beta)$ where $\sigma\alpha\tau$ is defined and $\sigma\alpha\tau = \beta$, or $dom(\alpha)$ and σ are not defined and $\alpha\tau$ is defined and $\alpha\tau = \beta$, or $cod(\alpha)$ and τ are not defined and $\sigma\alpha$ is defined and $\sigma\alpha = \beta$, or $dom(\alpha)$ and $cod(\alpha)$ are not defined and $\alpha = \beta$. The set A' thus defined and the partial operation

$((\alpha, \sigma, \tau, \beta), (\beta, \sigma', \tau', \gamma)) \mapsto (\alpha, \sigma'\sigma, \tau\tau', \gamma)$

form a category $occ(\mathbf{A})$, called the *category of occurrences of morphisms in morphisms* of $\mathbf{A}$.

Given a partial category $\mathbf{A} = (A, ;)$ and its morphism α, let A'_α be the set of triples (ξ_1, δ, ξ_2) such that $\xi_1 \delta \xi_2 = \alpha$. The set A'_α thus defined and the partial operation

$((\eta_1, \delta, \varepsilon\eta_2), (\eta_1\delta, \varepsilon, \eta_2)) \mapsto (\eta_1, \delta\varepsilon, \eta_2)$

form a category dec_α, called the *category of decompositions* of α. In this category each triple (ξ_1, δ, ξ_2) in which δ is an identity, and thus $\delta = cod(\xi_1) = dom(\xi_2)$, is essentially a decomposition of α into a pair (ξ_1, ξ_2) such that $\xi_1\xi_2 = \alpha$ and it can be identified with this decomposition.

Given partial categories $\mathbf{A} = (A, ;)$ and $\mathbf{A}' = (A', ;')$, a mapping $f : A \to A'$ such that $f(\alpha);' f(\beta)$ is defined and $f(\alpha);' f(\beta) = f(\alpha\beta)$ for every α and β such that $\alpha\beta$ is defined, and $f(\iota)$ is an identity for every identity ι, is called a *morphism* or a *functor* from $\mathbf{A}$ to $\mathbf{A}'$. Note that such a morphism becomes a functor in the usual sense if $\mathbf{A}$ and $\mathbf{A}'$ are categories.

Diagrams, limits and colimits in partial categories can be defined as in usual categories.

A *direct system* is a diagram $(a_i \stackrel{\alpha_{ij}}{\to} a_j : i \leq j, i, j \in I)$, where $(I, \leq)$ is a directed poset, α_{ii} is identity for every $i \in I$, and $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ for all $i \leq j \leq k$. The *inductive limit* of such a system is its colimit, i.e. a family $(a_i \stackrel{\alpha_i}{\to} a : i, j \in I)$ such that $\alpha_i = \alpha_{ij}\alpha_j$ for all $i \in I$ and for every family $(a_i \stackrel{\beta_i}{\to} b : i, j \in I)$ such that $\beta_i = \alpha_{ij}\beta_j$ for all $i \in I$ there exists a unique $a \stackrel{\beta}{\to} b$ such that $\beta_i = \alpha_i\beta$ for all $i \in I$.

A *projective system* is a diagram $(a_i \stackrel{\alpha_{ij}}{\leftarrow} a_j : i \leq j, i, j \in I)$, where $(I, \leq)$ is a directed poset, α_{ii} is identity for every $i \in I$, and $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ for all $i \leq j \leq k$. The *projective limit* of such a system is its limit, i.e. a family $(a_i \stackrel{\alpha_i}{\leftarrow} a : i, j \in I)$ such that $\alpha_i = \alpha_j\alpha_{ij}$ for all $i \in I$ and for every family $(a_i \stackrel{\beta_i}{\leftarrow} b : i, j \in I)$ such that $\beta_i = \beta_j\alpha_{ij}$ for all $i \in I$ there exists a unique $a \stackrel{\beta}{\leftarrow} b$ such that $\beta_i = \beta\alpha_i$ for all $i \in I$.

A *bicartesian square* is a diagram $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w, v \stackrel{\alpha'_2}{\to} u' \stackrel{\alpha'_1}{\leftarrow} w)$ such that $v \stackrel{\alpha'_2}{\to} u' \stackrel{\alpha'_1}{\leftarrow} w$ is a pushout of $v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w$ and $v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w$ is a pullback of $v \stackrel{\alpha'_2}{\to} u' \stackrel{\alpha'_1}{\leftarrow} w$, i.e. such that for every $v \stackrel{\beta_1}{\to} u'' \stackrel{\beta_2}{\leftarrow} w$ such that $\alpha_1\beta_1 = \alpha_2\beta_2$ there exists a unique $u' \stackrel{\beta}{\to} u''$ such that $\beta_1 = \alpha'_2\beta$ and $\beta_2 = \alpha'_1\beta$, and for every $v \stackrel{\gamma_1}{\leftarrow} t \stackrel{\gamma_2}{\to} w$ such that $\gamma_1\alpha'_2 = \gamma_2\alpha'_1$ there exists a unique $u \stackrel{\gamma}{\leftarrow} t$ such that $\gamma_1 = \gamma\alpha_1$ and $\gamma_2 = \gamma\alpha_2$.

Partial categories considered in this paper are related to some partial categories of isomorphism classes of labelled partially ordered sets.

A partial order on a set X is a binary relation $\leq$ between elements of X that is reflexive, anti-symmetric and transitive.

Given a partial order $\leq$ on a set X, the pair $P = (X, \leq)$ is called a *partially ordered set*, or briefly a *poset*. Given a partial order $\leq$ on a set X and a function $l : X \to W$ that assigns to every $x \in X$ a label $l(x)$ from a set W, the triple $L = (X, \leq, l)$ is called a *labelled partially ordered set*, or briefly an *lposet*. A subset $Y \subseteq X$ is said to be *downwards-closed* iff $x \leq y$ for some $y \in Y$ implies

$x \in Y$, *upwards-closed* iff $y \leq x$ for some $y \in Y$ implies $x \in Y$, *bounded* iff it has an upper bound, i.e. an element $z \in X$ such that $y \leq z$ for all $y \in Y$, *directed* iff for every $x, y \in Y$ there exists in Y an upper bound z of $\{x, y\}$, a *chain* iff $x \leq y$ or $y \leq x$ for all $x, y \in Y$, an *antichain* iff $x < y$ does not hold for any $x, y \in Y$, and L is said to be *directed complete* or a *directed complete partial order* (a DCPO) iff every of its directed subsets has a unique least upper bound. A Scott topology on the underlying set X of L is the topology in which a subset $U \subseteq X$ is open iff it is upward closed and does not contain the least upper bound of any directed subset of $X - U$.

A *cross-section* of L is a maximal antichain Z of $P = (X, \leq)$ such that, for every $x, y \in X$ for which $x \leq y$ and $x \leq z'$ and $z'' \leq y$ with some $z', z'' \in Z$, there exists $z \in Z$ such that $x \leq z \leq y$.

Note that if Z is a cross-section of L then the relation $\leq$ is the transitive closure of the union of the restrictions of the relation $\leq$ to the subsets $Z^- = \{x \in X : x \leq z \text{ for some } z \in Z\}$ and $Z^+ = \{x \in X : z \leq x \text{ for some } z \in Z\}$.

A cross-section Z' is said to *precede* a cross-section Z'', written as $Z' \preceq Z''$, iff $Z'^- \subseteq Z''^-$. The relation $\preceq$ is a partial order on the set of cross-sections of L.

For every two cross-sections Z' and Z'' of L there exist the greatest lower bound $Z' \wedge Z''$ and the least upper bound $Z' \vee Z''$ of Z' and Z'' with respect to $\preceq$, where

$Z' \wedge Z'' =$
$\quad \{z \in Z' \cup Z'' : z \leq z' \text{ and } z \leq z'' \text{ for some } z' \in Z' \text{ and } z'' \in Z''\}$,
$Z' \vee Z'' =$
$\quad \{z \in Z' \cup Z'' : z' \leq z \text{ and } z'' \leq z \text{ for some } z' \in Z' \text{ and } z'' \in Z''\}$.

Moreover, the set of cross-sections of L with the operations thus defined is a distributive lattice.

A partially ordered subset K of L such that K is the restriction $L|[Z', Z'']$ of L to subset $[Z', Z''] = Z''^- - Z'^-$ for some cross-sections Z' and Z'' such that $Z' \preceq Z''$, or the restriction $L|Z^-$ to the subset Z^- for a cross-section Z, or to the restriction $L|Z^+$ to the subset Z^+ for a cross-section Z, is said to be a *segment* of L. A segment $L|Z^-$ (resp.: $L|Z^+$) is said to be *initial* (resp.: *final*). If $K = L|[Z', Z'']$ then it is said to be a *closed* segment of L. A segment of a segment K of L is said to be a *subsegment* of L. Given a function f defined on L, an *initial segment* of f is defined as the restriction of f an initial segment of L.

Given a cross-section c of L, the restrictions of L to the subsets
$c^- = \{x \in X : x \leq z \text{ for some } z \in c\}$ and $c^+ = \{x \in X : z \leq x \text{ for some } z \in c\}$ are called respectively the *head* and the *tail* of L with respect to c, and written respectively as $head(L, c)$ and $tail(L, c)$.

The *sequential decomposition* of L at a cross-section c is the pair $s(c) = (head(L, c), tail(L, c))$ and L is said to *consist* of $head(L, c)$ *followed by* $tail(L, c)$.

A *parallel decomposition* of L is a pair $p = (p^F, p^S)$ of disjoint subsets p^F and p^S of X such that $p^F \cup p^S = X$ and $x' \leq x''$ only if x' and x'' are both in one of these subsets.

Given a parallel decomposition $p = (p^F, p^S)$ of L, the restrictions of L to the subsets p^F and p^S are called respectively the *first component* and the *second component* of L with respect to p, they are written respectively as *first*(L, p) and *second*(L, p) and called *independent components* of L, and L is said to *consist* of *parallel first*(L, p) and *second*(L, p). Note that L itself is an independent component of L.

A *fragment* or a *component* of L is an independent component C of a segment S of L such that the set of minimal elements of C is a cross-section of C and it is contained in the cross-section of P that consists of minimal elements of S.

An lposet L' is said to *occur* in L if it is a fragment of L.

If the set of elements of $L = (X, \leq, l)$ that are are minimal (resp., maximal) with respect to $\leq$ is a cross-section of L then we call the restriction of L to this set the *origin* (resp., the *end*) of L, write it as *origin*(L) (resp., as *end*(L)). If *origin*(L) and *end*(L) exist then L is said to be *closed*.

By **LPOSETS** we denote the category of lposets and their morphisms, where a *morphism* from an lposet $L = (X, \leq, l)$ to an lposet $L' = (X', \leq', l')$ is defined as a mapping $b : X \to X'$ such that, for all x and y, $x \leq y$ iff $b(x) \leq' b(y)$, and, for all x, $l(x) = l'(b(x))$. In the category **LPOSETS** a morphism from $L = (X, \leq, l)$ to $L' = (X', \leq', l')$ is an *isomorphism* iff it is bijective, and it is an *automorphism* iff it is bijective and $L = L'$. If there exists an isomorphism from an lposet L to an lposet L' then we say that L and $L'$are *isomorphic*. A *partially ordered multiset*, or briefly a *pomset*, is defined as an isomorphism class ξ of lposets. Each lposet that belongs to such a class ξ is called an *instance* of ξ. The pomset corresponding to an lposet L is written as $[L]$.

A pomset γ is said to *consist* of a pomset α *followed* by a pomset β, written as $\gamma|\alpha;\beta$, iff γ has an instance G with a cross-section c and a sequential decomposition of this instance at c into $G_1 \in \alpha$ and $G_2 \in \beta$.

A pomset γ is said to *consist* of two *parallel* pomsets: a pomset α and a pomset β, written as $\gamma = \alpha \parallel \beta$, iff γ has an instance G with a parallel decomposition into $G_1 \in \alpha$ and $G_2 \in \beta$.

3 Basic notions

Basic axioms characterizing multiplicative transition systems can be formulated regarding transitions as abstract entities and expressing their properties with the aid of a partial operation of composing transitions, called *multiplication*.

Let A be the set of transitions representing runs of a system, some transitions possibly without an initial or a final state. Then $\mathbf{A} = (A, ;)$ is a partial algebra that consists of the set A and of the multiplication $(\alpha, \beta) \mapsto \alpha\beta$, where $\alpha\beta$ denotes $\alpha; \beta$. It is reasonable to assume that this algebra is a partial category and that it enjoys some natural properties.

First, it is natural to expect that the multiplication satisfies the following cancellation laws.

(A1) If $\sigma\alpha$ and $\sigma'\alpha$ are defined, their targets are defined, and $\sigma\alpha = \sigma'\alpha$ then $\sigma = \sigma'$.

(A2) If $\alpha\tau$ and $\alpha\tau'$ are defined, their sources are defined, and $\alpha\tau = \alpha\tau'$ then $\tau = \tau'$.

Second, identities are expected to represent states and to be indecomposable into transitions which do not represent states.

(A3) If $\sigma\tau$ is an identity then σ and τ are also identities.

Third, transitions which are not identities are expected to be essentially different from their proper segments.

(A4) If $\sigma\alpha\tau$ is defined, it has a source and a target, and the category $dec_{\sigma\alpha\tau}$ of decompositions of $\sigma\alpha\tau$ is isomorphic to the category dec_{α} of decompositions of α then σ and τ are identities.

Fourth, the independence of transitions α_1 and α_2, transitions α_1 and α_2', and transitions α_2 and α_1' is expected to be represented by the existence of a bicartesian square $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$, and it is expected to imply the independence of transitions represented by segments of α_1 and α_2.

(A5) If $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square then for every decomposition $u \overset{\alpha_1}{\rightarrow} v = u \overset{\alpha_{11}}{\rightarrow} v_1 \overset{\alpha_{12}}{\rightarrow} v$ (resp. $w \overset{\alpha_1'}{\rightarrow} u' = w \overset{\alpha_{11}'}{\rightarrow} w_1 \overset{\alpha_{12}'}{\rightarrow} u'$) there exist a unique decomposition $w \overset{\alpha_1'}{\rightarrow} u' = w \overset{\alpha_{11}'}{\rightarrow} w_1 \overset{\alpha_{12}'}{\rightarrow} u'$ (resp. $u \overset{\alpha_1}{\rightarrow} v = u \overset{\alpha_{11}}{\rightarrow} v_1 \overset{\alpha_{12}}{\rightarrow} v$), and a unique $v_1 \overset{\alpha_2''}{\rightarrow} w_1$, such that $(v_1 \overset{\alpha_{11}}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v_1 \overset{\alpha_2''}{\rightarrow} w_1 \overset{\alpha_{11}'}{\leftarrow} w)$ and $(v \overset{\alpha_{12}}{\leftarrow} v_1 \overset{\alpha_2''}{\rightarrow} w_1, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_{12}'}{\leftarrow} w_1)$ are bicartesian squares.

Fifth, the independence of segments of a transition is expected to be the only reason of a representation of such a transition by two different expressions.

(A6) For all ξ_1, ξ_2, η_1, η_2 such that $\xi_1\xi_2 = \eta_1\eta_2$ there exist unique σ_1, σ_2, and a unique bicartesian square $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$, such that $\xi_1 = \sigma_1\alpha_1$, $\xi_2 = \alpha_2'\sigma_2$, $\eta_1 = \sigma_1\alpha_2$, $\eta_2 = \alpha_1'\sigma_2$.

Finally, every transition is expected to be an inductive limit of its closed segments.

(A7) Every direct system D in the category $occ(\mathbf{A})$ of occurrences of morphisms in morphisms in $\mathbf{A}$ such that elements of D are closed in the sense that they possess sources and targets has an inductive limit (a colimit).

(A8) Every $\alpha \in A$ is the inductive limit of the direct system of its closed segments.

Thus we have come to the following definition.

3.1. Definition. A *multiplicative transition system*, or briefly an MTS, is a partial category $\mathbf{A} = (A, ;)$ with a set A of morphisms and with a composition $(\alpha_1, \alpha_2) \mapsto \alpha_1;\alpha_2$ such that the axioms (A1) - (A8) hold. $\sharp$

In **A** two partial unary operations $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable that assign to an element a source and a target, if they exist.

An element α of A is said to be a *atom* of **A** iff it is not an identity, has a source and a target, and for every $\alpha_1 \in A$ and $\alpha_2 \in A$ the equality $\alpha = \alpha_1\alpha_2$ implies that either α_1 is an identity and $\alpha_2 = \alpha$ or α_2 is an identity and $\alpha_1 = \alpha$.

We say that **A** is *discrete* if every $\alpha \in A$ that is not an identity can be represented in the form $\alpha = \alpha_1 ... \alpha_n$, where α_1,...,α_n are atoms.

Note that if **A** is discrete then its every element has a source and a target and thus **A** is a category.

By a *cut* of $\alpha \in A$ we mean a pair (α_1, α_2) such that $\alpha_1\alpha_2 = \alpha$.

Cuts of every $\alpha \in A$ are partially ordered by the relation $\sqsubseteq_\alpha$, where $x \sqsubseteq_\alpha y$ with $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ means that $\eta_1 = \xi_1\delta$ with some δ. Due to (A1) - (A2) for $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \sqsubseteq_\alpha y$ there exists a unique δ such that $\eta_1 = \xi_1\delta$, written as $x \to y$.

The partial order $\sqsubseteq_\alpha$ makes the set of cuts of α a lattice LT_α.

Indeed, let $\alpha = \xi_1\xi_2 = \eta_1\eta_2$, $\xi_1 = \sigma_1\alpha_1$, $\xi_2 = \alpha'_2\sigma_2$, $\eta_1 = \sigma_1\alpha_2$, $\eta_2 = \alpha'_1\sigma_2$ with α_1, α'_1, α_2, α'_2, σ_1, σ_2 as in (A6). The least upper bound of $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ can be defined as $z = (\xi_1\alpha'_2, \sigma_2) = (\eta_1\alpha'_1, \sigma_2)$. To see this consider any $u = (\zeta_1, \zeta_2)$ such that $x \sqsubseteq_\alpha u$ and $y \sqsubseteq_\alpha u$. Then $\zeta_1 = \xi_1\delta$ and $\zeta_1 = \eta_1\epsilon$ for some δ and ϵ. As α'_1 and α'_2 form a pushout of α_1 and α_2, there exists a unique φ such that $\delta = \alpha'_2\varphi$ and $\epsilon = \alpha'_1\varphi$. Hence $\zeta_1 = \xi_1\alpha'_2\varphi = \eta_1\alpha'_1\varphi$ and, consequently, $z \sqsubseteq_\alpha u$.

Similarly, due to the fact that α_1 and α_2 form a pullback of α'_1 and α'_2, we obtain that $t = (\sigma_1, \alpha_1\alpha'_2\sigma_2)$ is the greatest lower bound of x and y.

The lattice LT_α is obviously an MTS.

Given two cuts x and y, by $x \sqcup_\alpha y$ and $x \sqcap_\alpha y$ we denote respectively the least upper bound and the greatest lower bound of x and y. From (A6) it follows that $(x \leftarrow x \sqcap_\alpha y \to y, x \to x \sqcup_\alpha y \leftarrow y)$ is a bicartesian square.

Given $\alpha \in A$ and its cuts $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \sqsubseteq_\alpha y$, by a *segment* of α from x to y we mean $\beta \in A$ such that $\xi_2 = \beta\eta_2$ and $\eta_1 = \xi_1\beta$, written as $\alpha|[x, y]$. A segment $\alpha|[x', y']$ of α such that $x \sqsubseteq_\alpha x' \sqsubseteq_\alpha y' \sqsubseteq_\alpha y$ is called a *subsegment* of $\alpha|[x, y]$. If $x = x'$ (resp. if $y = y'$) then we call it an *initial* (resp. a *final*) subsegment of $\alpha|[x, y]$. An initial segment ι of α is called also a *prefix* of α, written as ι *pref* α.

In the set $A_{s-closed}$ of those $\alpha \in A$ which are semiclosed in the sense that they have a source $dom(\alpha)$ one can define as follows a relation $\sqsubseteq$, where $\alpha \sqsubseteq \beta$ whenever every prefix of α is a prefix of β, and this relation is a partial order, i.e. $(A_{s-closed}, \sqsubseteq)$ is a poset.

Elements of A are called *transitions* of **A**. Transitions of **A** which are identities of **A** are called *states* of **A**. Transitions which are atomic identities are called *atomic states*. A transition α is said to be *closed* if it has the source $dom(\alpha)$ and the target $cod(\alpha)$. For every transition α, the existing states $u = dom(\alpha)$ and $v = cod(\alpha)$ are called respectively the *initial state* and the *final state* of α and we write α as $u \xrightarrow{\alpha} v$. The composition $(\alpha_1, \alpha_2) \mapsto \alpha_1\alpha_2$ is called a *multiplication*. The independence of closed transitions can be defined as follows (cf. [3]).

3.2. Definition. Transitions $u \overset{\alpha_1}{\to} v$ and $u \overset{\alpha_2}{\to} w$ are said to be *parallel independent* iff there exist unique transition $v \overset{\alpha_2'}{\to} u'$ and $w \overset{\alpha_1'}{\to} u'$ such that $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\to} w, v \overset{\alpha_2'}{\to} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square. ♯

3.3. Definition. Transitions $u \overset{\alpha_1}{\to} v$ and $v \overset{\alpha_2'}{\to} u'$ are said to be *sequential independent* iff there exist unique transition $u \overset{\alpha_2}{\to} w$ and $w \overset{\alpha_1'}{\to} u'$ such that $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\to} w, v \overset{\alpha_2'}{\to} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square. ♯

These definitions are adequate in subalgebras of multiplicative transition systems provided that bicartesian squares in such subalgebras are bicartesian squares in the original multiplicative transition systems. This appears to be true if the respective subalgebras are inheriting in the following sense.

3.4. Definition. A subalgebra $\mathbf{A}'$ of an MTS $\mathbf{A}$ is said to be *inheriting* if it is closed with respect to components of its elements in the sense that arrows α and β of $\mathbf{A}$ are also arrows of $\mathbf{A}'$ whenever $\alpha\beta$ is an arrow of $\mathbf{A}'$. ♯

This following proposition reflects the crucial property of inheriting subalgebras of multiplicative transition systems.

3.5. Proposition. If $\mathbf{A}'$ is an inheriting subalgebra of an MTS $\mathbf{A}$ then:

(1) each bicartesian square of $\mathbf{A}$ whose arrows are in $\mathbf{A}'$ is a bicartesian square in $\mathbf{A}'$,
(2) each bicartesian square in $\mathbf{A}'$ is a bicartesian square in $\mathbf{A}$. ♯

Proof. The first part of this proposition is immediate. For the second part it suffices to exploit the property (A6) of $\mathbf{A}$ and the fact that $\mathbf{A}'$ is an inheriting subalgebra of $\mathbf{A}$.

Multiplicative transition systems are models of concurrent system richer than usual transition systems in the sense that they specify not only states, transitions, and independence of transitions of the modelled systems, but also their runs and how runs compose. Moreover, independence becomes a definable notion, and it can be defined not only for indecomposable transitions, but also for compound transitions.

3.6. Example. Define a transition system without a distinguished initial state as $M = (S, E, T)$ such that S is a set of states, E is a set of events, and $T \subseteq S \times E \times S$ is a set of transitions, where $(s, e, s') \in T$ stands for the transition from the state s to the state s' due to the event e. Assume that E contains a distinguished element $*$ standing for "no event", and assume that for every state $s \in S$ the set T contains an idle transition $(s, *, s)$ standing for "stay in

s". Then M can be represented by the structure $G(M) = (T, dom, cod)$, where $dom(s, e, s') = (s, *, s)$ and $cod(s, e, s') = (s', *, s')$ for every $(s, e, s') \in T$.

Write $s \stackrel{e}{\rightarrow} s'$ to indicate that $(s, e, s') \in T$. Denote by Lts the set of triples of the form $\alpha = s \stackrel{x}{\rightarrow} s'$ where x is any finite word over the alphabet $E - \{*\}$ such that $x = e_1 ... e_m$ for $\alpha = s_0 \stackrel{e_1}{\rightarrow} s_1 \stackrel{e_2}{\rightarrow} s_2 ... s_{m-1} \stackrel{e_m}{\rightarrow} s_m$ with $s_0 = s$ and $s_m = s'$, or x is the empty word represented by $*$ and $s' = s$. Define $dom(s \stackrel{x}{\rightarrow} s') = s \stackrel{*}{\rightarrow} s$ and $cod(s \stackrel{e_1}{\rightarrow} s') = s' \stackrel{*}{\rightarrow} s'$. For triples $\alpha_1 = s_1 \stackrel{x_1}{\rightarrow} s'_1$ and $\alpha_2 = s_2 \stackrel{x_2}{\rightarrow} s'_2$ such that $s'_1 = s_2$ define the result of composing α_1 and α_2 as $\alpha_1\alpha_2 = s_1 \stackrel{x_1x_2}{\rightarrow} s'_2$.
The set Lts with the composition thus defined is an MTS $LTS(M)$ in the sense of definition 3.1. In this MTS each order $\sqsubseteq_\alpha$ is linear and $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w, v \stackrel{\alpha'_2}{\rightarrow} u' \stackrel{\alpha'_1}{\leftarrow} w)$ is a bicartesian square iff α_1 and α'_1 are identities or α_2 and α'_2 are identities. $\sharp$

3.7. Example. Consider the transition system M from example 3.6. Consider a symmetric irreflexive relation $I \subseteq (E - \{*\})^2$, called an independence relation, and the least equivalence relation $\|_I$ between words over the alphabet $E - \{*\}$ such that words $uabv$ and $ubav$ are equivalent whenever $(a, b) \in I$. The equivalence classes of such a relation are known in the literature as Mazurkiewicz traces with respect to I (see [5]). Denote by Ts the set of triples as in example 3.6 but with words over the alphabet $E - \{*\}$ replaced by traces with respect to I. Define dom and cod and the composition as in example 3.6, but with the concatenation of words replaced by the induced concatenation of traces.

The set Ts with the composition thus defined is an MTS $TS(M, I)$ in the sense of definition 3.1, and that this MTS is a homomorphic image of the MTS from example 3.6. However, in this system there exist nontrivial bicartesian squares, namely, the squares $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w, v \stackrel{\alpha'_2}{\rightarrow} u' \stackrel{\alpha'_1}{\leftarrow} w)$ such that $\alpha_1 = u \stackrel{x_1}{\rightarrow} v$, $\alpha_2 = u \stackrel{x_2}{\rightarrow} w$, $\alpha'_1 = w \stackrel{x_1}{\rightarrow} u'$, $\alpha'_2 = v \stackrel{x_2}{\rightarrow} u'$ with $(a, b) \in I$ for all (a, b) such that a occurs in x_1 and b occurs in x_2. $\sharp$

3.8. Example. Imagine a tank v to keep a liquid. Imagine transitions of the level of liquid in intervals of real time. Suppose that the flow of real time cannot be observed. Then only the flow of intrinsic time that can be derived from what happens in the tank is available. Consequently, a concrete transition during which the level of liquid at a moment t in an interval $[t', t'']$ of real time is $f(t)$ must be represented by a modified version $s \mapsto \widehat{f}(s)$ of the correspondence $t \mapsto f(t)$, where $\widehat{f}(s)$ is the variation of f in the interval $[t', t]$, and where the variation of f in the interval $[u', u'']$, $var(f; u', u'')$, is defined as the least upper bound of the set of quantities $|f(t_1) - f(t_0)| + ... + |f(t_n) - f(t_{n-1})|$, each quantity corresponding to a partition $t_0 = u' < t_1 < ... < t_n = u''$ of the interval $[u', u'']$. Such a transition can be represented by the labelled ordered set $P = (X_P, \leq_P, l_P)$, where $X_P = \{v\} \times domain(\widehat{f})$, $(v, s) \leq_P (v, s')$ iff $s \leq s'$, and $l_P(v, s) = \widehat{f}(s)$. When considered up to isomorphism and then called an abstract transition it can be represented by the isomorphism class $[P]$ that contains P.

An abstract transition $\pi = [P]$ in v and an abstract transition $\rho = [R]$ in another tank v' are illustrated in figure 3.1.

Let A_v be the set of abstract transitions in v of this kind. The composition of abstract transitions in v is a partial operation $(\pi_1, \pi_2) \mapsto \pi_1;_v \pi_2$ where $\pi_1;_v \pi_2$ is defined as $[P]$ for a concrete transition P that consists of a segment $P_1 \in [P_1] = \pi_1$ and a segment $P_2 \in [P_2] = \pi_2$ such that the maximal element of P_1 is the minimal element of P_2. The partial category $\mathbf{A}_v = (A_v, ;_v)$ is a multiplicative transition system.

In the case of the system of two tanks v and v' such that there is no pouring of liquid from v to v' or from v' to v the set of abstract transitions, $A_{v,v'}$, consists of the sets A_v and $A_{v'}$ of abstract transitions in v and v', and of the set of abstract transitions τ where τ is defined as $[T] = \pi \parallel \rho$ for a concrete transition T that consists of two parallel concrete transitions: a concrete transition $P \in [P] = \pi \in A_v$ and a concrete transition $R \in [R] = \rho \in A_{v'}$. The composition is the partial operation $(\tau_1, \tau_2) \mapsto \tau_1;_{v.v'} \tau_2$ where $(\tau_1, \tau_2) \mapsto \tau_1;_{v.v'} \tau_2$ is $(\tau_1, \tau_2) \mapsto \tau_1;_v \tau_2$ if $\tau_1, \tau_2 \in A_v$, $(\tau_1, \tau_2) \mapsto \tau_1;_{v.v'} \tau_2$ is $(\tau_1, \tau_2) \mapsto \tau_1;_{v'} \tau_2$ if $\tau_1, \tau_2 \in A_{v'}$, and $(\tau_1, \tau_2) \mapsto \tau_1;_{v.v'} \tau_2 = [T]$ for a concrete transition T that consists of two parallel concrete transitions: a concrete transition $P \in [P] \in [P_1]; [P_2]$ and a concrete transition $R \in [R] \in [R_1]; [R_2]$ if $\tau_1, \tau_2 \in A_{v,v'}$, $\tau_1 = [T_1]$ with T_1 consisting of two parallel concrete transitions $P_1 \in [P_1] \in A_v$ and $R_1 \in [R_1] \in A_{v'}$, and $\tau_2 = [T_2]$ with T_2 consisting of two parallel concrete transitions $P_2 \in [P_2] \in A_v$ and $R_2 \in [R_2] \in A_{v'}$. The partial category $\mathbf{A}_{v,v'} = (A_{v,v'}, ;_{v,v'})$ is a multiplicative transition system.

In the case of the system of two tanks v and v' such that from time to time a quantity of liquid is poured from v to v' the set of abstract transitions, $A'_{v,v'}$, consists of the sets A_v and $A_{v'}$ of abstract transitions in v and v', of the set $A_{v,v'}$ of abstract transitions of the system of v and v' running independently, and of the set of abstract transitions corresponding concrete transitions $K = (X_K, \leq_K, l_K)$, each K consisting of a sequence $..., T_1, S_1, T_2, S_2, ...$ of segments $..., T_1, T_2, ...$ and $..., S_1, S_2, ...$ such that x is a maximal element of T_i iff it is a minimal element of S_i and y is a maximal element of S_i iff it is a minimal element of T_{i+1}, where each S_i is a concrete pouring of an amount m of liquid represented by an lposet $S = (X_S, \leq, l_S)$ with $X_S = \{x_1, x_2, x_3, x_4\}$, $x_1 <_S x_3$, $x_1 <_S x_4$, $x_2 <_S x_3$, $x_2 <_S x_4$, $l_S(x_1) = (d, r)$, $l_S(x_2) = (p, q)$, $l_S(x_3) = (d, r+m)$, $l_S(x_4) = (p, q-m)$.

Abstract transitions corresponding to segments of a concrete transition K are illustrated in figure 3.2. The abstract transition corresponding to a concrete transition K is illustrated in Figure 3.3.

The composition is the extention $(\kappa_1, \kappa_2) \mapsto \kappa_1;'_{v,v'} \kappa_2$ of $(\kappa_1, \kappa_2) \mapsto \kappa_1;_{v.v'} \kappa_2$ such that $\kappa_1;'_{v,v'} \kappa_2$ is defined as $[K]$ with a concrete transition K that consists of $K_1 \in \kappa_1$ followed by $K_2 \in \kappa_2$. The partial category $\mathbf{A}'_{v,v'} = (A'_{v,v'}, ;'_{v,v'})$ is a multiplicative transition system. In this system an abstract transition which corresponds to a concrete transition K which consists of $S' = S_1$ followed by T and followed by $S'' = S_2$, where T consists of parallel P and R, can be represented as $\sigma';''(\pi \parallel \rho);''\sigma''$, where $\sigma' = [S']$, $\pi = [P]$, $\rho = [R]$, and $\sigma'' = [S'']$. $\sharp$

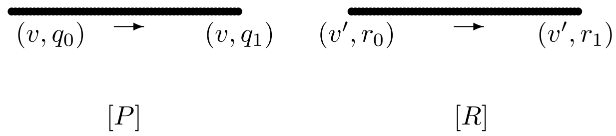


Figure 3.1: $[P]$, $[R]$

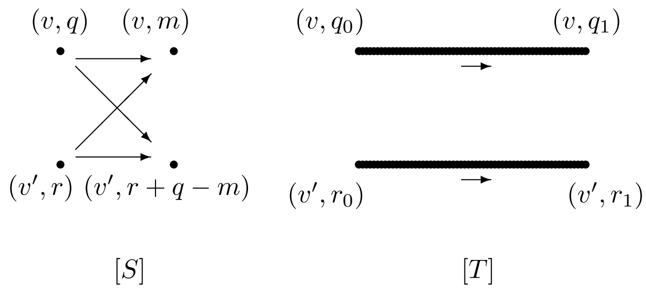


Figure 3.2: $[S]$, $[T]$

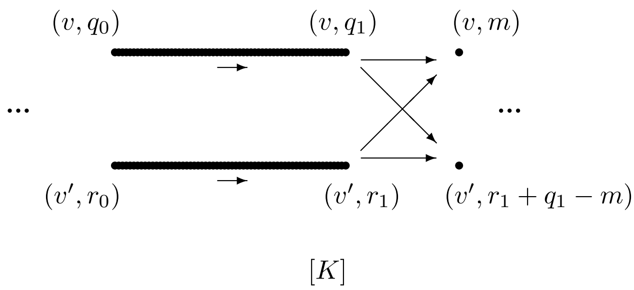


Figure 3.3: $[K]$

4 Equivalence of transitions

In the definitions 3.2 and 3.3 we have characterized the natural concepts of sequential and parallel independence of transitions, similar to the concepts introduced in [3], as the existence in the respective MTS of appropriate bicartesian squares. Now we shall use this characterization to define independence and a natural equivalence of elements of multiplicative transition systems similar to the considered in [15] independence and equivalence of transitions in transition systems with independence. This will allow us to adapt and study the concept of a region similar to that introduced in [2].

4.1. Examples. In the MTS $LTS(M)$ in example 3.6 transitions $u \stackrel{\alpha_1}{\to} v$ and $u \stackrel{\alpha_2}{\to} w$ are *parallel independent* only if one of them is an identity. Similarly, transitions $u \stackrel{\alpha_1}{\to} v$ and $v \stackrel{\alpha_2'}{\to} u'$ are sequential independent only if one of them is an identity. In the MTS $TS(M)$ in example 3.7 transitions $u \stackrel{\alpha_1}{\to} v$ and $u \stackrel{\alpha_2}{\to} w$ are *parallel independent* iff $(a, b) \in I$ for all a occurring in α_1 and all b occurring in α_2. Similarly, transitions $u \stackrel{\alpha_1}{\to} v$ and $v \stackrel{\alpha_2'}{\to} u'$ are sequential independent iff $(a, b) \in I$ for all (a, b) such that a occurs in α_1 and b occurs in α_2'. In the MTS $\mathbf{A}'_{v,v'}$ in example 3.8 transitions $\pi \parallel dom(\rho)$ and $dom(\pi) \parallel \rho$ are parallel independent, transitions $\pi \parallel dom(\rho)$ and $cod(\pi) \parallel \rho$ are sequential independent, and transitions $dom(\pi) \parallel \rho$ and $\pi \parallel cod(\rho)$ are sequential independent. $\sharp$

4.2. Definition. By the *natural equivalence* of elements of an MTS $\mathbf{A} = (A, ;)$ we mean the least equivalence relation $\equiv$ in A such that $\alpha_1 \equiv \alpha_1'$ whenever in this MTS there exists a bicartesian square $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w, v \stackrel{\alpha_2'}{\to} u' \stackrel{\alpha_1'}{\leftarrow} w)$. $\sharp$

4.3. Examples. In the MTS $\mathbf{A}'_{v,v'}$ in example 3.8 transitions $\pi \parallel dom(\rho)$ and $cod(\rho) \parallel \pi$ are equivalent in the sense of definition 4.2. In the MTS $LTS(M)$ in example 3.6 the natural equivalence coincides with the identity relation. In the MTS $TS(M)$ in example 3.7 we have $\alpha_1 \equiv \alpha_1'$ whenever $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w, v \stackrel{\alpha_2'}{\to} u' \stackrel{\alpha_1'}{\leftarrow} w)$ with α_1 and α_1' representing the same trace t_1, and α_2 and α_2' representing the same trace t_2, for $(a, b) \in I$ for all (a, b) such that a occurs in t_1 and b occurs in t_2. $\sharp$

5 Regions

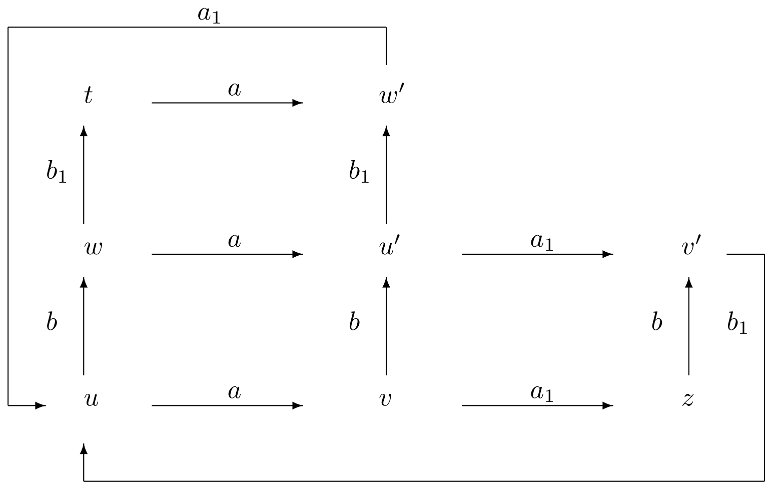
The existence in behaviour-oriented partial categories of the natural equivalence of transitions allows us to adapt and exploit the concept of a region similar to those described in [1] and [2].

5.1. Definition. By a *region* of an MTS $\mathbf{A} = (A, ;)$ we mean a nonempty subset r of the set of states of $\mathbf{A}$ such that:

$$
\begin{aligned}
&dom(\alpha) \in r \text{ and } cod(\alpha) \notin r \text{ and } \alpha' \equiv \alpha \\
&\qquad\qquad \text{implies } dom(\alpha') \in r \text{ and } cod(\alpha') \notin r, \\
&dom(\alpha) \notin r \text{ and } cod(\alpha) \in r \text{ and } \alpha' \equiv \alpha \\
&\qquad\qquad \text{implies } dom(\alpha') \notin r \text{ and } cod(\alpha') \in r. \ \sharp
\end{aligned}
$$

5.2. Example. Consider the MTS $\mathbf{A}'_{v,v'}$ in example 3.8. In this MTS the sets $[(v,q)] = \{(v,q)\} \cup (\{v'\} \times [0,+\infty))$ with $q \geq 0$,
the sets $[(v',r)] = \{(v',r)\} \cup (\{v\} \times [0,+\infty))$ with $r \geq 0$, and disjoint unions of such sets are regions. $\sharp$

5.3. Example . Consider the transition system M' in figure 5.1.



M'

Figure 5.1

Consider the independence relation $I' = \{(a,b),(a,b_1),(a_1,b),(a_1,b_1)\}$ and the MTS $TS(M',I')$. In this MTS we have transitions
$\alpha = u \overset{[a]}{\to} v$, $\beta = u \overset{[b]}{\to} w$, $\alpha' = w \overset{[a]}{\to} u'$, $\beta' = v \overset{[b]}{\to} u'$ $\alpha'' = t \overset{[a]}{\to} w'$,
$\beta'' = z \overset{[b]}{\to} v'$, $\alpha_1 = u' \overset{[a_1]}{\to} v'$, $\beta_1 = u' \overset{[b_1]}{\to} w'$, $\alpha'_1 = w' \overset{[a_1]}{\to} u$, $\beta'_1 = v' \overset{[b_1]}{\to} u$
$\alpha''_1 = v \overset{[a_1]}{\to} z$, $\beta''_1 = w \overset{[b_1]}{\to} t$,
where $[a]$,$[a_1]$,$[b]$,$[b_1]$ are traces correspondig to a, a_1, b, b_1, and compositions of these transitions. For example,
$\alpha\beta' = \beta\alpha' = \gamma = u \overset{[ab]}{\to} u'$, $\alpha_1\beta'_1 = \beta_1\alpha'_1 = \gamma_1 = u' \overset{[a_1b_1]}{\to} u$,
transitions α,α' are equivalent, transitions β, β' are equivalent, and we have regions
$E = \{u,w,t,v',z\}$, $F = \{u,v,z,t,w'\}$, $G = \{v,u',w'\}$, $H = \{w,u',v'\}$, $E \cup G$, $F \cup H$, and $\{u,v,w,z,t,u',v',w'\}$. $\sharp$

From the definition of a region we obtain the following proposition.

5.4. Proposition. If $\mathbf{A} = (A, ;)$ is an MTS, r is a region of $\mathbf{A}$, and $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in $\mathbf{A}$, then $v \in r$ implies that $u \in r$ or $u' \in r$. $\sharp$

Due to the property (A5) of multiplicative transition systems we obtain the following proposition.

5.5. Proposition. If $\mathbf{A} = (A, ;)$ is an MTS, r is a region of $\mathbf{A}$,
and $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in $\mathbf{A}$ with morphisms which are not identities, then for every decomposition $u \overset{\alpha_1}{\rightarrow} v = u \overset{\alpha_{11}}{\rightarrow} v_1 \overset{\alpha_{12}}{\rightarrow} v$ such that $u, v \in r$ we have $v_1 \in r$, and for every decomposition
$w \overset{\alpha_1'}{\rightarrow} u' = w \overset{\alpha_{11}'}{\rightarrow} w_1 \overset{\alpha_{12}'}{\rightarrow} u'$ such that $w, u' \in r$ we have $w_1 \in r$. $\sharp$

The following three propositions follow from the definition of a region.

5.6. Proposition. The set of all states of $\mathbf{A}$ is a region of $\mathbf{A}$. $\sharp$

5.7. Proposition. If p and q are disjoint regions of $\mathbf{A}$ then $p \cup q$ is a region of $\mathbf{A}$. $\sharp$

5.8. Proposition. If p and q are different regions of $\mathbf{A}$ such that $p \subseteq q$ then $q - p$ is a region of $\mathbf{A}$. $\sharp$

Moreover, we are also able to prove the following proposition.

5.9. Proposition. Every region of $\mathbf{A}$ contains a minimal region. $\sharp$

Proof. Let r be a region of $\mathbf{A}$ and let x be an element of r. Given a chain $(r_i : i \in I)$ of regions of $\mathbf{A}$ that are contained in r and contain and element x, for $r' = \bigcap(r_i : i \in I)$ and a transition α such that $dom(\alpha) \in r'$ and $cod(\alpha) \notin r'$, there exists $i_0 \in I$ such that $dom(\alpha) \in r_i$ and $cod(\alpha) \notin r_i$ for $i > i_0$. Consequently, for every transition α' such that $\alpha' \equiv \alpha$ we have $dom(\alpha') \in r_i$ and $cod(\alpha') \notin r_i$ for $i > i_0$, and thus $dom(\alpha') \in r'$ and $cod(\alpha') \notin r'$. Similarly, for α such that $dom(\alpha) \notin r'$ and $cod(\alpha) \in r'$ and for $\alpha' \equiv \alpha$. So, r' is a region. Consequently, in the set of regions that are contained in r and contain x there exists a minimal region. $\sharp$

The propositions 5.8 and 5.9 imply the following properties.

5.10. Proposition. Every state of $\mathbf{A}$ belongs to a minimal region. $\sharp$

5.11. Proposition. If a state s of $\mathbf{A}$ does not belong to a region r then there exists a minimal region r' such that $r \cap r' = \emptyset$ and s belongs to r'. $\sharp$

5.12. Proposition. Every region of $\mathbf{A}$ can be represented as a disjoint union of minimal regions. $\sharp$

Proof. Let m be the disjoint union of a family M of minimal regions of $\mathbf{A}$. Then m is a region of $\mathbf{A}$ and if it does not cover A then $A - m$ is a region of $\mathbf{A}$ and the family M can be extended by a minimal region of $\mathbf{A}$ that contains a given element of $A - m$ as in the proof of Proposition 5.9. Consequently, a family of disjoint minimal regions of $\mathbf{A}$ can be defined such that its union covers A. $\sharp$

6 Transitions as labelled posets

Now we shall show that elements of multiplicative transition systems can be interpreted as posets.

Let $\mathbf{A} = (A, ;)$ be an MTS.

6.1. Definition. Given $\alpha \in A$ and a cut $x = (\xi_1, \xi_2)$ of α, by a *state* corresponding to such a cut x we mean $cod(\xi_1)$, and we write such a state as $state_\alpha(x)$. $\sharp$

It is easy to see that the lattice LT_α of cuts of α viewed as a category is an MTS and that the obvious extension of the correspondence $x \mapsto state_\alpha(x)$ to the mapping mp_α from LT_α to $\mathbf{A}$ preserves the composition. Given two cuts x and y, by $x \sqcup_\alpha y$ and $x \sqcap_\alpha y$ we denote respectively the least upper bound and the greatest lower bound of x and y. The diagram $(x \leftarrow x \sqcap_\alpha y \rightarrow y, x \rightarrow x \sqcup_\alpha y \leftarrow y)$ is a bicartesian square in LT_α. From (A6) it follows that the image under the mapping mp_α of such a diagram is a bicartesian square in $\mathbf{A}$.

6.2. Example. Consider the MTS $\mathbf{A}'_{v,v'}$ in example 3.8. For the transition $\kappa = \sigma'(\pi \parallel \rho)\sigma''$ of this MTS we have the MTS LT_κ shown in figure 6.1 and its minimal regions

$i = \{(u, \kappa)\}$,

$j = \{(\sigma', (\pi \parallel \rho)\sigma''), ..., (\sigma'(\pi \parallel dom(\rho)), (cod(\pi) \parallel \rho)\sigma'')\}, ...,$

$j' = \{(\sigma'(dom(\pi) \parallel \rho), (\pi \parallel cod(\rho))\sigma''), ..., (\sigma'(\pi \parallel \rho), \sigma'')\}, ...,$

$k = \{(\sigma', (\pi \parallel \rho)\sigma''), ..., (\sigma'(dom(\pi) \parallel \rho), (\pi \parallel cod(\rho))\sigma'')\}, ...,$

$k' = \{(\sigma'(\pi \parallel dom(\rho)), (cod(\pi) \parallel \rho)\sigma''), ..., (\sigma'(\pi \parallel \rho), \sigma'')\}$,

$l = \{(\kappa, u)\}$. $\sharp$

$$\begin{tikzcd}
 & (\sigma'(dom(\pi) \parallel \rho), (\pi \parallel cod(\rho))\sigma'') \arrow[r] & \cdots \arrow[r] & (\sigma'(\pi \parallel \rho), \sigma'') \arrow[r] & (\kappa, u) \\
(u, \kappa) \arrow[r] & (\sigma', (\pi \parallel \rho)\sigma'') \arrow[r] \arrow[u] & \cdots \arrow[r] & (\sigma'(\pi \parallel dom(\rho)), (cod(\pi) \parallel \rho)\sigma'') \arrow[u] &
\end{tikzcd}$$

LT_κ

Figure 6.1

6.3. Example. Consider the MTS $TS(M', I')$ in example 5.3. For the transition $\delta = \gamma\gamma_1 = \alpha\beta'\alpha_1\beta_1'$ of this system we have the MTS LT_δ shown in figure 6.2 and its minimal regions

$e = \{(u, \delta), (\beta, \alpha'\gamma_1), (\beta\beta_1'', \alpha''\alpha_1')\}$, $g = \{(\alpha, \beta'\gamma_1), (\gamma, \gamma_1), (\gamma\beta_1, \alpha_1')\}$,
$e' = \{(\alpha\alpha_1'', \beta''\beta_1'), (\gamma\alpha_1, \beta_1'), (\delta, u)\}$, $f = \{(u, \delta), (\alpha, \beta'\gamma_1), (\alpha\alpha_1'', \beta''\beta_1')\}$,
$h = \{(\beta, \alpha'\gamma_1), (\gamma, \gamma_1), (\gamma\alpha_1, \beta_1')\}$, $f' = \{(\beta\beta_1'', \alpha''\alpha_1'), (\gamma\beta_1, \alpha_1'), (\delta, u)\}$. $\sharp$

$$\begin{tikzcd}
(\beta\beta_1'', \alpha''\alpha_1') \arrow[r, "\alpha''"] & (\gamma\beta_1, \alpha_1') \arrow[r, "\alpha_1'"] & (\delta, u) \\
(\beta, \alpha'\gamma_1) \arrow[r, "\alpha'"] \arrow[u, "\beta_1''"] & (\gamma, \gamma_1) \arrow[r, "\alpha_1"] \arrow[u, "\beta_1"] & (\gamma\alpha_1, \beta_1') \arrow[u, "\beta_1'"] \\
(u, \delta) \arrow[r, "\alpha"] \arrow[u, "\beta"] & (\alpha, \beta'\gamma_1) \arrow[r, "\alpha_1''"] \arrow[u, "\beta'"] & (\alpha\alpha_1'', \beta''\beta_1') \arrow[u, "\beta''"]
\end{tikzcd}$$

LT_δ

Figure 6.2

Let $\mathbf{A} = (A, ;)$ be an arbitrary MTS.

Given an element α of $\mathbf{A}$, by R_α we denote the set of minimal regions of the multiplicative transition system LT_α.

Using regions of $\mathbf{A}$ we want to assign to each transition α of $\mathbf{A}$ a labelled partially ordered set (an lposet) $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$. Each element $x \in X_\alpha$ is supposed to play the role of an occurrence in α of a minimal region $l_\alpha(x)$ of $\mathbf{A}$. The partial order $\leq_\alpha$ is supposed to reflect how occurrences of minimal regions arise from other minimal occurrences.

The underlying set X_α of L_α is supposed to be defined referring to the set R_α of minimal regions of the MTS LT_α and to a relation $\vdash_\alpha$ between minimal regions of LT_α and minimal regions of $\mathbf{A}$.

We are going to show how to define the respective lposet L_α for every element of $\mathbf{A}$.

6.4. Proposition. Every minimal region $r \in R_\alpha$ is *convex* in the sense that $w \in r$ for every w such that $u \sqsubseteq_\alpha w \sqsubseteq_\alpha v$ for some $u \in r$ and $v \in r$. $\sharp$

Proof. Suppose that $r \in R_\alpha$ and $a \sqsubseteq_\alpha c \sqsubseteq_\alpha b$ for $a, b \in r$ and $c \notin r$. Define r^- to be the set of $u \in r$ such that $u \sqsubseteq_\alpha c$ or $u' \sqsubseteq_\alpha c$ for some u' that can be connected with u by a side of a bicartesian square with the nodes of the opposite side not in r. Define r^+ to be the set of $u \in r$ such that $c \sqsubseteq_\alpha u$ or $c \sqsubseteq_\alpha u'$ for some u' that can be connected with u by a side of a bicartesian square with the nodes of the opposite side not in r. There is no bicartesian square with a side connecting some $u \in r$ and $v \in r$ such that $u \sqsubseteq_\alpha c \sqsubseteq_\alpha v$ and with the nodes of the opposite side not in r because by (A5) it would imply $c \in r$. By (A5) there are no bicartesian squares with sides connecting some u' with $u \in r$ and $v \in r$ such that $u \sqsubseteq_\alpha c \sqsubseteq_\alpha v$ and with the nodes of the opposite sides not in r. Consequently, the sets r^- and r^+ are disjoint. On the other hand, r is a minimal region of LT_α and thus $r \subseteq r^- \cup r^+$. Moreover, there is no bicartesian square connecting an element of r^- with an element of r^+ and with the nodes of the opposite side not in r. Consequently, r cannot be a minimal region of LT_α as supposed. $\sharp$

In R_α there exists a partial order that can be defined as follows.

6.5. Definition. Given $x, y \in R_\alpha$, we write $x \preceq_\alpha y$ iff for every $v \in y$ there exists $u \in x$ such that $u \sqsubseteq_\alpha v$, for every $u \in x$ there exists $v \in y$ such that $u \sqsubseteq_\alpha v$, and the following conditions are satisfied:

(1) $t \in x$ iff $w \in y$, for every bicartesian square $(u \leftarrow t \rightarrow w, u \rightarrow v \leftarrow w)$ with $u \in x$ and $v \in y$,
(2) $t' \in x$ iff $w' \in y$, for every bicartesian square $(t' \leftarrow u \rightarrow v, t' \rightarrow w' \leftarrow v)$ with $u \in x$ and $v \in y$. $\sharp$

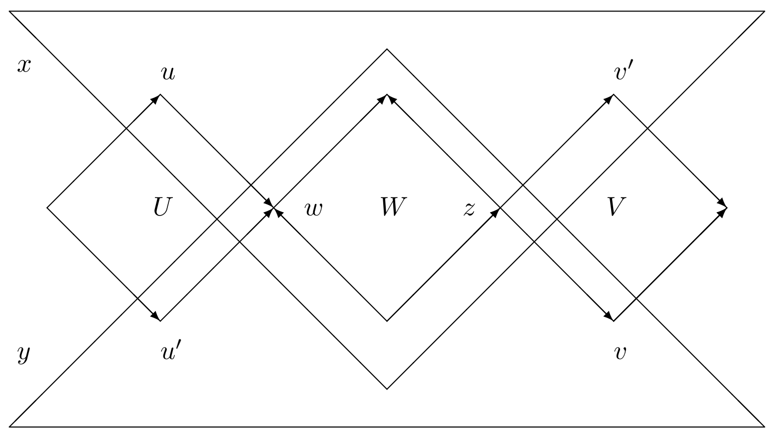
6.6. Proposition. If minimal regions $x, y \in R_\alpha$ are not disjoint and different then neither $x \preceq_\alpha y$ nor $y \preceq_\alpha x$. $\sharp$

Proof. Suppose that x and y are different minimal regions of LT_α such that $x \cap y \neq \emptyset$. Then $x - y$ and $y - x$ are nonempty and there exist $u \in x - y$,

$v \in y - x$, and $w, z \in x \cap y$ such that u and w are adjacent nodes of a bicartesian square U, z and v are adjacent nodes of a bicartesian square V, and the nodes of the bicartesian square $W = (w \leftarrow w \sqcap_\alpha z \rightarrow z, w \rightarrow w \sqcup_\alpha z \leftarrow z)$ are in $x \cap y$.

Consider the case in which $w = u \sqcup_\alpha u'$ for some u' not in x and $z = v \sqcap_\alpha v'$ for some v' not in y, as it is depicted in figure 6.3. Then $u' \in y$, $v' \in x$, and the condition (1) is not satisfied for $z \sqsubseteq_\alpha v$ and the bicartesian square $(v \leftarrow z \rightarrow v', v \rightarrow v \sqcup_\alpha v' \leftarrow v')$. Consequently, $x \preceq_\alpha y$ does not hold.

Similarly, in the other possible cases we come to the conclusion that neither $x \preceq_\alpha y$ nor $y \preceq_\alpha x$. ♯



$x, y \in R_\alpha$

Figure 6.3

6.7. Proposition. If minimal regions $x, y \in R_\alpha$ are disjoint then either $x \preceq_\alpha y$ or $y \preceq_\alpha x$. ♯

Proof. It is impossible that u and v are incomparable for all $u \in x$ and $v \in y$ since one of the regions x or y contains $u \sqcap_\alpha v$ or $u \sqcup_\alpha v$.

Suppose that $u \sqsubseteq_\alpha v$ for $u \in x$ and $v \in y$. As x and y are disjoint and convex, it suffices to prove that every element of y has a predecessor in x. Consider $w \in y$. If $v \sqsubseteq_\alpha w$ then $u \sqsubseteq_\alpha w$. If $w \sqsubseteq_\alpha v$ then $u' \sqsubseteq_\alpha w$ for $u' = u \sqcap_\alpha w$ and by considering the bicartesian square $(u \leftarrow u' \rightarrow w, u \rightarrow w' \leftarrow w)$ we obtain that $w' \in y$ because y is convex. Hence $u' \in x$. If w and v are incomparable then either $v \sqcap_\alpha w \in y$ and we may replace w by $v \sqcap_\alpha w$ and proceed as in the previous case, or $v \sqcup_\alpha w \in y$ and we may replace v by $v \sqcup_\alpha w \in y$ and proceed as in the previous case. On the other hand, $u \sqsubseteq_\alpha v$ for $u \in x$ and $v \in y$ excludes $v' \sqsubseteq_\alpha u'$ for $u' \in x$ and $v' \in y$ since x and y are convex. Hence $x \preceq_\alpha y$.

Similarly, in the case $v \sqsubseteq_\alpha u$ we obtain $y \preceq_\alpha x$. ♯

6.8. Proposition. The relation $\preceq_\alpha$ is a partial order on R_α.

Proof. The transitivity of the relation $\preceq_\alpha$ follows from the definition of this relation. The antisymmetry follows from the transitivity and from the propositions 6.6 and 6.7. $\sharp$

The relation $\vdash_\alpha$ between minimal regions of LT_α and minimal regions of $\mathbf{A}$ can be defined as follows.

6.9. Proposition. For every minimal region m of LT_α there exists a minimal region r of $\mathbf{A}$ such that the set $state_\alpha(m) = \{state_\alpha(u) : u \in m\}$ is contained in r, and we write $m \vdash_\alpha r$. $\sharp$

Proof. Given a minimal region m of LT_α, let r be a minimal element of the set of regions of $\mathbf{A}$ containing the set $state_\alpha(m)$. As the image of every bicartesian square of LT_α under the mapping mp_α from LT_α to $\mathbf{A}$ is a bicartesian square in $\mathbf{A}$, and for every partition of m into two disjoint nonempty subsets m' and m'' there exists in LT_α a bicartesian square connecting m' and m'', the same holds true for r. Consequently, r is a minimal region of $\mathbf{A}$. $\sharp$

Finally, the lposet $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ can be defined by defining X_α as the set of pairs (m, r) such that $m \in R_\alpha$ and $m \vdash_\alpha r$, the relation $\leq_\alpha$ as the partial order on X_α such that $x \leq_\alpha x'$ for $x = (m, r)$ and $x' = (m', r')$ whenever $m \preceq_\alpha m'$, and $l_\alpha(x)$ as r for $x = (m, r) \in X_\alpha$.

6.10. Example. Consider the MTS $\mathbf{A}'_{v,v'}$ described in example 3.8, its minimal regions $[(v, q)]$, $[(v', r)]$ described in example 5.2, and the minimal regions i, j,...,j', k,...,k', l of LT_κ for $\kappa = \sigma'(\pi \parallel \rho)\sigma''$ as in example 6.2. We obtain $L_\kappa = (X_\kappa, \leq_\kappa, l_\kappa)$, where

$$X_\kappa = \{(i, [(p, q_0 + m)]), (i, [(d, r_0 - m)]), (j, [(p, q_0)]), ..., (j', [(p, q_1)]),$$
$$(k, [(d, r_0)]), ..., (k', [(d, r_1)]), (l, [(p, q_1 - m')]), (l, [(d, r_1 + m')])\},$$

$$(i, [(p, q_0 + m)]), (i, [(d, r_0 - m)]) \leq_\kappa$$
$$\{(j, [(p, q_0)]) \leq_\kappa ... \leq_\kappa (j', [(p, q_1)])\}, \{(k, [(d, r_0)]) \leq_\kappa ...$$
$$\leq_\kappa (k', [(d, r_1)])\}$$
$$\leq_\kappa (l, [(p, q_1 - m')]), (l, [(d, r_1 + m')]),$$

$$l_\kappa((i, [(p, q_0 + m)])) = [(p, q_0 + m)],\ l_\kappa((j, [(p, q_0)])) = [(p, q_0)],$$
$$l_\kappa((j', [(p, q_1)])) = [(p, q_1)],\ l_\kappa((k, [(d, r_0)])) = [(d, r_0)], ...,$$
$$l_\kappa((k', [(d, r_1)]) = [(d, r_1)],\ l_\kappa((l, [(p, q_1 - m')])) = [(p, q_1 - m')],$$
$$l_\kappa((l, [(d, r_1 + m')])) = [(d, r_1 + m')].$$

The corresponding $[L_\kappa]$ is essentially as that in figure 3.3. $\sharp$

6.11. Example. Consider the MTS $TS(M', I')$ described in example 5.3, its minimal regions E, F, G, H, and the minimal regions e, g, e', f, h, f' of LT_δ for $\delta = \gamma\gamma_1 = \alpha\beta'\alpha_1\beta_1'$ as in example 6.3. We obtain $L_\delta = (X_\delta, \leq_\delta, l_\delta)$, where

$X_\delta = \{(e, E), (g, G), (e', E), (f, F), (h, H), (f', F)\}$,
$(e, E) \leq_\delta (g, G) \leq_\delta (e', E)$, $(f, F) \leq_\delta (h, H) \leq_\delta (f', F)$,
$l_\delta((e, E)) = l_\delta((e', E) = E$, $l_\delta((g, G)) = G$,
$l_\delta((f, F)) = l_\delta((f', F)) = F$, $l_\delta((h, H)) = H$.

The corresponding $[L_\delta]$ is presented in figure 6.4. ♯

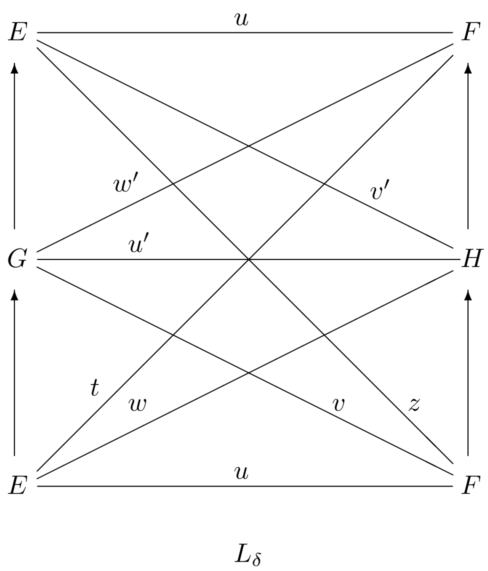


Figure 6.4

6.12. Proposition. For every element u of LT_α, and for every $x, y \in R_\alpha$ such that $x \preceq_\alpha y$, and $x \preceq_\alpha x'$ for some $x' \in X_\alpha$ such that $u \in x'$, and $y' \preceq_\alpha y$ for some $y' \in X_\alpha$ such that $u \in y'$, there exists $z \in X_\alpha$ such that $u \in z$, and $x \preceq_\alpha z$, and $z \preceq_\alpha y$. ♯

Proof. For $x' = x$ it suffices to define z as x. For $y' = y$ it suffices to define z as y. Consider the case in which $x' \neq x$ and $y' \neq y$. By proposition 6.6 in this case x and y are disjoint, x' and x are disjoint, and y' and y are disjoint. Consequently, u does not belong to x, u does not belong to y, and, by proposition 5.11, there exists $z \in X_\alpha$ that is disjoint both with x and with y, as required. ♯

Crucial for a representation of behaviour-oriented partial categories are the properties of **A** described in proposition 6.12 and in the following propositions.

6.13. Proposition. Every two different minimal regions x and y of LT_α such that $x \vdash_\alpha r$ and $y \vdash_\alpha r$ for a minimal region r of $\mathbf{A}$ are disjoint. $\sharp$

Proof. The correspondence between $u \xrightarrow{\delta} v$ such that $u = (\xi_1, \xi_2)$, $v = (\eta_1, \eta_2)$, $\eta_1 = \xi_1\delta$, $\xi_2 = \delta\eta_2$ and $mp_\alpha(u) \xrightarrow{\delta} mp_\alpha(v)$ is a functor F_α from LT_α to $\mathbf{A}$. Due to (A6) this functor preserves bicartesian squares and, consequently, $mp_\alpha^{-1}(r)$ is a region in LT_α. Indeed, the image of a bicartesian square
$D = (v \leftarrow t \rightarrow w, v \rightarrow u \leftarrow w)$ of LT_α under F_α is a bicartesian square
$E = (v' \leftarrow t' \rightarrow w', v' \rightarrow u' \leftarrow w')$ of $\mathbf{A}$ since otherwise due to (A5) there would be a bicartesian square
$E' = (v' \leftarrow t'' \rightarrow w', v' \rightarrow u'' \leftarrow w')$ that would be the image of a diagram
$D' = (v \leftarrow \overline{t} \rightarrow w, v \rightarrow \overline{u} \leftarrow w)$ with $\overline{t} \neq t$ or $\overline{u} \neq u$, what is impossible in LT_α.

Say that elements $u, v \in mp_\alpha^{-1}(r)$ are connected if in LT_α there exists a bicartesian square S with one side with the vertices u and v and with the opposite side with the images of vertices under F_α not in r. Divide $mp_\alpha^{-1}(r)$ into parts such that different parts have no connected vertices and consider maximal decreasing chains of parts thus obtained. Each part is a region of LT_α and for every element x of this part the intersection of a chain of regions contained in this part and containing x is a region as in the proof of Proposition 5.9. Consequently, there exists a minimal region of LT_α that is contained in the considered part and contains x. Consequently, $mp_\alpha^{-1}(r)$ can be represented in a unique way as the union of disjoint minimal regions of LT_α. As these are the only minimal regions contained in $mp_\alpha^{-1}(r)$, the required conclusion follows. $\sharp$

6.14. Proposition. For every α in $\mathbf{A}$ and for $x, y \in X_\alpha$, the equality $l_\alpha(x) = l_\alpha(y)$ implies $x \leq_\alpha y$ or $y \leq_\alpha x$. $\sharp$

Proof. It suffices to take into account propositions 6.7 and 6.13. $\sharp$

7 Towards a representation

In [14] a *universe of objects* has been defined as a structure $\mathbf{U} = (V, W, ob)$ where V is a set of *objects*, W is a set of *instances* of objects from V, and ob is a mappings that assigns the respective object to each of its instances. A *concrete process* in such a universe $\mathbf{U}$ has been defined as a labelled partially ordered set $L = (X, \leq, ins)$, where

(1) X is a set (of *occurrences* of objects from V),
(2) $ins : X \rightarrow W$ is a mapping (a *labelling* that assigns an object instance to each occurrence of the respective object),
(3) $\leq$ is a partial order on X (the *flow order* or the *causal dependency relation* of L) such that

(3.1) for every object $v \in V$, the set $X|v = \{x \in X : ob(ins(x)) = v\}$ is either empty or it is a maximal chain and has an element in every cross-section,
(3.2) every element of X belongs to a cross-section,
(3.3) no segment of L is isomorphic to its proper subsegment.

and an *abstract process* has been defined as an isomorphism class of concrete processes. It has been shown that for every abstract processes α and β such that the source of β is the target of α there exists exactly one abstract process γ such that $\alpha;\beta$ defined as γ *consist* of α *followed* by β, and that the set of processes in $\mathbf{U}$ with the operation $(\alpha,\beta) \mapsto \alpha;\beta$ is a partial category (a *behaviour oriented partial category*). It has been shown that for every abstract processes α and β in disjoint sets of objects there exists exactly one abstract process γ such that $\alpha \parallel \beta$ defined as γ consists of parallel abstract processes α and β.

The construction of the labelled poset $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ for every element α of an MTS $\mathbf{A}$ is such that due to the properties (A1) - (A4) of $\mathbf{A}$ we obtain that no segment of L_α is isomorphic to its subsegment. This suggests that elements of MTSs represent processes in a universe of objects in the sense of [14].

To see this, consider the universe $U(\mathbf{A}) = (V(\mathbf{A}), W(\mathbf{A}), ob(\mathbf{A}))$ of objects, where $V(\mathbf{A})$ is the set of decompositions of the set of states of $\mathbf{A}$ into disjoint unions of minimal regions of $\mathbf{A}$, $W(\mathbf{A})$ is the set of pairs $w = (v, r)$ consisting of a decomposition v of the set of states of $\mathbf{A}$ into a disjoint union of minimal regions of $\mathbf{A}$ and of a minmal region $r \in v$, and $(ob(\mathbf{A}))(w) = v$ for every $w = (v, r) \in W(\mathbf{A})$. Due to proposition 5.12 the sets $V(\mathbf{A})$ and $W(\mathbf{A})$ are nonempty. Given $\alpha \in A$, consider the lposet $L^*_\alpha = (X^*_\alpha, \leq^*_\alpha, l^*_\alpha)$, where X^*_α is the set of triples (m, v, r) such that such that $m \in R_\alpha$ and $m \vdash_\alpha r$ and $(v, r) \in W(\mathbf{A})$, the relation $\leq^*_\alpha$ is the partial order on X^*_α such that $x \leq^*_\alpha x'$ for $x = (m, r, v)$ and $x' = (m', r', v')$ whenever $m \preceq_\alpha m'$ and $r = r'$ implies $v = v'$ and $m = m'$ implies $r = r'$, and $l^*_\alpha(x) = (v, r)$ for $x = (m, r, v) \in X^*_\alpha$. As the minimal regions of every decomposition $v \in V(\mathbf{A})$ are disjoint, due to propositions 5.6, 5.12, 6.6, 6.7 we obtain that the set $X^*_\alpha|v = \{x \in X^*_\alpha : (ob(\mathbf{A}))(l^*_\alpha(x)) = v\}$ is a chain and has an element in every cross-secton of L^*_α. Moreover, $X^*_\alpha|v$ is a maximal chain since otherwise every $x = (m, r, v) \in X^*_\alpha|v$ would be comparable with $x' = (m', r', v')$ for some $v' \neq v$ and, consequently, there would be $r = r'$ for every $x = (m, r, v) \in X^*_\alpha|v$ and this would imply $v = v'$. Hence, taking into account (A4), we obtain that L^*_α is a concrete process in $U(\mathbf{A})$.

Thus we obtain the following proposition.

7.1. Proposition. Given a multiplicative transition system $\mathbf{A}$, the correspondence $\alpha \mapsto [L^*_\alpha] = [(X^*_\alpha, \leq^*_\alpha, l^*_\alpha)]$ between elements of $\mathbf{A}$ and pomsets is a mapping from $\mathbf{A}$ to the partial category of processes in the universe $U(\mathbf{A}) = (V(\mathbf{A}), W(\mathbf{A}), ob(\mathbf{A}))$ in the sense of [14]. $\sharp$

7.2. Example. Consider the MTS represented by the diagram in figure 7.1, where $\alpha\beta' = \beta\alpha' \neq \varphi$. In this diagram $(q \xleftarrow{\alpha} p \xrightarrow{\beta} r, q \xrightarrow{\beta'} s \xleftarrow{\alpha'} r)$ is a bicartesian square, the sets $pq = \{p, q\}$, $pr = \{p, r\}$, $qs = \{q, s\}$, $rs = \{r, s\}$ are minimal regions, and $X = \{pq, rs\}$, $Y = \{pr, qs\}$ are decompositions of the set of states into disjoint unions of minimal regions. For the transition φ the lattice LT_φ of decompositions of this transition consists of the least element $a = (p, \varphi)$ and the greatest element $b = (\varphi, s)$. Consequently, L^*_φ is a transition as shown in figure 7.2 and it is identical with L^{**}_φ. $\sharp$

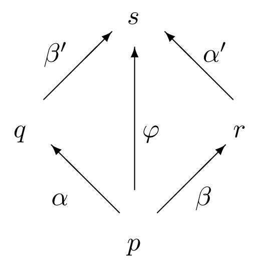


Figure 7.1

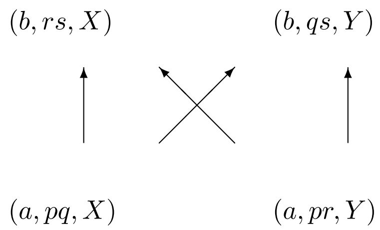


Figure 7.2

Note that the correspondence $\alpha \mapsto [L^*_\alpha] = [(X^*_\alpha, \leq^*_\alpha, l^*_\alpha)]$ need not be a homomorphism. To see this it suffices to consider a MTS $\mathbf{A}$ that is the reduct of an algebra of transitions, and in this MTS a transition $\gamma = \alpha\beta$, where $\alpha = dom(\varphi) \parallel \psi$ and $\beta = \varphi \parallel cod(\psi)$. It is easy to see that $[L^*_\gamma] \neq [L^*_\alpha][L^*_\beta]$.

However, every transition L^*_α can be transformed into a process L^{**}_α such that the correspondence $\alpha \mapsto [L^{**}_\alpha]$ is a homomorphism. This can be done as follows.

The fact that all $(m, r, v) \in X^*_\alpha$ with the same r and v form a chain implies the following proposition.

7.3. Proposition. The following relation between elements of X^*_α is an equivalence relation: $(m, r, v) \simeq_\alpha (m', r', v')$ iff $v' = v$, $r' = r$, $m \vdash_\alpha r$, $m' \vdash_\alpha r$, and $m'' \vdash_\alpha r$ for all m'' such that $m \sqsubseteq_\alpha m'' \sqsubseteq_\alpha m'$ or $m' \sqsubseteq_\alpha m'' \sqsubseteq_\alpha m$. $\sharp$

Due to this proposition we obtain the following proposition.

7.4. Proposition. The triple $L_\alpha^{**} = (X_\alpha^{**}, \leq_\alpha^{**}, l_\alpha^{**})$ with $X_\alpha^{**} = X_\alpha^*/\simeq_\alpha$, $x \leq_\alpha^{**} x'$ iff $(m,r,v) \leq_\alpha^* (m',r',v')$ for all $(m,r,v) \in x$ and $(m',r',v') \in x'$, and $l_\alpha^{**}(x) = l_\alpha^*(m,r,v)$ for $(m,r,v) \in x$, is a concrete process in $U(\mathbf{A})$. $\sharp$

7.5. Example. Consider a system M consisting of machines M_1 and M_2 which work independently as shown in figure 7.3 and execute jointly an action γ that leads M_1 to the state a and M_2 to the state c if M_1 comes to the state b and M_2 comes to the state d.

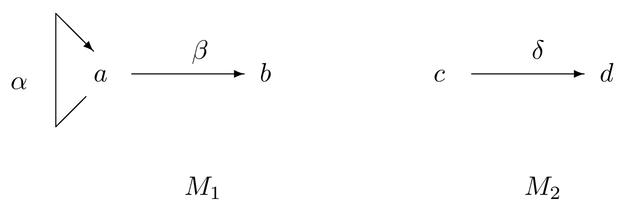


Figure 7.3

In this system we have among others the following transitions:

- a, b, c, d are transitions reducing to their initial (and final) states,
- $a \parallel c$, $a \parallel d$, $b \parallel c$, $b \parallel d$ are transitions identical with their initial and final states,
- α is an atomic transition with the initial state a and the final state a,
- β is an atomic transition with the initial state a and the final state b,
- γ is an atomic transition with the initial state $b \parallel c$ and the final state $a \parallel d$,
- δ is an atomic transition with the initial state c and the final state d,
- $\alpha \parallel \delta$ is a transition with the initial state $a \parallel c$ and the final state $a \parallel d$ that consists of parallel transions α and δ,
- an execution of α twice is a transition with the initial state a and the final state a that consists of α followed by α,
- an infinite repetition of α that begins but never ends is a transition with the initial state a and no final state, etc.

In particular, we have transitions $a \parallel c$, $a \parallel d$, $b \parallel c$, $b \parallel d$, $\alpha_c = \alpha \parallel c$, $\alpha_d = \alpha \parallel d$, $\beta_c = \beta \parallel c$, $\beta_d = \beta \parallel d$, γ, $\delta_a = \delta \parallel a$, $\delta_b = \delta \parallel b$,

The system is an MTS with bicartesian squares
$(a \parallel c \overset{\alpha_c^m}{\leftarrow} a \parallel c \overset{\delta_a}{\to} a \parallel d, a \parallel c \overset{\delta_a}{\to} a \parallel d \overset{\alpha_d^m}{\leftarrow} a \parallel d)$,
$(b \parallel c \overset{\beta_c}{\leftarrow} a \parallel c \overset{\delta_a}{\to} a \parallel d, b \parallel c \overset{\delta_b}{\to} b \parallel d \overset{\beta_d}{\leftarrow} a \parallel d)$,
minimal regions
$A = \{a \parallel c, a \parallel d\}$, $B = \{b \parallel c, b \parallel d\}$, $C = \{a \parallel c, b \parallel c\}$, $D = \{a \parallel d, b \parallel d\}$,
and decompositions $P = \{A, B\}$, $Q = \{C, D\}$ of the set of states into disjoint unions of minimal regions.

The respective universe of objects is $\mathbf{U}(\mathbf{A_1}) = (V(\mathbf{A_1}), W(\mathbf{A_1}), ob(\mathbf{A_1}))$,
where $W(\mathbf{A_1}) = \{A, B, C, D\}$, $V(\mathbf{A_1}) = \{P, Q\}$,
$(ob(\mathbf{A_1}))(A) = (ob(\mathbf{A_1}))(B) = P$, $(ob(\mathbf{A_1}))(C) = (ob(\mathbf{A_1}))(D) = Q$.
For every transition π of we have the corresponding lattice LT_π of decompositions of π, the corresponding set R_π of minimal regions of this lattice, the corresponding partial order $\preceq_\pi$ on R_π, and the corresponding transition L^*_π in $\mathbf{U_1}$. For example, for $\pi = \alpha_c\beta_c\delta_b\gamma\beta_c$ we have the lattice of decompositions of π shown in figure 7.4, the set
$R_\pi = \{x, y, z, p, q, r, s\}$ of minimal regions, where

$x = \{(a \parallel c, \pi)\} \vdash_\pi A, C,$
$y = \{(\alpha_c, \beta_c\delta_b\gamma\beta_c), (\alpha_c\delta_a, \beta_d\gamma\beta_c)\} \vdash_\pi A$
$z = \{(\alpha_c\beta_c, \delta_b\gamma\beta_c), (\alpha_c\beta_c\delta_b, \gamma\beta_c)\} \vdash_\pi B$
$p = \{(\alpha_c, \beta_c\delta_b\gamma\beta_c), (\alpha_c\beta_c, \delta_b\gamma\beta_c)\} \vdash_\pi C$
$q = \{(\alpha_c\delta_a, \beta_d\gamma\beta_c), (\alpha_c\beta_c\delta_b, \gamma\beta_c)\} \vdash_\pi D$
$r = \{(\alpha_c\beta_c\delta_b\gamma, \beta_c)\} \vdash_\pi A, C$
$s = \{(\pi, b \parallel c))\} \vdash_\pi B, C$

the process L^*_π in $\mathbf{U_1}$ shown in figure 7.5, and the corresponding process L^{**}_π in $\mathbf{U_1}$ shown in figure 7.6. $\sharp$

$$\begin{tikzcd}
 & (\alpha_c\delta_a, \beta_d\gamma\beta_c) \arrow[r, "\beta_d"] & (\alpha_c\beta_d\delta_b, \gamma\beta_c) \arrow[r, "\gamma"] & (\alpha_c\beta_c\delta_b\gamma, \beta_c) \arrow[r, "\beta_c"] & (\pi, b \parallel c) \\
(a \parallel c, \pi) \arrow[r, "\alpha_c"'] & (\alpha_c, \beta_c\delta_b\gamma\beta_c) \arrow[u, "\delta_a"] \arrow[r, "\beta_c"'] & (\alpha_c\beta_c, \delta_b\gamma\beta_c) \arrow[u, "\delta_b"] & &
\end{tikzcd}$$

Figure 7.4

$$\begin{tikzcd}
(x, C, Q) \arrow[r] \arrow[dr] & (p, C, Q) \arrow[r, "\delta"] & (q, D, Q) \arrow[r] \arrow[dr] & (r, C, Q) \arrow[r] \arrow[dr] & (s, C, Q) \\
(x, A, P) \arrow[r, "\alpha_c"'] \arrow[ur] & (y, A, P) \arrow[r, "\beta"'] & (z, B, P) \arrow[r, "\gamma"'] \arrow[ur] & (r, A, P) \arrow[r, "\beta_c"'] \arrow[ur] & (s, B, P)
\end{tikzcd}$$

Figure 7.5

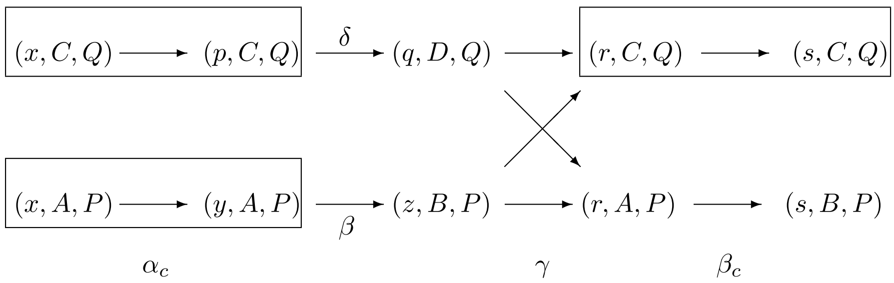


Figure 7.6

Now we want to prove that the correspondence
$\alpha \mapsto [L^{**}_\alpha] = [(X^{**}_\alpha, \leq^{**}_\alpha, l^{**}_\alpha)]$ between elements of an MTS $\mathbf{A}$ and processes in the universe $U(\mathbf{A}) = (V(\mathbf{A}), W(\mathbf{A}), ob(\mathbf{A}))$ of objects enjoys the following property.

7.6. Proposition. If $\gamma = \alpha\beta$ with $cod(\alpha) = dom(\beta) = c$ then L^{**}_γ is the pushout object in the category **LPOSETS** of the injections of L^{**}_c in L^{**}_α and in L^{**}_β. $\sharp$

Proof. Let $d \in LT_\gamma$ be the cut (α, β) of γ. The correspondence $i_\alpha : (\alpha_1, \alpha_2) \mapsto (\alpha_1, \alpha_2\beta)$ is an isomorphism between the lattice LT_α and the sublattice $LT_{\gamma,\alpha}$ of LT_γ consisting of the cuts between $(dom(\gamma), \gamma)$ and (α, β). Similarly, the correspondence $i_\beta : (\beta_1, \beta_2) \mapsto (\alpha\beta_1, \beta_2)$ is an isomorphism between the lattice LT_β and the sublattice $LT_{\gamma,\beta}$ of LT_γ consisting of the cuts between (α, β) and $(\gamma, cod(\gamma))$.

Let r be a region of LT_γ and let r_α and r_β be respectively the part of r in $LT_{\gamma,\alpha}$ and the part of r in $LT_{\gamma,\beta}$. Every bicartesian square that is contained in $LT_{\gamma,\alpha}$ and has a side outside of r_α must be disjoint with r_α or must have the entire opposite side in r_α. Consequently, r_α is a region of $LT_{\gamma,\alpha}$. Similarly, r_β is a region of $LT_{\gamma,\beta}$.

Due to (A6) every bicartesian square that is contained in LT_γ and has a side in r_α and the opposite side disjoint with r can be decomposed into two bicartesian squares of which one has a side in r_α and the opposite side disjoint with r_α. Consequently, r_α is a minimal region of $LT_{\gamma,\alpha}$ whenever r is a minimal region of LT_γ, and $r_\alpha \subseteq m$ for every minimal region of LT_γ that contains m. Similarly, every bicartesian square that is contained in LT_γ and has a side in r_β and the opposite side disjoint with r can be decomposed into two bicartesian squares of which one has a side in r_β and the opposite side disjoint with r_β.

Consequently, r_β is a minimal region of $LT_{\gamma,\beta}$ whenever r is a minimal region of LT_γ, and $r_\alpha \subseteq n$ for every minimal region of LT_γ that contains n.

Thus every minimal region r of LT_γ has a part r_α in $LT_{\gamma,\alpha}$ and a part r_β in $LT_{\gamma,\beta}$, these parts are minimal regions of $LT_{\gamma,\alpha}$ and $LT_{\gamma,\beta}$, respectively, and they determine r uniquely. Moreover, if both r_α and r_β are nonempty then, due to the convexity of minimal regions of LT_γ, the cut $d = (\alpha, \beta)$ belongs to r.

Exploiting these facts we can verify that $(L^{**}_\alpha \stackrel{k_{\gamma,\alpha}}{\rightarrow} L^{**}_\gamma \stackrel{k_{\gamma,\beta}}{\leftarrow} L^{**}_\beta)$ is a pushout of $(L^{**}_\alpha \stackrel{j_{\alpha,c}}{\leftarrow} L^{**}_c \stackrel{j_{\beta,c}}{\rightarrow} L^{**}_\beta)$ with
$j_{\alpha,c} : [m, r, v] \mapsto [m', r, v]$ for m containing (c, c) and m' containing (α, c)
$j_{\beta,c} : [m, r, v] \mapsto [m', r, v]$ for m containing (c, c) and m' containing (c, β)
$k_{\gamma,\alpha} : [m, r, v] \mapsto [m', r, v]$ for m containing (α_1, α_2) and m' containing $(\alpha_1, \alpha_2\beta)$
$k_{\gamma,\beta} : [m, r, v] \mapsto [m', r, v]$ for m containing (β_1, β_2) and m' containing $(\alpha\beta_1, \beta_2)$
$\sharp$

Consequently, we obtain the following result.

7.7. Proposition. Given a multiplicative transition system $\mathbf{A}$, the correspondence $\alpha \mapsto [L^{**}_\alpha] = [(X^{**}_\alpha, \leq^{**}_\alpha, l^{**}_\alpha)]$ between elements of $\mathbf{A}$ and processes in the universe $U(\mathbf{A}) = (V(\mathbf{A}), W(\mathbf{A}), ob(\mathbf{A}))$ of objects is a homomorphism from $\mathbf{A}$ to the partial category of processes in $U(\mathbf{A})$. $\sharp$

8 Partial order of transitions

The representation of each transition α of a multiplicative transition system $\mathbf{A}$ by an lposet $L^*_\alpha = (X^*_\alpha, \leq^*_\alpha, l^*_\alpha)$ can be exploited as a basis of a formal definition of the parallel composition of transitions and of the corresponding partial order of transitions. To this end it suffices to define the parallel composition of abstract transitions in disjoint sets of objects from $V(\mathbf{A})$ as the partial operation $(\alpha, \beta) \mapsto \alpha \parallel \beta$, where $\alpha \parallel \beta$ consists of parallel pomsets α and β. Then the inclusion of an abstract transition α in an abstract transition β can be defined as the relation $\ll$ that is satisfied iff α is an independent component of β. Due to 6.14 the relation $\ll$ is a partial order such that every two elements α and β have the greatest lower bound $\alpha \bigtriangleup \beta$, and for every α_1, α_2, β_1, β_2 such that $\alpha_1; \alpha_2$ and $\beta_1; \beta_2$ are defined also $(\alpha_1 \bigtriangleup \beta_1); (\alpha_2 \bigtriangleup \beta_2)$ is defined and $(\alpha_1; \alpha_2) \bigtriangleup (\beta_1; \beta_2) = (\alpha_1 \bigtriangleup \beta_1); (\alpha_2 \bigtriangleup \beta_2)$.

9 Concluding remarks

Making use of the fact that runs of a system and a composition of such runs form a partial algebra satisfying a set of axioms, we have defined a multiplicative transition system, MTS, as an arbitrary partial algebra satisfying this set of axioms, and we have shown that every MTS can be viewed as a partial category of processes in a universe of objects. As elements of an MTS may represent decomposable runs, algebras of this type become a universal framework for describing systems that may exhibit any combination of discrete and continuous behaviour. As every MTS can be viewed as a partial category of processes in a universe of objects, such processes become universal basic structures for representing arbitrary system runs.

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