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An Algebraic Framework for Concurrent Systems

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Contents

Preface

This book contributes with a concept of a process viewed as a model of a run of a system (discrete, continuous, or of a mixed type), with operations allowing to define complex processes in terms of their components, and with the idea of using the formal tools thus obtained to define the behaviours of concurrent systems.

A process may have an initial state (a source), a final state (a target), or both. Processes of which one is a continuation of the other can be composed sequentially. Independent processes, i.e. processes which do not disturb each other, can be composed in parallel. Processes may be prefixes, i.e. independent components of initial segments of other processes. Processes and operations on processes are represented by partially ordered multisets of a certain type and operations on such multisets.

Processes in a universe of objects and the sequential composition of processes form a partial category, called a partial category of processes. Processes in a universe of objects and the operations of composing processes sequentially and in parallel form a partial algebra, called an algebra of processes. Partial categories and algebras of processes belong to axiomatically defined classes of partial algebras, called behaviour-oriented partial categories and behaviour-oriented algebras. Some of behaviour-oriented partial categories and behaviour-oriented algebras can be represented as partial categories of processes and algebras of processes.

Partial categories and algebras of processes can be used to define behaviours of concurrent systems. Namely, the behaviour of a system can be defined as the set of possible processes of this system with a structure on this set. The structure reflexes the prefix order and makes the set of possible processes a directed complete poset.

Partial categories and algebras of processes can also be used to define behaviours with states and processes provided with specific structures, to define operations on behaviours similar to those in the existing calculi of behaviours, and to define random behaviours.

1

Introduction

Motivation

In this book an algebraic approach to defining behaviours of concurrent systems is presented with the intention to develop an approach universal enough to cope with systems that may exhibit arbitrary combination of discrete and continuous behaviour. There are reasons for which we need such a universal approach.

In order to describe and analyse systems including computer components, which operate in discrete steps, and real-world components, which operate in a continuous way, an approach is needed that includs ideas from both compute science and control theory (cf. [LSV 07]). Consequently, a simple language is needed to describe in the same way and to relate behaviours of systems of any nature, including discrete, continuous, and hybrid systems. This will allow one to avoid inventing a special way in every particular case.

The presented idea of a universal approach to defining behaviours of concurrent systems consists in regarding such systems as generalized transition systems.

Usual transition systems are models of systems which operate in discrete steps (cf. [RT 86] and [NRT 90]). They specify system states and transitions between states, the latter supposed to be indivisible. Consequently, they have means to represent implicitly partial and complete system runs viewed as sequences of successive transitions. They can be provided in a natural way with a composition of runs of which one starts from the final state of the other, and this results in the structure of a partial category.

In the case of systems with continuous behaviour runs cannot be viewed as sequences of discrete steps. Nevertheless, the concept of a run still makes sense, and there is a natural composition of runs of which one starts from the resulting state of the other (a sequential composition). Moreover, the continuity can be expressed as infinite divisibility of runs with respect to such a composition. Moreover, we have not only global states and runs of entire system, but also local states and runs of system components and their sequential composition, and also a natural composition of local runs which do not disturb each other (a parallel composition).

Consequently, the behaviour of a concurrent system can be defined as the set of possible partial and complete runs of the system and system components, and the structure on this set that follows from the existence of the compositions. We call such runs processes and represent them and their compositions as elements as elements and operations of some algebras.

Note that by processes we mean runs of the system or its subsystems, or segments of such runs.[1]

Every process may have an initial state (a source), a final state (a target), or both. Every process with an initial state and a final state is said to be bounded. Processes of which one is a continuation of the other can be composed sequentially. Independent processes, i.e. processes of subsystems which do not disturb each other, can be composed in parallel. Processes may be prefixes, i.e. independent components of initial segments of other processes, and this relation is a partial order. The set of possible system processes is prefix-closed and directed complete. The structure on this set reflects how processes compose and the prefix order.

1.1. Example. Consider a system M consisting of machines M_1 and M_2 which work independently as shown in figure 1.1 and execute jointly an action γ that leads M_1 to the state a and M_2 to the state c if M_1 comes to the state b and M_2 comes to the state d.

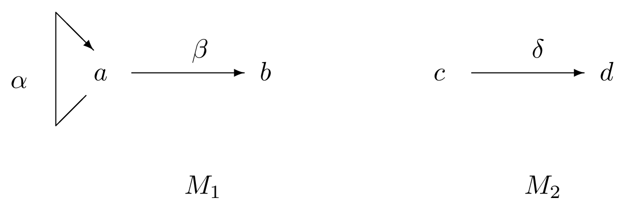
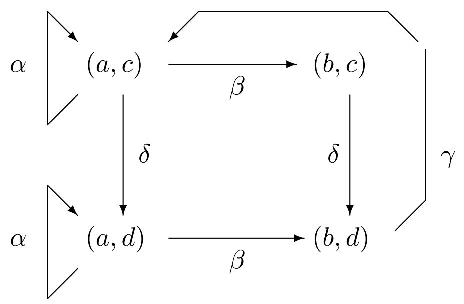


Figure 1.1

The transition system representing the possible states and actions of M is shown in figure 1.2.

[1] Note that our understanding of a process as a run of a system, as in the theory of Petri nets (cf. for example [BD 87], [RT 86], [DMM 89]), is different from that in the known calculi of behaviours (cf. for example [BK 84], [Miln 80], [Miln 96]), where a process means a behaviour.



M

Figure 1.2

The behaviour of M consists of processes of M_1 and M_2 represented by paths in the transition systems of M_1 and M_2 in figure 1.1, and of processes of entire system M, each process represented by a path in the transition system of M in figure 1.2. In particular, the behaviour of M contains the following processes:

- a, b, c, d are processes reducing to their initial (and final) states,
- (a, c), (a, d), (b, c), (b, d) (or, equivalently, the results $a + c$, $a + d$, $b + c$, $b + d$ of composing in parallel respectively a and c, a and d, b and c, b and d) are processes identical with their initial and final states,
- α is a process with the initial state a and the final state a,
- β is a process with the initial state a and the final state b,
- γ is a process with the initial state (b, c) and the final state (a, d),
- δ is a process with the initial state c and the final state d,
- an independent execution of α and δ is process with the initial state (a, c) and the final state (a, d) (the result $\alpha + \delta$ of composing α and δ in parallel),
- an execution of α twice is a process with the initial state a and the final state a (the result $\alpha\alpha$ of composing α and α sequentially),
- an infinite repetition of α that begins but never ends is a process with the initial state a and no final state (the result α^{ω} of composing α sequentially infinitely many times with a start),
- an infinite repetition of α without beginning that ends is a process with the final state a and no initial state (the result $\alpha^{-\omega}$ of composing α sequentially infinitely many times with an end),
- an infinite repetition of α that never begins and never ends is a process with no initial state and no final state (the result $\alpha^{-\omega,\omega}$ of composing α sequentially infinitely many times without a start and without an end), etc.

Moreover, process $\alpha+\delta$ has prefixes a, c, α, δ, the result $a+c$ of composing in parallel a and c, the result $\alpha+c$ of composing in parallel α and c, the result $a+\delta$ of composing in parallel a and δ, and entire $\alpha+\delta$. $\sharp$

Processes and algebras of processes

In order to develop our approach we formulate first a general, system independent definition of processes, define partial operations of composing processes, and define the respective algebras of processes.

Processes are thought as activities in a universe of objects, each object with a set of possible internal states and instances corresponding to these states, each activity changing states of some objects, where changes are viewed as replacements of the existing occurrences of active objects by new occurrences. They are independent whenever they represent activities in disjoint subsets of the universe.

For example, processes of the system M of machines M_1 and M_2 can be thought as activities in the universe that consists of M_1 and M_2.

We propose to represent processes of any kind (discrete, continuous, and partially discrete - partially continuous) as specific labelled partially ordered sets (lposets), where a partial order represents causality. In order to define operations on processes we identify isomorphic processes and represent them by the respective isomorphism classes, called partially ordered multisets (pomsets).

Processes in a universe of objects, and operations of composing such processes, constitute a partial algebra
$\mathbf{A} = (A, ;, +)$, where A is a set of processes, $(\alpha_1, \alpha_2) \mapsto \alpha_1;\alpha_2$, where $\alpha_1;\alpha_2$ is written also as $\alpha_1\alpha_2$, is the partial operation of composing sequentially processes of which α_1 leads to a state from which α_2 starts, $(\alpha_1, \alpha_2) \mapsto \alpha_1+\alpha_2$ is the partial operation of composing in parallel independent processes (see [Wink 09a]).

For example, processes α, β, γ, δ of the system M of machines from example 1.1 can be represented as pomsets shown in figure 1.3.

The independent execution of α and δ followed by an execution of α in presence of the state d of M_2 can be represented as the pomset $(\alpha+\delta)(\alpha+d)$ shown in figure 1.4. Similarly, the independent execution of β and δ followed by an execution of γ by M_1 and M_2 can be represented as the pomset $(\beta+\delta)\gamma$ shown in figure 1.4.

The parallel composition of processes reflects the independence of processes. Moreover, it allows one also to represent processes in arbitrary contexts. In particular, processes in which only some objects are involved can be represented with any degree of locality due to the possibility of composing them in parallel with states of sets of objects that are not involved. For example, the process α of machine M_1 can be represented both as a local process of M_1 and as a global process $\alpha+c$ of entire system M.

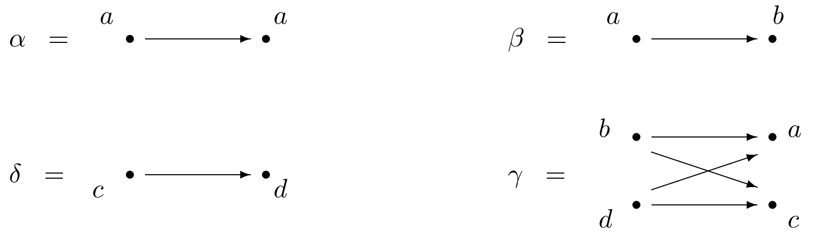


Figure 1.3

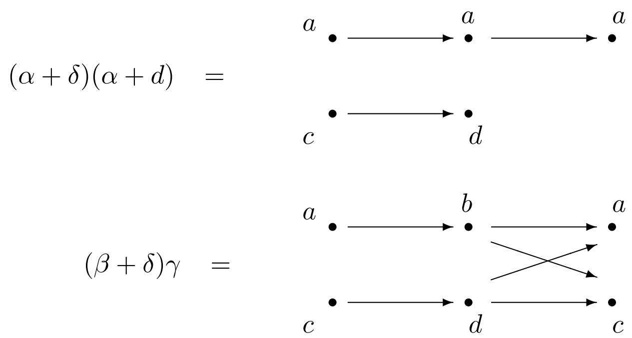


Figure 1.4

The introduced notions allow us to define the respective prefix relation and represent the behaviours of concurrent systems as prefix-closed directed complete partially ordered subsets of algebras of processes in suitable universa of objects, the subsets equipped with structures which reflect how processes compose, the prefix order, and possibly specific features of the represented behaviours. Following the existing in computer science terminology, we call constructs thus obtained behaviours, and follow the idea of [WiMa 87] to define typical operations on such constructs.

Moreover, we show how to apply our approach to systems which show random behaviours. In order to characterize such behaviours we define for each system an adequate probability space.

Due to the universal nature of our process concept and due to the characterization of behaviours of systems in terms of processes, our approach applies not only to discrete systems but also to continuous and hybrid systems.

Algebras of processes as axiomatically defined algebras

For every algebra of processes in a universe of objects the subalgebra of bounded processes is a member of an axiomatically defined class of partial algebras with axioms allowing to define independence of elements, called in [Wink 07a] behaviour algebras. These algebras generalize asynchronous systems of [Sh 85] and [Bedn 88], and transition systems with independence of [WN 95]. They are richer than the mentioned models in the sense that they allow one to represent not only states, transitions, and independence of transitions of discrete systems, but also long runs of arbitrary systems, the internal structures of runs, and how runs compose. Moreover, the independence becomes a definable notion, and it can be defined not only for transitions, but also for arbitrary runs.

In [Wink 05] it has been shown that in the case of behaviour algebras that are discrete in a sense the sets of indecomposable elements of reducts of such algebras to their categories form, together with the existing information on independence, structures close to transition systems with independence of [WN 95]. In particular, it has been shown that such structures generate freely the respective categories.

In [Wink 07a] it has been shown that the partial monoid of a behaviour algebra can be embedded homomorphically in the partial monoid of preclasses of a tolerance relation with the set theoretical union of disjoint preclasses as the operation, and that under some conditions the behaviour algebra itself can be embedded homomorphically in the algebra of bounded processes in a universe of objects.

It has been shown also that every element of a behaviour algebra defines a unique set (the canonical underlying set) and a unique structure on this set (the canonical structure) that consists of a partial order (the canonical partial order) and of a labelling (the canonical labelling). The structures thus defined are consistent with operations on elements. In many cases they can be enriched consistently with some additional structures. This allows one to represent behaviours of systems with rich structures of states and processes. Moreover, the approach applies not only to discrete systems, but also to continuous and hybrid systems, and the continuity of a processes can be reflected as infinite divisibility of the representing element of the respective algebra.

In [Wink 09a] and [Wink 09b] the concept of behaviour algebras has been generalized. In particular, elements have been admitted which may be lacking sources or targets or both sources and targets, it has been shown how to define behaviours and probabilistic models of random behaviours, and a general concept of behaviour-oriented algebras has been introduced.

For $\mathbf{A} = (A, ;, +)$ being a behaviour-oriented algebra the reduct $(A, ;)$ is a partial category $\mathbf{pcat}(\mathbf{A})$ with definable unary partial operations $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ assigning to a morphism α respectively the source and the target of this morphism, if such a source or a target exists, and the reduct $(A, +)$ is a partial commutative monoid $\mathbf{pmon}(\mathbf{A})$. For $\mathbf{A}$ corresponding to a behaviour algebra in the sense of [Wink 07a] the reduct $\mathbf{pcat}(\mathbf{A})$ is a category of processes, $dom(\alpha)$ and $cod(\alpha)$ are defined for all processes, and they represent the initial and the final states of the respective processes.

In [Wink 11] simplified behaviour-oriented algebras, called multiplicative transition systems, have been introduced and studied, with the intention of expressing all the interesting properties of behaviours in terms of global processes and one only partial operation of composing processes sequentially. Such algebras are partial categories that enjoy the properties of the reducts of behaviour-oriented algebras to partial categories. Modifying the concept of a region as in [ER 90] and exploiting the existence of minimal regions, it has been shown that the multiplicative transition systems of a broad class can be represented as partial categories of processes.

In the present paper, whose parts have been presented in [Wink 09a], [Wink 09b], and [Wink 11], we extend and summarize these results.

Relation to other work

The presented approach concentrates on algebras whose elements and operations are supposed to represen partial and complete processes (runs) of concurrent systems and natural operations on such processes. The decision to deal with such algebras rather than with concrete systems has been taken in order to deal with a space of processes that admits the well recognized algebraic structure of a category or a partial category, and the structure of a partial monoid. This does not limit the possibilities of applications since the behaviours of systems, and systems with a distinguished initial state can be represented as subsets of those processes of the respective algebra that contain only processes of a given system, or a given system starting in a given initial state. Processes in such subsets may be prefixes of other processes, which results in a natural partial order similar to the partial order in configuration structures as those in [GP 95]. In particular, for systems with finitary processes we can derive from processes occurrences of their atomic components and next deal the sets of such occurrences as configurations of a configuration structure. However, configuration structures thus obtained are specific since the indeterminism in the underlying sets of processes is fully expressible in terms of state components.

For systems represented by Petri nets as described in Appendix F processes in our sense correspond to executions of the representing nets in the sense of the theory of Petri nets. More precisely, they correspond to executions reduced to occurrences of local situations, and thus to executions in which occurrences of transitions are represented only implicitly.

In our approach runs of a system represented by a Petri net are viewed as processes in a universe of objects, each instance of an object representing a local situation in the net. Usually, such processes form a subalgebra of the algebra of all processes in this universe, and the representing net can be viewed as a specification of the set of generators of this subalgebra.

In the case of elementary and Condition/Event net systems, that is systems whose states are given by sets of conditions, and whose transitions correspond to events which depend on and affect only some conditions, concrete executions of a net can be defined as deterministic occurrence nets, called causal nets, with a homomorphism to the so called safe completion of the original net, and isomorphic concrete executions can be identified (cf. [Wink 03] and [Wink 06] for details). In the present formulation such executions can be defined as activities in the respective universum of conditions, each condition with two instances corresponding to the states "satisfied" and "not satisfied". This way of defining processes extends easily on contextual Petri nets as those considered in [MR 95] and [BBM 02]. However, the notion of independence of processes is more subtle for contextual Petri nets since processes which share a context may be independent.

In the case of net systems based on Place/Transition Petri nets it is not enough to define concrete executions of a net as causal nets with a homomorphism to this net since the corresponding abstract executions do not contain information sufficient for defining the operations on executions and independence of executions. In [MMS 96] it has been shown that the notion of concatenable decorated processes is what one needs. This notion takes into account to some extent the identities of tokens taking part in an execution, and it makes possible to define the corresponding operations on executions and independence of executions. An essential feature of this approach is that the identification of tokens in an execution is an intrinsic property of this execution. In our approach we propose instead to regard executions as running in a fixed universe of objects which may become tokens, and such a universe is external with respect to the considered executions (see [Wink 05] for details). In the case of executions of Place/Transition nets this solution is less elegant that that in [MMS 96], but in general it may be more universal. For instance, it does not require explicit references to events as in [MMS 96] and thus is more natural for continuous systems.

Processes equipped with graph structures are close to graph processes of [CMR 96], and thus to derivations of graph grammars in the sense of the so called double pushout approach. A grammar generating derivations represented by processes from a given set of processes can be recovered by decomposing processes of this set into atoms and by defining productions as instances of atoms thus obtained. However, our approach is less flexible than the existing standard approach because it limits the set of objects (nodes and edges) which may appear in processes representing derivations of a grammar to a universe that must be fixed in advance. On the other hand, we need not restrict ourselves only to graph structures.

Our methods of representing systems and their processes and behaviours seem to be well suited for modelling object oriented computations like those that can be programmed in Java or in other similar languages. This is however a subject that requires a special presentation, and we do not resume it in the present paper.

The fact that systems and their behaviours are modelled in the framework of algebras allows one to describe in a natural way such relations between systems and their behaviours as various similarities and equivalences. But also this requires a special presentation which we do not resume in the present paper.

Summary

In chapter 2 we formalize the concept of a process. In chapter 3 we introduce operations on processes, describe their properties and define the respective algebras of processes. In chapter 4 we define behaviours of systems and we describe typical operations on behaviours. In chapter 5, we describe how the approach can be used to describe random behaviours. In chapter 6 we define abstract behaviour-oriented algebras and describe their relation to algebras of processes. In chapter 7 we describe how elements of behaviour-oriented algebras can used to represent processes provided with some structures. In chapter 8 we define behaviour-oriented partial categories and describe how they are related to partial categories of processes. In chapter 9 we describe how behaviour-oriented partial categories generated by atomic elements are related to transition systems with independence. Chapter 5 is included in order to illustrate how the approach applies to random behaviours and it is not necessary to follow the remaining parts of the material.

2

Processes

We think of processes as of activities in a universe of objects, each object with a set of possible internal states and instances corresponding to these states, each activity changing states of some objects.

Universes of objects

A universe of objects and processes in such a universe can be defined as follows.

2.1. Definition. By a *universe of objects* we mean a structure $\mathbf{U} = (V, W, ob)$, where V is a set of *objects*, W is a set of *instances* of objects from V (a set of *object instances*), and ob is a mapping that assigns the respective object to each of its instances. $\sharp$

2.2. Example. For machines M_1 and M_2 as in example 1.1, let $V_1 = \{M_1, M_2\}$, $W_1 = \{a, b, c, d\}$, $ob_1(a) = ob_1(b) = M_1$, $ob_1(c) = ob_1(d) = M_2$. Then $\mathbf{U}_1 = (V_1, W_1, ob_1)$ is a universe of objects. $\sharp$

2.3. Example. Suppose that a producer p produces some material for a distributor d. Define an instance of p to be a pair (p, q), where $q \geq 0$ is the amount of material at disposal of p. Define an instance of d to be a pair (d, r), where $r \geq 0$ is the amount of material at disposal of d. Define $V_2 = \{p, d\}$, $W_2 = W_p \cup W_d$, where $W_p = \{(p, q) : q \geq 0\}$, $W_d = \{(d, r) : r \geq 0\}$. Define $ob_2(w) = p$ for $w = (p, q) \in W_p$ and $ob_2(w) = d$ for $w = (d, r) \in W_d$. Then $\mathbf{U}_2 = (V_2, W_2, ob_2)$ is a universe of objects. $\sharp$

2.4. Example. Tokens used to mark places of a Place/Transition Petri net with a set P of places can be regarded as instances of objects from a universe $\mathbf{U}_{tokens} = (V_{tokens}, W_{tokens}, ob_{tokens})$, where $V_{tokens} = \{v_1, v_2, ...\}$ is an infinite set of objects which may serve as tokens in places of the net, each object v with the possible instances $w = (v, p)$ for p being $position(w)$, the actual position of v from a set *Positions* that contains P and two distinguished elements *source* and *sink*, and where W_{tokens} is the set of instances of objects from V_{tokens} and $ob_{tokens} : W_{tokens} \rightarrow V_{tokens}$ is the mapping that assigns the respective object to its instances, i.e., $ob_{tokens}((v, p)) = v$. $\sharp$

2.5. Example. Units of data that occur in a world can be regarded as instances of objects from a universe $\mathbf{U}_{data} = (V_{data}, W_{data}, ob_{data})$, where V_{data} is an infinite set of objects which may serve as units of data, each object v with the possible instances $w = (v, c, p)$ for c being $content(w)$, the actual content of v from a set *Contents* that contains a distinguished element *none*, and for p being $position(w)$, the actual position of v from a set *Positions* that contains two distinguished elements *source* and *sink*, and where W_{data} is the set of instances of objects from V_{data} and $ob_{data} : W_{data} \to V_{data}$ is the mapping that assigns the respective object to its instances, i.e., $ob_{data}((v, c, p)) = v$. $\sharp$

Processes

2.6. Definition. Given a universe $\mathbf{U} = (V, W, ob)$ of objects, by a *concrete process* in $\mathbf{U}$ we mean a labelled partially ordered set $L = (X, \leq, ins)$, where

(1) X is a set (of *occurrences* of objects from V, called *object occurrences*),
(2) $ins : X \to W$ is a mapping (a *labelling* that assigns an object instance to each occurrence of the respective object),
(3) $\leq$ is a partial order (the *causal dependency relation* of L) such that
  (3.1) for every object $v \in V$, the set $X|v = \{x \in X : ob(ins(x)) = v\}$ is either empty or it is a maximal chain and has an element in every cross-secton,
  (3.2) every element of X belongs to a cross-section,
  (3.3) no segment of L is isomorphic to its proper subsegment. $\sharp$

The notion of a cross-section is defined in Appendix A. Condition (3.1) means that L contains all information on the behaviour within L of every object which has in L an occurrence, and that every potential global state of L contains an element of this information The author would like to take the opportunity to explain that in the paper "Behaviour Algebras" (item [Wink 07a] of the references) the corresponding condition is too weak since it does not require every maximal chain $X|v$ to have an element in every cross-sectiona and it implies the present condition (3.1) only if the flow order is strongly K-dense. Condition (3.2) guarantees that every occurrence of an object in L belongs to a potential global state of L. Condition (3.3) allows one to distinguish every segment of L even if L is considered up to isomorphism. Note that (3.3) holds if for an object v with nonempty $X|v$ there is no flow order and labelling preserving bijection from an interval of $X|v$ to its proper subinterval.

2.7. Example. Let $\mathbf{U}_1 = (V_1, W_1, ob_1)$ be the universe described in example 2.2.

An execution of action α by the machine M_1 is a concrete process $A = (X_A, \leq_A, ins_A)$ in $\mathbf{U}_1$, where
$X_A = \{x_1, x_2\}$,
$x_1 <_A x_2$,
$ins_A(x_1) = ins_A(x_2) = a$.

An execution of action β by the machine M_1 is a concrete process $B = (X_B, \leq_B, ins_B)$ in $\mathbf{U}_1$, where
$X_B = \{x_1, x_2\}$,
$x_1 <_B x_2$,
$ins_B(x_1) = a$, $ins_B(x_2) = b$.

Joint execution of action γ by the machines M_1 and M_2 is a concrete process $C = (X_C, \leq_C, ins_C)$ in $\mathbf{U}_1$, where
$X_C = \{x_1, x_2, x_3, x_4\}$,
$x_1 <_C x_3, \quad x_1 <_C x_4, \quad x_2 <_C x_3, \quad x_2 <_C x_4,$
$ins_C(x_1) = b$, $ins_C(x_2) = d$, $ins_C(x_3) = a$, $ins_C(x_4) = c$.

An execution of action δ by the machine M_2 is a concrete process $D = (X_D, \leq_D, ins_D)$ in $\mathbf{U}_1$, where
$X_D = \{x_1, x_2\}$,
$x_1 <_D x_2$,
$ins_D(x_1) = c$, $ins_D(x_2) = d$.

Independent execution of α and δ followed by an execution of α is a concrete process $E = (X_E, \leq_E, ins_E)$ in $\mathbf{U}_1$, where
$X_E = X_{A'} \cup X_{D'} \cup X_{A''}$,
$\leq_E$ is the transitive closure of $\leq_{A'} \cup \leq_{D'} \cup \leq_{A''}$,
$ins_E = ins_{A'} \cup ins_{D'} \cup ins_{A''}$,
for variants A' and A'' of A and a variant D' of D such that the maximal element of $X_{A'}$ coincides with the minimal element of $X_{A''}$, and these are the only common elements of pairs of sets from among $X_{A'}$, $X_{D'}$, $X_{A''}$.

Independent execution of β and δ followed by an execution of γ is a concrete process $F = (X_F, \leq_F, ins_F)$ in $\mathbf{U}_1$, where
$X_F = X_{B'} \cup X_{D'} \cup X_{C'}$,
$\leq_F$ is the transitive closure of $\leq_{B'} \cup \leq_{D'} \cup \leq_{C'}$,
$ins_F = ins_{B'} \cup ins_{D'} \cup ins_{C'}$,
for a variant B' of B, a variant D' of D, and a variant C' of C such that the maximal element of $X_{B'}$ coincides with the minimal element of $X_{C'}$ with the same label, the maximal element of $X_{D'}$ coincides with the minimal element of $X_{C'}$ with the same label, and these are the only common elements of pairs of sets from among $X_{B'}$, $X_{D'}$, $X_{C'}$.

The lposets representing the concrete processes A, B, C, D, E, F are represented graphically in figure 2.1.

The isomorphism classes of lposets corresponding to the concrete processes A, B, C, D are represented graphically in figure 1.3 as α, β, γ, δ, respectively. The isomorphism classes of lposets corresponding to the concrete processes E and F are represented graphically in figure 1.4 as $(\alpha+\delta)(\alpha+d)$ and $(\beta+\delta)\gamma$, respectively. $\sharp$

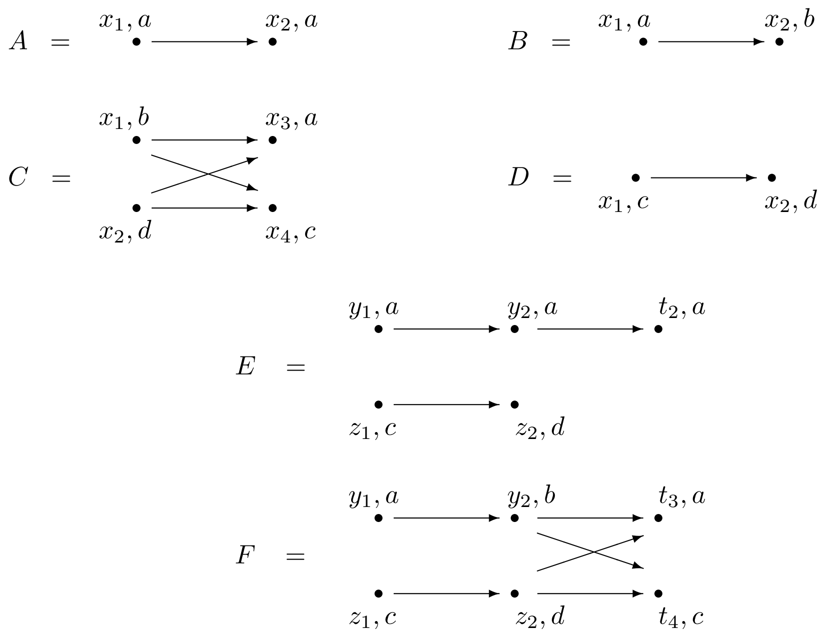


Figure 2.1

2.8. Example. Let $\mathbf{U}_2 = (V_2, W_2, ob_2)$ be the universe described in example 2.3.

Undisturbed production of material by the producer p in an interval $[t', t'']$ of global time is a concrete process $Q = (X_Q, \leq_Q, ins_Q)$ in $\mathbf{U}_2$, where X_Q is the set of values of variations $var(t \mapsto q(t); t', t)$ in intervals $[t', t] \subseteq [t', t'']$ of the real valued function $t \mapsto q(t)$ which specifies the amount of material at disposal of p at every moment of $[t', t'']$, $\leq_Q$ is the restriction of the usual order of numbers to X_Q, and $ins_Q(x) = (p, q(t))$ for $x = var(t \mapsto q(t); t', t)$. The number $var(t \mapsto q(t); t', t'')$, written as $length(Q)$, is called the length of Q. The set X_Q with the order $\leq_Q$ represents the intrinsic local time of the producer. If the material is produced in a continuous way than the function $t \mapsto q(t)$ is continuous and X_Q is a closed interval. Otherwise it ma consist of a set of disjoint intervals. If there is no uncontrolled lose of the material then the function $t \mapsto q(t)$ is increasing and $q(t'') - q(t') = length(Q)$. Otherwise $q(t'') - q(t') < length(Q)$. (We remind that the variation of a real-valued function f on an interval $[a,b]$, written as $var(f;a,b)$, is the least upper bound of the set of numbers $|f(a_1)-f(a_0)|+...+|f(a_n)-f(a_{n-1})|$ corresponding to subdivisions $a=a_0<a_1<...<a_n=b$ of $[a,b]$. In the case of more than one real-valued function the concept of variation turns into the concept of the length of the curve defined by these functions.)

Undisturbed distribution of material by the distributor d in an interval $[t',t'']$ of global time is a concrete process $R=(X_R,\leq_R,ins_R)$ in $\mathbf{U}_2$, where X_R is the set of values of variations $var(t\mapsto r(t);t',t)$ in intervals $[t',t]\subseteq[t',t'']$ of the real valued function $t\mapsto r(t)$ which specifies the amount of material at disposal of d at every moment of $[t',t'']$, $\leq_R$ is the restriction of the usual order of numbers to X_R, $and ins_R(x)=(d,q(t))$ for $x=var(t\mapsto q(t);t',t)$. The number $var(t\mapsto r(t);t',t'')$, written as $length(R)$, is called the length of R. The set X_R with the order $\leq_R$ represents the intrinsic local time of the distributor. If the material is distributed in a continuous way than the function $t\mapsto r(t)$ is continuous and X_R is a closed interval. Otherwise it ma consist of a set of disjoint intervals. If there is no uncontrolled supply of the material then the function $t\mapsto r(t)$ is decreasing and $r(t')-r(t'')=length(R)$. Otherwise $r(t')-r(t'')<length(R)$.

Transfer of an amount m of material from the producer p to the distributor d is a concrete process $S=(X_S,\leq_S,ins_S)$ in $\mathbf{U}_2$, where $X_S=\{x_1,x_2,x_3,x_4\}$, $x_1<_S x_3$, $x_1<_S x_4$, $x_2<_S x_3$, $x_2<_S x_4$, $ins_S(x_1)=(d,r)$, $ins_S(x_2)=(p,q)$, $ins_S(x_3)=(d,r+m)$, $ins_S(x_4)=(p,q-m)$. The set X_R with the order $\leq_R$ represents the intrinsic global time of the system consisting of the producer and the distributor.

Transfer of an amount of material from the producer p to the distributor d followed by independent behaviour of p and d and by another transfer of material from p to d is a concrete process $T=(X_T,\leq_T,ins_T)$ in $\mathbf{U}_2$, where $X_T=X_{Q'}\cup X_{R'}\cup X_{S'}\cup X_{S''}$, $\leq_T$ is the transitive closure of $\leq_{Q'}\cup\leq_{R'}\cup\leq_{S'}\cup\leq_{S''}$, $ins_T=ins_{Q'}\cup ins_{R'}\cup ins_{S'}\cup ins_{S''}$, for a variant Q' of Q, a variant R' of R, and variants S' and S'' of S, such that one maximal element of $X_{S'}$ coincides with the minimal element of $X_{Q'}$ with the same label and the other maximal element coincides with the minimal element of $X_{R'}$ with the same label, one minimal element of $X_{S''}$ coincides the maximal element of $X_{Q'}$ with the same label and the other minimal element coincides with the maximal element of $X_{R'}$ with the same label, and these are the only common elements of pairs of sets from among $X_{Q'}$, $X_{R'}$, $X_{S'}$, $X_{S''}$.

The abstract processes corresponding to the concrete processes Q, R, S, and T, are represented graphically in figure 2.2. $\sharp$

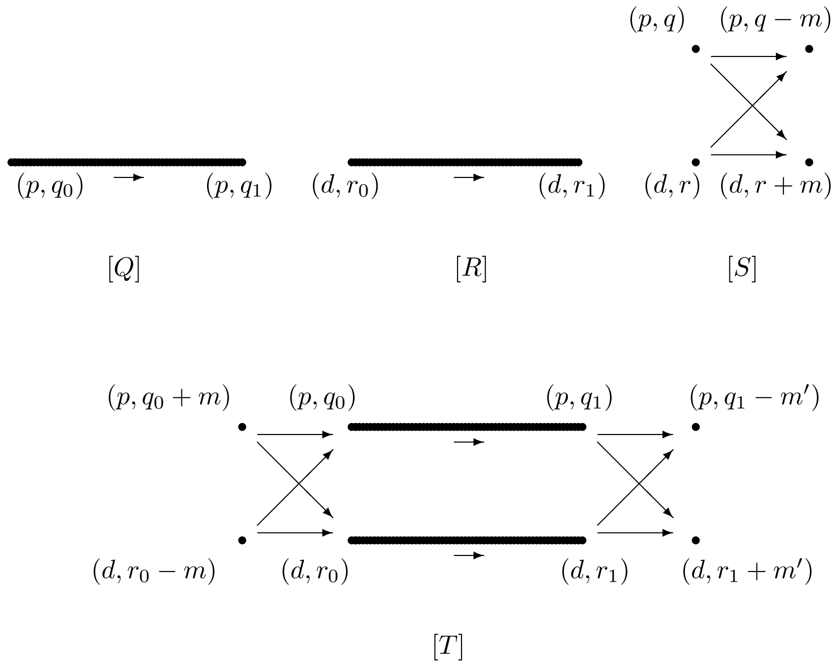


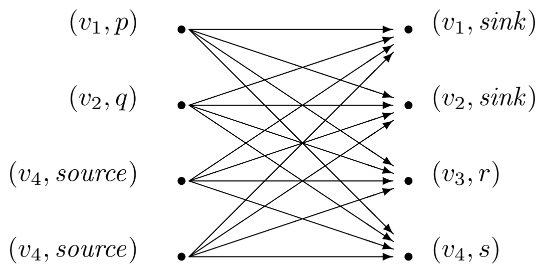
Figure 2.2: $[Q]$, $[R]$, $[S]$, $[T]$

2.9. Example. The marking of a Place/Transition Petri net that consists of a single token v in a single place p or, equivalently, the presence of v in p, can be regarded as a concrete process $p' = (X_{p'}, \leq_{p'}, ins_{p'})$ in $\mathbf{U}_{tokens}$ from example 2.4, where $X_{p'} = \{x\}$, $\leq_{p'}$ is the identity, and $ins_{p'}(x) = (v, p)$.

The marking that consists of a single token v_1 in p and a single token in q can be regarded as as a concrete process $M = (X_M, \leq_M, ins_M)$, where $X_M = \{x_1, x_2\}$, $\leq_M$ is the identity, $ins_M(x_1) = (v_1, p)$ and $ins_M(x_2) = (v_2, q)$.

Execution of a transition of a Place/Transition Petri net with input places p, q and output places r, s can be regarded as a concrete process $Z = (X_Z, \leq_Z, ins_Z)$ in $\mathbf{U}_{tokens}$, where
$X_Z = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$,
$x_1, x_2, x_3, x_4 \leq_Z x_5, x_6, x_7, x_8$,
$ins_Z(x_1) = (v_1, p)$, $ins_Z(x_5) = (v_1, sink)$,
$ins_Z(x_2) = (v_2, q)$, $ins_Z(x_6) = (v_2, sink)$,
$ins_Z(x_3) = (v_3, source)$, $ins_Z(x_7) = (v_3, r)$,
$ins_Z(x_4) = (v_4, source)$, $ins_Z(x_8) = (v_4, s)$.

The isomorphism class of lposets corresponding to the process Z is represented graphically in figure 2.3. $\sharp$

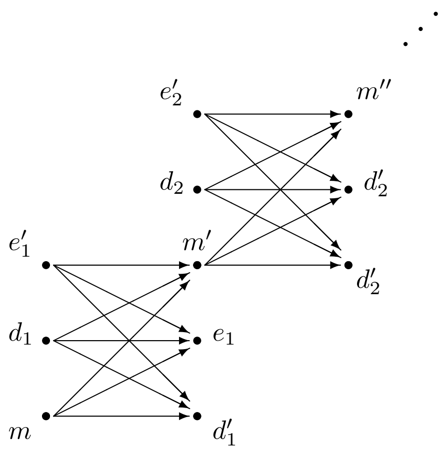


$[Z]$

Figure 2.3: $[Z]$

2.10. Example. Let $\mathbf{U}_{data} = (V_{data}, W_{data}, ob_{data})$ be the universe of data as in example 2.5. Consider an automaton $\mathcal{A}$ with a set Q of states, an input alphabet I, an output alphabet J, a transition function $f : I \times Q \to Q$, an output function $g : I \times Q \to J$, and an initial state q_0. The run of this automaton with the initial state $q \in Q$, the sequence $\mu = d_1 d_2 ...$ of input data $d_1 = (v_{11}, i_1, input)$, $d_2 = (v_{12}, i_2, input)$,... and the sequence $\nu = e_1 e_2 ...$ of output data $e_1 = (v_{21}, j_1, output)$, $e_2 = (v_{22}, j_2, output)$,... can be regarded as a concrete process $P = (X_P, \leq_P, ins_P)$ over $\mathbf{U}_{data}$, where
$x_P = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, ...\}$,
$x_1, x_2, x_3 <_P x_4, x_5, x_6$,
$x_4, x_7, x_8 <_P x_9, x_{10}, x_{11}$, and so on,
$ins_P(x_1) = m = (\mathcal{A}, q, memory)$,
$ins_P(x_2) = d_1 = (v_{11}, i_1, input)$,
$ins_P(x_3) = e'_1 = (v_{21}, none, source)$,
$ins_P(x_4) = m' = (\mathcal{A}, q' = f(i_1, q), memory)$,
$ins_P(x_5) = e_1 = (v_{21}, j_1 = g(i_1, q), output)$,
$ins_P(x_6) = d'_1 = (v_{11}, i_1, sink)$,
$ins_P(x_7) = d_2 = (v_{12}, i_2, input)$,
$ins_P(x_8) = e'_2 = (v_{22}, none, source)$,
$ins_P(x_9) = m'' = (\mathcal{A}, q'' = f(i_2, q'), memory)$,
$ins_P(x_{10}) = e_2 = (v_{22}, j_2 = g(i_2, q'), output)$,
$ins_P(x_{11}) = d'_2 = (v_{12}, i_2, sink)$, and so on.

The isomorphism class of lposets corresponding to the process P is represented graphically in figure 2.4. $\sharp$



$[P]$

Figure 2.4

Some properties of processes. Abstract processes

As concrete processes are lposets, their morphisms are defined as morphisms of lposets, that is as mappings that preserve the ordering and the labelling (see Appendix A).

Let $\mathbf{U} = (V, W, ob)$ be a universe of objects.

Let $L = (X, \leq, ins)$ be a concrete process in $\mathbf{U}$.

Every cross-section of $(X, \leq)$ contains an occurrence of each object v with nonempty $X|v$, and it is called a *cross-section* of L. By $csections(L)$ we denote the set of cross-sections of L. This set is partially ordered by the relation $\preceq$, and for every two cross-sections Z' and Z'' from $csections(L)$ there exist in $csections(L)$ the greatest lower bound $Z' \wedge Z''$ and the least upper bound $Z' \vee Z''$ of Z' and Z'' with respect to $\preceq$. From (3.1) and (3.2) of definition 2.6 it follows that the set of objects occurring in a cross-section is the same for all cross-sections of L. We call it the *range* of L and write it as $objects(L)$. We say that L is *global* if $objects(L) = V$. We say that L is *bounded* if the set of elements of L that are minimal with respect to $\leq$ and the set of elements of L that are maximal with respect to $\leq$ are cross-sections; the respective cross-sections are then called the *origin* and the *end* of L, and they are written as $origin(L)$ and $end(L)$. We say that L is *semibounded* if the set of elements of L that are minimal with respect to $\leq$ is a cross-section, i.e. if $origin(L)$ is defined. We say that L is *locally complete* if for every segment of L (which is bounded by definition) the poset of cross-sections of this segment is a complete lattice.

The following proposition is a direct consequence of process definition.

2.11. Proposition. For each cross-section c of L, the restrictions of L to the subsets $X^-(c) = \{x \in X : x \leq z \text{ for some } z \in c\}$ and $X^+(c) = \{x \in X : z \leq x \text{ for some } z \in c\}$ are concrete processes, called respectively the *head* and the *tail* of L with respect to c, and written respectively as $head(L, c)$ and $tail(L, c)$. $\sharp$

The following proposition reflects an important property of concrete processes.

2.12. Proposition. For every cross-section c of L, every isomorphism between initial segments of $tail(L, c)$ (resp.: between final segments of $head(L, c)$) is an identity. $\sharp$

Proof. Let Q be the restriction of L to $X^+(c)$ and let R and S be two initial segments of Q.

Suppose that $f : R \to S$ is an isomorphism that it is not an identity. Then there exists an initial subsegment T of R such that the image of T under f, say T', is different from T. By (3.3) of definition 2.6 neither T' is a subsegment of T nor T is a subsegment of T'. Define T'' to be the least segment containing both T and T', and consider $f' : T \to T''$, where $f'(x) = f(x)$ for $x \leq f(x)$ and $f'(x) = x$ for $f(x) < x$. In order to derive a contradiction, and thus to prove that f is an identity, it suffices to verify, that f' is an isomorphism. It can be done as follows.

For injectivity suppose that $f'(x) = f'(y)$. If $x \leq f(x)$ and $y \leq f(y)$ then $f(x) = f'(x) = f'(y) = f(y)$ and thus $x = y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) = f'(y) = y$. The case $x \leq f(x)$ and $f(y) < y$ is excluded by $f'(x) = f'(y)$ since $x \leq f(x) = f'(x) = f'(y) = y$ and, on the other hand, $f(y) < y = f(x)$ implies $y < x$. Similarly, the case $f(x) < x$ and $y \leq f(y)$ is excluded. Consequently, f' is injective.

For surjectivity suppose that y is in T''. If $y \leq f(y)$ then $y = f(t)$ for some $t \leq y$ and thus $y = f'(t)$ since $t \leq y = f(t)$ and thus $f'(t) = f(t)$. If $f(y) < y$ then $y = f'(y)$. Consequently, f' is surjective.

For monotonicity suppose that $x \leq y$. If $x \leq f(x)$ and $y \leq f(y)$ then $f'(x) = f(x) \leq f(y) = f'(y)$. If $f(x) < x$ and $f(y) < y$ then $f'(x) = x \leq y = f'(y)$. If $x \leq f(x)$ and $f(y) < y$ then $f'(x) = f(x) \leq f(y) < y = f'(y)$. If $f(x) < x$ and $y \leq f(y)$ then $f'(x) = x \leq y \leq f(y) = f'(y)$. Consequently, f' is monotonic.

For monotonicity of the inverse suppose that $f'(x) < f'(y)$. If $x \leq f(x)$ and $y \leq f(y)$ then $f(x) = f'(x) < f'(y) = f(y)$ and thus $x < y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) < f'(y) = y$. If $x \leq f(x)$ and $f(y) < y$ then $x \leq f(x) = f'(x) < f'(y) = y$. If $f(x) < x$ and $y \leq f(y)$ then $f(x) <$

$x = f'(x) < f'(y) = f(y)$ and thus $x < y$. Consequently, the inverse of f' is monotonic.

Verification for final subsegments of the restriction of L to $X^-(c)$ is similar. $\sharp$

2.13. Corollary. For every segment Q of L, every automorphism of Q is an identity. $\sharp$

2.14. Corollary. If L is bounded then for every bounded concrete process L' there may be at most one isomorphism from L to L'. $\sharp$

The following theorem gives sufficient conditions of local completeness of L.

2.15. Theorem. L is locally complete if the following conditions are satisfied:

(1) For every object v that occurs in L the set $X|v$ of its occurrences in L is a locally complete chain.
(2) The relation of incomparability with respect to the flow order $\leq$ is a closed subset of the product $X \times X$ for X provided with the interval topology, i.e., the weakest topology in which all intervals $\{x \in X : a < x < b\}$ are open sets. $\sharp$

Proof. Let Z_1 and Z_2 be cross-sections of L such that $Z_1 \preceq Z_2$ and let S be the set of cross-sections of L such that $Z_1 \preceq s \preceq Z_2$. Due to (1) for every $v \in V$ that occurs in L there exists the least upper bound x_v of those elements of $X|v$ which belong to some $s \in S$. Due to (2) the set Z of all such elements is an antichain. This set is a maximal antichain of L and it is easy to verify that it is also a cross-section of L. $\sharp$

2.16. Definition. An *abstract process* is an isomorphism class of concrete processes. $\sharp$

For every concrete process L' such that L and L' are isomorphic we have $\mathit{objects}(L') = \mathit{objects}(L)$. Consequently, for the abstract process $[L]$ that corresponds to a concrete process L we define $\mathit{objects}([L]) = \mathit{objects}(L)$.

We say that an abstract process is *global* (resp.: *bounded*, *locally complete*, *K-dense*, *weakly K-dense*) if the instances of this process are global (resp.: bounded, locally complete, K-dense, weakly K-dense).

Collecting concrete processes into isomorphism classes, i.e. making abstract processes, allows one to define some operations on the latter. In what follows, the word "process" means "abstract process".

By $PROC(\mathbf{U})$ we denote the set of all processes in $\mathbf{U}$. By $gPROC(\mathbf{U})$, $glcPROC(\mathbf{U})$, and $KPROC(\mathbf{U})$, we denote respectively the set of all global, global locally complete, weakly K-dense processes in $\mathbf{U}$.

3

Algebras of processes

For each process α from $PROC(\mathbf{U})$ with an instance $L \in \alpha$ that has the cross-section $origin(L)$ (resp.: the cross-section $end(L)$) there exists the unique process $[origin(L)]$, called the *source* or the *domain* or the *initial state* of α and written as $dom(\alpha)$ (resp.: the unique process $[end(L)]$, called the *target* or the *codomain* or the *final state* of α and written as $cod(\alpha)$). If $origin(L)$ (resp. $end(L)$) is not defined for L then $dom(\alpha)$ (resp. $cod(\alpha)$) is not defined for α.

In $PROC(\mathbf{U})$ there are two partial operations of composing processes: a sequential composition and a parallel composition.

The sequential composition

The sequential composition allows one to combine two processes whenever one of them is a continuation of the other. It can be defined due to the proposition 2.11 according to which for each cross-section c of a concrete process $L = (X, \leq, ins)$, the restrictions $head(L, c)$ and $tail(L, c)$ of L to the subsets $X^-(c) = \{x \in X : x \leq z \text{ for some } z \in c\}$ and $X^+(c) = \{x \in X : z \leq x \text{ for some } z \in c\}$ are concrete processes.

3.1. Definition. A process α is said to *consist* of a process α_1 *followed* by a process α_2 iff an instance L of α has a cross-section c such that $head(L, c)$ is an instance of α_1 and $tail(L, c)$ is an instance of α_2. $\sharp$

For example, the process ϕ in figure 3.1 consists of the process λ followed by the process γ.

3.2. Proposition. For every two processes α_1 and α_2 such that $cod(\alpha_1)$ and $dom(\alpha_2)$ are defined and $cod(\alpha_1) = dom(\alpha_2)$ there exists a unique process, written as $\alpha_1;\alpha_2$, or as $\alpha_1\alpha_2$, that consists of α_1 followed by α_2. If α_1 and α_2 are locally complete then so is $\alpha_1\alpha_2$. If α_1 and α_2 are global or weakly K-dense then so is $\alpha_1\alpha_2$. $\sharp$

Proof. Take $L_1 = (X_1, \leq_1, ins_1) \in \alpha_1$ and $L_2 = (X_2, \leq_2, ins_2) \in \alpha_2$ with $X_1 \cap X_2 = end(L_1) = origin(L_2)$ and with the restriction of L_1 to $end(L_1)$ identical with the restriction of L_2 to $origin(L_2)$, and provide $X_1 \cup X_2$ with the least common extension of the flow orders and labellings of L_1 and L_2.

Let L be the lposet thus obtained. It suffices to prove that L is a process and notice that $head(L, c) = L_1$ and $tail(L, c) = L_2$.

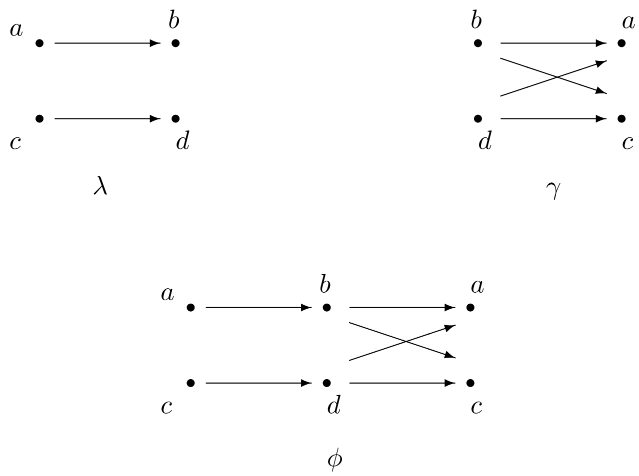


Figure 3.1

In order to prove that L is a process it suffices to show that L does not contain a segment with isomorphic proper subsegment. To this end suppose the contrary.

Suppose that $f : Q \to R$ is an isomorphism from a segment Q of L to a proper subsegment R of Q, where Q consists of a part Q_1 contained in L_1 and a part Q_2 contained in L_2. By applying twice the method described in the proof of proposition 2.12 we can modify f to an isomorphism $f' : Q \to R$ such that the image of Q_1 under f', say R_1, is contained in Q_1, and the image of Q_2 under f', say R_2, is contained in Q_2. As R is a proper subsegment of Q, one of these images, say R_1, is a proper part of the respective Q_i. By taking the greatest lower bounds and the least upper bounds of appropriate cross-sections we can extend Q_1 and R_1 to segments Q_1' and R_1' of P_1 such that R_1' is a proper subsegment of Q_1' and there exists an isomorphism from Q_1' to R_1'. This is in a contradiction with the fact that L_1 is a process. Consequently, L is a process. If α_1 and α_2 are locally complete then $L_1 = head(L, c)$ and $L_2 = tail(L, c)$ are locally complete. Given a segment Q of L and a subset S of cross-sections of L contained in Q, let c^- be the least upper bound of the set of cross-sections $s \wedge c$ with $s \in S$ and c^+ the least upper bound of cross-sections $s \vee c$ with $s \in S$. Then for every $v \in V$ define x_v as the greater of the two elements of $X|v$ in c^- and in c^+, and define d as the set of all x_v. As c^- and c^+ are cross-sections, d does not contain comparable elements and is an antichain. As all $v \in V$ have in d occurrences, d is a maximal antichain. It is also straightforward to verify that d is a cross-section and the least upper bound of S. In a similar way we can define a cross-section that is the greatest lower bound of S.

The cases of globality and weak K-density are obvious. ♯

3.3. Definition. The operation $(\alpha_1, \alpha_2) \mapsto \alpha_1; \alpha_2$ is called the *sequential composition* of processes. ♯

In the sequel the symbol ; will be omitted and $\alpha_1; \alpha_2$ will be written as $\alpha_1 \alpha_2$.

Each process which is a source or a target of a process is an identity, i.e. a process ι such that $\iota\phi = \phi$ whenever $\iota\phi$ is defined and $\psi\iota = \psi$ whenever $\psi\iota$ is defined. Moreover, if $dom(\alpha)$ is defined then it is the unique identity ι such that $\iota\alpha$ is defined, and if $cod(\alpha)$ is defined then it is the unique identity κ such that $\alpha\kappa$ is defined. Consequently, $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable partial operations on processes.

Identities are bounded processes with flow orders reducing to identity relations. They are called *states*, or *identities*, and we can identify them with the sets of instances of occurring objects.

The parallel composition

The parallel composition allows one to combine processes with disjoint sets of involved objects. It can be defined as follows.

3.4. Definition. Given a concrete process $L = (X, \leq, ins)$, by a *splitting* of L we mean an ordered pair $s = (X^F, X^S)$ of two disjoint subsets X^F and X^S of X such that $X^F \cup X^S = X$, $x' \leq x''$ only if x' and x'' are both in one of these subsets. ♯

3.5. Proposition. For each splitting $s = (X^F, X^S)$ of a concrete process $L = (X, \leq, ins)$, the restrictions of L to the subsets X^F and X^S are concrete processes, called respectively the *first part* and the *second part* of L with respect to s, and written respectively as $first(L, s)$ and $second(L, s)$. ♯

A proof is straightforward.

3.6. Definition. A process α is said to *consist* of two *parallel* processes α_1 and α_2 iff an instance L of α has a splitting s such that $first(L, s)$ is an instance of α_1 and $second(L, s)$ is an instance of α_2. ♯

For example, λ in figure 3.2 consists of parallel processes β and δ.

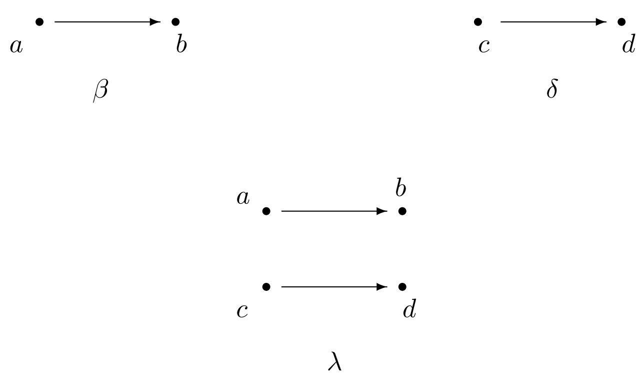


Figure 3.2

3.7. Proposition. For every two processes α_1 and α_2 such that $objects(\alpha_1) \cap objects(\alpha_2) = \emptyset$ there exists a process α with an instance L that has a splitting s such that $first(L, s)$ is an instance of α_1 and $second(L, s)$ is an instance of α_2. If such a process α exists then it is unique, we write it as $\alpha_1 + \alpha_2$, and we say that the processes α_1 and α_2 are *parallel*. If α_1 and α_2 are locally complete then so is $\alpha_1 + \alpha_2$. If α_1 and α_2 are global or weakly K-dense then so is $\alpha_1 + \alpha_2$. $\sharp$

For a proof it suffices to take $L_1 = (X_1, \leq_1, ins_1) \in \alpha_1$ and $L_2 = (X_2, \leq_2, ins_2) \in \alpha_2$ with $X_1 \cap X_2 = \emptyset$, and to provide $X_1 \cup X_2$ with the least common extension of the flow orders and labellings of L_1 and L_2.

3.8. Definition. The operation $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ is called the *parallel composition* of processes. $\sharp$

In the set $PROC(\mathbf{U})$ of processes in $\mathbf{U}$ there exists a process 0 such that $\alpha + 0 = \alpha$ for every α, namely the process with the empty set of object instances, called the *empty process*.

The operations of composing processes allow one to represent complex processes in terms of their components.

3.9. Examples. In the case of processes in example 2.8 we can represent $[T]$ as $[S']([Q'] + [R'])[S'']$.

All bounded executions of a Place/Transition Petri net with a set of places and a set of transitions can be regarded as processes which can be obtained by composing processes corresponding to presences of tokens in places of this net and executions of its transitions as described in example 2.9. Bounded executions starting from an initial marking can be regarded as those processes whose initial state corresponds to the initial marking. ♯

The operations of composing processes allow one also to turn the sets $PROC(\mathbf{U})$, $gPROC(\mathbf{U})$, $glcPROC(\mathbf{U})$, $KPROC(\mathbf{U})$ into partial algebras.

Partial categories of processes and their properties

Taking into account the definitions of operations on processes we obtain the following proposition (see Appendix D for the notions).

3.10. Proposition. The partial algebra $(PROC(\mathbf{U}), ;)$ is a partial category $\mathbf{pcatPROC}(\mathbf{U})$. For every $\alpha \in PROC(\mathbf{U})$, if $dom(\alpha)$ is defined then it is the source of α in this partial category, and if $cod(\alpha)$ is defined then it is the target of α in this partial category ♯

An important property of the partial category $\mathbf{pcatPROC}(\mathbf{U})$ is that for its composition we have the following cancellation laws.

3.11. Proposition. If $\sigma\alpha$ and $\sigma'\alpha$ are defined, their targets are defined, and $\sigma\alpha = \sigma'\alpha$ then $\sigma = \sigma'$. If $\alpha\tau$ and $\alpha\tau'$ are defined, their sources are defined, and $\alpha\tau = \alpha\tau'$ then $\tau = \tau'$. ♯

Proof. Suppose that $\sigma\alpha$ and $\sigma'\alpha$ are defined, their targets are defined, and $\sigma\alpha = \sigma'\alpha$. Suppose that L and L' are instances of $\sigma\alpha$ and $\sigma'\alpha$, that c and c' are cross-sections of L and L' such that $\sigma = [head(L,c)]$, $\sigma' = [head(L',c')]$, $\alpha = [tail(L,c)] = [tail(L',c')]$, and that f and f' are isomorphisms from L to L' such that $f(c) = c'$. Then $f|tail(L,c) = f'|tail(L,c)$ and $f'(c) = c'$ since otherwise $f \circ (f')^{-1}$ would be an automorphism from L to L whose restriction to $tail(L,c)$ would be different from identity isomorphism of final segments of L, and this would contradict to proposition 2.12. Thus f consists of two disjoint mappings $f|tail(L,c) : tail(L,c) \to tail(L',c')$ and $f|head(L,c) : head(L,c) \to head(L',c')$, Being disjoint restrictions of the isomorphism f both these mappings are isomorphisms. Consequently, $\sigma = [head(L,c)] = [head(L',c')] = \sigma'$.

The proof of the second law is similar. ♯

Another important property of the partial category $\mathbf{pcatPROC}(\mathbf{U})$ is that bicartesian squares in this partial category can be characterized as follows.

3.12. Proposition. A diagram $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in **pcatPROC(U)** if and only if there exist c, φ_1, φ_2 such that c is an identity, there is no identity $d \neq 0$ such that $\varphi_1 = d + \varphi_1'$ for some φ_1' or $\varphi_2 = d + \varphi_2'$ for some φ_2', $c + \varphi_1 + \varphi_2$ is defined, $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$, $\alpha_2 = c + dom(\varphi_1) + \varphi_2$, $\alpha_1' = c + \varphi_1 + cod(\varphi_2)$, $\alpha_2' = c + cod(\varphi_1) + \varphi_2$. ♯

Proof. Suppose that $D = (v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square, that L is an instance of $\alpha_1\alpha_2' = \alpha_2\alpha_1'$, and that Z_1, Z_2 are cross-sections of L such that $[head(L, Z_1)] = \alpha_1$, $[tail(L, Z_1)] = \alpha_2'$, $[head(L, Z_2)] = \alpha_2$, $[tail(L, Z_2)] = \alpha_1'$. Suppose that X' is the set of common elements of Z_1 and Z_2.

We have $Z_1 \vee Z_2 = end(L)$ since otherwise D could not be a pushout diagram, and $Z_1 \wedge Z_2 = origin(L)$ since otherwise D could not be a pullback diagram. Consequently, we can define c as the set of object instances of elements of X', φ_1 as $[L_1]$ for the restriction of L to the set $X_1 = \{x \in X - X' : z_2 \leq x \leq z_1 \text{ for some } z_1 \in Z_1 \text{ and } z_2 \in Z_2\}$, and φ_2 as $[L_2]$ for the restriction of L to the set $X_2 = \{x \in X - X' : z_1 \leq x \leq z_2 \text{ for some } z_1 \in Z_1 \text{ and } z_2 \in Z_2\}$.

Conversely, suppose that there exist c, φ_1, φ_2 such that c is an identity, $c + \varphi_1 + \varphi_2$ is defined, $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$, $\alpha_2 = c + dom(\varphi_1) + \varphi_2$, $\alpha_1' = c + \varphi_1 + cod(\varphi_2)$, $\alpha_2' = c + cod(\varphi_1) + \varphi_2$, and consider the diagram $D = (v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$.

Suppose that $\alpha_1\rho_2 = \alpha_2\rho_1 = \sigma$. Then in each instance L of σ there are cross-sections Z_1 and Z_2 such that $head(L, Z_1)$ is an instance of α_1 and $head(L, Z_2)$ is an instance of α_2. Consequently, $head(L, Z_1 \vee Z_2)$ is an instance of $\alpha = c + \varphi_1 + \varphi_2$ and $tail(L, Z_1 \vee Z_2)$ is an instance of a process ρ such that $\alpha\rho = \sigma$. By proposition 3.11 such a process is unique. Thus $v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w$ is a pushout of $v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w$.

Suppose that $\xi_1\alpha_2' = \xi_2\alpha_1' = \tau$. Then in each instance T of τ there are cross-sections Y_1 and Y_2 such that $tail(T, Y_1)$ is an instance of α_1' and $tail(T, Y_2)$ is an instance of α_2'. Consequently, $tail(T, Y_1 \wedge Y_2)$ is an instance of α and $head(T, Y_1 \wedge Y_2)$ is an instance of a process ξ such that $\xi\alpha = \tau$. By proposition 3.11 such a process is unique. Thus $v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w$ is a pullback of $v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w$.

Hence D is a bicartesian square. The uniqueness of α_1' and α_2' follows from the fact that in **pcatPROC(U)** only identity processes are isomorphisms. ♯

3.13. Proposition. If $\mathbf{A} = (A, ;)$ is the partial category of processes in a universe of objects then it enjoys the following properties:

(A1) If $\sigma\alpha$ and $\sigma'\alpha$ are defined, their targets are defined, and $\sigma\alpha = \sigma'\alpha$ then $\sigma = \sigma'$.

(A2) If $\alpha\tau$ and $\alpha\tau'$ are defined, their sources are defined, and $\alpha\tau = \alpha\tau'$ then $\tau = \tau'$.

(A3) If $\sigma\tau$ is an identity then σ and τ are also identities.

(A4) If $\sigma\alpha\tau$ is defined, it has a source and a target, and the category $dec_{\sigma\alpha\tau}$ of decompositions of $\sigma\alpha\tau$ is isomorphic to the category dec_α of decompositions of α then σ and τ are identities.

(A5) For all ξ_1, ξ_2, η_1, η_2 such that $\xi_1\xi_2 = \eta_1\eta_2$ there exist unique σ_1, σ_2, and a unique bicartesian square $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$, such that $\xi_1 = \sigma_1\alpha_1$, $\xi_2 = \alpha_2'\sigma_2$, $\eta_1 = \sigma_1\alpha_2$, $\eta_2 = \alpha_1'\sigma_2$.

(A6) If $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square then for every decomposition $u \overset{\alpha_1}{\rightarrow} v = u \overset{\alpha_{11}}{\rightarrow} v_1 \overset{\alpha_{12}}{\rightarrow} v$ (resp. $w \overset{\alpha_1'}{\rightarrow} u' = w \overset{\alpha_{11}'}{\rightarrow} w_1 \overset{\alpha_{12}'}{\rightarrow} u'$) there exist a unique decomposition $w \overset{\alpha_1'}{\rightarrow} u' = w \overset{\alpha_{11}'}{\rightarrow} w_1 \overset{\alpha_{12}'}{\rightarrow} u'$ (resp. $u \overset{\alpha_1}{\rightarrow} v = u \overset{\alpha_{11}}{\rightarrow} v_1 \overset{\alpha_{12}}{\rightarrow} v$), and a unique $v_1 \overset{\alpha_2''}{\rightarrow} w_1$, such that $(v_1 \overset{\alpha_{11}}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v_1 \overset{\alpha_2''}{\rightarrow} w_1 \overset{\alpha_{11}'}{\leftarrow} w)$ and $(v \overset{\alpha_{12}}{\leftarrow} v_1 \overset{\alpha_2''}{\rightarrow} w_1, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_{12}'}{\leftarrow} w_1)$ are bicartesian squares.

(A7) Given a family $\alpha = (u \overset{\alpha_i}{\rightarrow} v_i : i \in \{1, ..., n\})$, $n \geq 2$, the existence for all $i, j \in \{1, ..., n\}$ such that $i \neq j$ of bicartesian squares of the form $(v_i \overset{\alpha_i}{\leftarrow} u \overset{\alpha_j}{\rightarrow} v_j, v_i \overset{\alpha_j'}{\rightarrow} u_{ij}' \overset{\alpha_i'}{\leftarrow} v_j)$ implies the existence in $\mathbf{A}$ of a unique bicartesian n-cube with α being the family of its initial morphisms.

(A8) Every decomposition of $\alpha \in A$ into a pair $c = (\xi_1, \xi_2)$ of $\xi_1 \in A$ and $\xi_2 \in A$ such that $\xi_1\xi_2 = \alpha$ separates bicartesian squares in the category dec_α of decompositions of α in the sense that every two bicartesian squares in dec_α, one with $a = (\eta, \delta\xi_2)$ such that $\eta \neq \xi_1$ among the nodes, and another with $b = (\xi_1\varepsilon, \zeta)$ such that $\zeta \neq \xi_2$ among the nodes, do not share a node whenever they cannot be decomposed into bicartesian squares such that some of their bicartesian squares share a common side with the node c.

(A9) Every direct system D in the category $occ(\mathbf{A})$ of occurrences of morphisms in morphisms in $\mathbf{A}$ such that elements of D are bounded in the sense that they possess sources and targets has an inductive limit (a colimit).

(A10) Every $\alpha \in A$ is the inductive limit of the direct system of its bounded segments, that is of bounded $\xi \in A$ such that $\alpha = \alpha_1\xi\alpha_2$ for some α_1 and α_2. $\sharp$

Proof. The properties (A1) - (A2) have been proved as proposition 3.11.

(A3) is a direct consequence of process definition.

For (A4) suppose that there exists an isomorphism b between the restriction of $\mathbf{A}$ to the set of components of α and the restriction of $\mathbf{A}$ to the set of components of $\sigma\alpha\tau$, and consider an instance L of α and an instance L' of $\sigma\alpha\tau$. The isomorphism b induces an isomorphism $\bar{b}$ between the lattice of cross-sections of L and the lattice of cross-sections of L'. As every object has a unique instance in every cross-section of L and a unique instance in every cross-section of L', by considering for every occurrence of an object in L the cross-sections containing this occurrence and by using the isomorphism $\bar{b}$ we can construct an isomorphism between L and L'. To this end it suffices to notice that an occurrence of an object instance p in a cross-section c_1 of L and an occurrence of p in a cross-section c_2 of L correspond to the same occurrence of p in L iff $[tail(head(L, c_1 \vee c_2), c_1 \wedge c_2)] = p + \delta$ for some δ, and that for L' we have a similar property.

Consequently, L cannot be a proper segment of L', and we obtain (A4).

For (A5) we refer to the characterization of bicartesian squares in the partial category $\mathbf{A} = \mathbf{pcatPROC(U)}$ as described in proposition 3.12. With this characterization a proof of (A5) can be carried out as follows. Consider an instance L of $\xi_1\xi_2 = \eta_1\eta_2$ and its cross-sections c_1 and c_2 such that $\xi_1 = [head(L, c_1)]$, $\xi_2 = [tail(L, c_1)]$, $\eta_1 = [head(L, c_2)]$, $\xi_1 = [tail(L, c_2)]$.
Define $\sigma_1 = [head(L, c_1 \wedge c_2)]$, $\sigma_2 = [tail(L, c_1 \vee c_2)]$, $\alpha_1 = [head(tail(L, c_1 \wedge c_2), c_1)]$,
$\alpha_2 = [head(tail(L, c_1 \wedge c_2), c_2)]$, $\alpha'_1 = [head(tail(L, c_2), c_1 \vee c_2)]$,
$\alpha'_2 = [head(tail(L, c_1), c_1 \vee c_2)]$. Follow the proof of 3.12 to show that the diagram
$D = (v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w, v \stackrel{\alpha'_2}{\rightarrow} u' \stackrel{\alpha'_1}{\leftarrow} w)$ is a bicartesian square.

For (A6) it suffices to take into account the characterization 3.12 of bicartesian squares and notice that a decomposition of α_1 induces a decomposition of φ_1.

The property (A7) follows easily from proposition 3.12

The property (A8) follows easily from proposition 3.12.

For (A9) it suffices to take into account corollary 2.14 and consider the respective colomits in the category **LPOSETS**.

The property (A10) follows from the condition (2) of definition 2.6. $\sharp$

Taking into account proposition 3.2 we obtain the following result.

3.14. Proposition. The restrictions $\mathbf{pcatgPROC(U)}$, $\mathbf{pcatglcPROC(U)}$, and $\mathbf{pcatKPROC(U)}$, of the partial category $\mathbf{pcatPROC(U)}$ to the subsets $gPROC(\mathbf{U})$, $glcPROC(\mathbf{U})$, and $KPROC(\mathbf{U})$, respectively are subalgebras of $\mathbf{pcatPROC(U)}$, and they enjoy the properties (A1) - (A10). $\sharp$

Partial categories of processes in a universe of objects which enjoy the properties (A1) - (A10) are essentially specific mutiplicative transition systems (MTSs) in the sense of [Wink 11]. In the rest of the paper we call them partial categories of processes.

3.15. Definition. A *partial category of processes* is a partial category $\mathbf{A} = (A, ;)$ such that A is a set of processes in a universe of objects and $\mathbf{A}$ enjoys the properties (A1) - (A10). $\sharp$

The following proposition allows one to consider every partial category of processes as the union of a family of partial categories of processes, each partial category containing only processes from a fixed universe of objects.

3.16. Proposition. For every universe $\mathbf{U}'$ of objects that is obtained by restricting $\mathbf{U}$ to a subset V' of objects, and to the subset W' of instances of objects from V', and for every partial category of processes $\mathbf{A} = (A, ;)$, the restriction of $\mathbf{A}$ to the set of elements $\alpha \in A$ with $objects(\alpha) = V'$ is a partial category of processes. $\sharp$

A proof is straightforward.

Due to (A1) - (A10) we obtain the following propositions.

3.17. Proposition. For every α, the relation $\sqsubseteq_\alpha$ between decompositions of α into pairs (ξ_1, ξ_2) such that $\xi_1 \xi_2 = \alpha$, where $(\xi_1, \xi_2) \sqsubseteq_\alpha (\eta_1, \eta_2)$ iff $\eta_1 = \xi_1 \delta$ and $\xi_2 = \delta \eta_2$ for some δ, is a partial order. $\sharp$

A proof follows from immediately from the properties (A1) - (A4).

3.18. Proposition. For every α, the partial order $\sqsubseteq_\alpha$ between decompositions of α into pairs (ξ_1, ξ_2) such that $\xi_1 \xi_2 = \alpha$ makes the set of such decompositions a lattice LT_α. $\sharp$

Proof. Let $\alpha = \xi_1 \xi_2 = \eta_1 \eta_2$, $\xi_1 = \sigma_1 \alpha_1$, $\xi_2 = \alpha_2' \sigma_2$, $\eta_1 = \sigma_1 \alpha_2$, $\eta_2 = \alpha_1' \sigma_2$ with α_1, α_1', α_2, α_2', σ_1, σ_2 as in (A5). The least upper bound of $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ can be defined as $z = (\xi_1 \alpha_2', \sigma_2) = (\eta_1 \alpha_1', \sigma_2)$. To see this consider any $u = (\zeta_1, \zeta_2)$ such that $x \sqsubseteq_\alpha u$ and $y \sqsubseteq_\alpha u$. Then $\zeta_1 = \xi_1 \delta$ and $\zeta_1 = \eta_1 \epsilon$ for some δ and ϵ. As α_1' and α_2' form a pushout of α_1 and α_2, there exists a unique φ such that $\delta = \alpha_2' \varphi$ and $\epsilon = \alpha_1' \varphi$. Hence $\zeta_1 = \xi_1 \alpha_2' \varphi = \eta_1 \alpha_1' \varphi$ and, consequently, $z \sqsubseteq_\alpha u$.

Similarly, due to the fact that α_1 and α_2 form a pullback of α_1' and α_2', we obtain that $t = (\sigma_1, \alpha_1 \alpha_2' \sigma_2)$ is the greatest lower bound of x and y. $\sharp$

Partial monoids of processes and their properties

The following two propositions are direct consequences of definitions.

3.19. Proposition. The partial algebra $(PROC(\mathbf{U}), +)$ is a partial commutative monoid **pmonPROC**$(\mathbf{U})$ with the empty process 0 such that $\alpha + 0 = \alpha$ for every α. $\sharp$

3.20. Proposition. If $\mathbf{A} = (A, +)$ is the partial monoid of processes in a universe of objects then it enjoys the following properties:

(B1) If $\alpha + \sigma$ and $\alpha + \sigma'$ are defined and $\alpha + \sigma = \alpha + \sigma'$ then $\sigma = \sigma'$.

(B2) $\alpha + \alpha$ is defined only for $\alpha = 0$.
(B3) The following relation $\lhd$ is a partial order:
$\alpha_1 \lhd \alpha_2$ iff α_2 contains α_1 in the sense that $\alpha_2 = \alpha_1 + \rho$ for some ρ.
(B4) Given a subset B of A, if $\alpha_1 + \alpha_2$ is defined for all $\alpha_1, \alpha_2 \in B$ such that $\alpha_1 \neq \alpha_2$ then in A there exists the least upper bound $\bigtriangledown B$ of B with respect to $\lhd$.
(B5) For all α_1 and α_2 there exists the greatest lower bound of α_1 and α_2 with respect to $\lhd$, written as $\alpha_1 \bigtriangleup \alpha_2$.
(B6) If $\alpha_1 + \alpha_2$ is defined then $(\alpha_1 \bigtriangleup \sigma) + (\alpha_2 \bigtriangleup \sigma)$ is defined and $(\alpha_1 \bigtriangleup \sigma) + (\alpha_2 \bigtriangleup \sigma) = (\alpha_1 + \alpha_2) \bigtriangleup \sigma$.
(B7) If $\alpha_1 \bigtriangleup \alpha_2 = 0$ and $\alpha_1 \lhd \alpha$ and $\alpha_2 \lhd \alpha$ for some α then $\alpha_1 + \alpha_2$ is defined.
(B8) Each $\alpha \neq 0$ contains some β that is a (+)-*atom* in the sense that $\beta \neq 0$ and $\beta = \alpha_1 + \alpha_2$ only if either $\alpha_1 = \beta$ and $\alpha_2 = 0$ or $\alpha_1 = 0$ and $\alpha_2 = \beta$.
(B9) Each α is determined uniquely by the set $h(\alpha)$ of (+)-atoms it contains in the sense that $h(\alpha_1) = h(\alpha_2)$ implies $\alpha_1 = \alpha_2$. $\sharp$

Algebras of processes and their properties

3.21. Proposition. The partial category **pcatPROC**(**U**) and the partial monoid **pmonPROC**(**U**) are related to each other as follows:

(C1) $dom(\alpha_1 + \alpha_2)$ and $dom(\alpha_1) + dom(\alpha_2)$ are defined and $dom(\alpha_1 + \alpha_2) = dom(\alpha_1) + dom(\alpha_2)$ whenever $\alpha_1 + \alpha_2$, $dom(\alpha_1)$, $dom(\alpha_2)$ are defined.
(C2) $cod(\alpha_1 + \alpha_2)$ and $cod(\alpha_1) + cod(\alpha_2)$ are defined and $cod(\alpha_1 + \alpha_2) = cod(\alpha_1) + cod(\alpha_2)$ whenever $\alpha_1 + \alpha_2$, $cod(\alpha_1)$, $cod(\alpha_2)$ are defined.
(C3) $dom(\alpha) = 0$ implies $\alpha = 0$ and $cod(\alpha) = 0$ implies $\alpha = 0$.
(C4) If $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22})$ is defined then $\alpha_{11} + \alpha_{21}$, $\alpha_{11} + \alpha_{22}$, $\alpha_{12} + \alpha_{21}$, $\alpha_{12}+\alpha_{22}$ are also defined and $(\alpha_{11}\alpha_{12})+(\alpha_{21}\alpha_{22}) = (\alpha_{11}+\alpha_{21})(\alpha_{12}+\alpha_{22})$.
(C5) If $\alpha_{11}\alpha_{12}$ and $\alpha_{21}\alpha_{22}$ are defined, and $\alpha_{11} + \alpha_{21}$ is defined, or $\alpha_{11} + \alpha_{22}$ is defined, or $\alpha_{12} + \alpha_{21}$ is defined, or $\alpha_{12} + \alpha_{22}$ is defined, then $(\alpha_{11}\alpha_{12}) + (\alpha_{21}\alpha_{22})$ is defined.
(C6) $\alpha_1 + \alpha_2 = \beta_1\beta_2$ implies the existence of unique α_{11}, α_{12}, α_{21}, α_{22} such that $\alpha_1 = \alpha_{11}\alpha_{12}$, $\alpha_2 = \alpha_{21}\alpha_{22}$, $\beta_1 = \alpha_{11} + \alpha_{21}$, $\beta_2 = \alpha_{12} + \alpha_{22}$.
(C7) In **pmonPROC**(**U**) there exists the least congruence $\sim$ such that $\alpha \sim \beta$ for all α and β such that $\alpha = \gamma\beta\delta$ or $\alpha = \gamma\beta$ or $\alpha = \beta\delta$ for some γ and δ, and this congruence is strong, that is $\alpha_1 \sim \alpha_1'$ and $\alpha_2 \sim \alpha_2'$ implies that $\alpha_1 + \alpha_2$ is defined iff $\alpha_1' + \alpha_2'$ is defined.
(C8) A diagram $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in **pcatPROC**(**U**) if and only if there exist c, φ_1, φ_2 such that c is an identity, there is no identity $d \neq 0$ such that $d \lhd \varphi_1$ or $d \lhd \varphi_1$, $c+\varphi_1+\varphi_2$ is defined, $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$, $\alpha_2 = c + dom(\varphi_1) + \varphi_2$, $\alpha_1' = c + \varphi_1 + cod(\varphi_2)$, $\alpha_2' = c + cod(\varphi_1) + \varphi_2$. $\sharp$

A proof is straightforward assuming $\alpha \sim \beta$ whenever $objects(\alpha) = objects(\beta)$ and taking into account proposition 3.12.

The obtained results can be summarized as follows.

3.22. Proposition. $\mathbf{PROC}(\mathbf{U}) = (PROC(\mathbf{U}), ;, +)$ is a partial algebra that enjoys the following properties:

(A)The reduct $(PROC(\mathbf{U}), ;)$ is a partial category $\mathbf{pcatPROC}(\mathbf{U})$ with the properties (A1) - (A10).

(B)The reduct $(PROC(\mathbf{U}), +)$ is a partial commutative monoid $\mathbf{pmonPROC}(\mathbf{U})$ with the properties (B1) - (B9).

(C)The reducts $(PROC(\mathbf{U}), ;)$ and $(PROC(\mathbf{U}), +)$ are related according to (C1) - (C8). $\sharp$

Taking into account proposition 3.2 and 3.14 we obtain the following result.

3.23. Proposition. The restrictions $\mathbf{gPROC}(\mathbf{U})$, $\mathbf{glcPROC}(\mathbf{U})$, $\mathbf{KPROC}(\mathbf{U})$ of the partial algebra $\mathbf{PROC}(\mathbf{U})$ to the subsets $gPROC(\mathbf{U})$, $glcPROC(\mathbf{U})$, $KPROC(\mathbf{U})$ respectively, are subalgebras of $\mathbf{PROC}(\mathbf{U})$, and they enjoy the properties (A), (B), (C). $\sharp$

Partial algebras of processes in a universe of objects which enjoy the properties (A), (B), (C) are essentially versions of algebras of processes in the sense of [Wink 09a]. In the rest of the paper we call them algebras of processes.

3.24. Definition. An *algebra of processes* is a partial algebra $\mathbf{A} = (A, ;, +)$ such that A is a set of processes in a universe of objects and $\mathbf{A}$ enjoys the properties (A), (B), (C). $\sharp$

The reducts $(A, ;)$ and $(A, +)$ of an algebra $\mathbf{A}$ of processes are denoted respectively $\mathbf{pcat}(\mathbf{A})$ and $\mathbf{pmon}(\mathbf{A})$.

Taking into account proposition 3.16 one can consider the reduct $\mathbf{pcat}(\mathbf{A})$ of an algebra of processes $\mathbf{A}$ as the union of a family of partial categories of processes $\mathbf{A}_i$, where each $\mathbf{A}_i$ contains only processes in a universe $\mathbf{U}_i$. The monoidal structure of $\mathbf{A}$ provides an algebraic relation between between partial categories $\mathbf{A}_i$, a structure that cannot be defined within $\mathbf{pcat}(\mathbf{A})$ itself. Due to this structure a process in a universe of objects can be lifted to a process in a larger universe by adding an identity or another process. This allows one to interprete local runs of a system in presence of independent states or processes as global runs.

The weak K-density of processes results in a special property of the respective algebras.

3.25. Proposition. If $\mathbf{A} = (A, ;, +)$ is an algebra of weakly K-dense processes in a universe of objects then it enjoys the following property:

(C9) Given α such that $dom(\alpha)$ contains an identity p which is a $(+)$-atom (an atomic identity), and $cod(\alpha)$ contains an identity q which is a $(+)$-atom (an atomic identity), if α cannot be represented as $(p + \alpha_1)(q + \alpha_2)$ then for every ξ and η such that $\alpha = \xi\eta$ the state $cod(\xi) = dom(\eta)$ contains an atomic identity m such that ξ cannot be represented as $(p + \xi_1)(m + \xi_2)$ and η cannot be represented as $(m + \eta_1)(q + \eta_2)$. $\sharp$

Proof. Let L be an instance of α, x the occurrence of p in $origin(L)$, and y the occurrence of q in $end(L)$. Consider a cross-section c of L such that $head(L, c)$ is an instance of ξ and $tail(L, c)$ an instance of η. The fact that α cannot be represented as $(p + \alpha_1)(q + \alpha_2)$ implies that there is no cross-section of L containing both x and y. Consequently, x precedes y and, due to the weak K-density of the partial order of L, between x and y there exists an occurrence z of an atomic identity m that belongs to c. Hence ξ and η cannot be represented as $(p + \xi_1)(m + \xi_2)$ and $(m + \eta_1)(q + \eta_2)$. $\sharp$

A partial order of processes

The operations of composing processes can be used to define prefixes of processe and use the prefix concept to define a partial order of processes.

Let $\mathbf{A} = (A, ;, +)$ be an algebra of processes.

3.26. Definition. A process α is said to be a *full prefix* of a process β, and we write α *fpref* β, if $\beta = \alpha\gamma$ for some γ. $\sharp$

For example, the process λ in figure 4.1 is a full prefix of the process ϕ.

3.27. Definition. A process α is said to be a *prefix* of a process β, and we write α *pref* β, if $\beta = (\alpha + \delta)\gamma$ for some γ and δ. $\sharp$

For example, the processes β and δ in figure 3.2 are prefixes of the processes λ and of ϕ in figure 3.1.

Note that a process α is gobal iff $\alpha + \beta$ is defined only for $\beta = 0$.

Note that, due to (B4), (B11), for all α and β in A we can define $\alpha - \beta$ as the least upper bound $\bigtriangledown C$ of the set C of those $(+)$-atoms contained in α which are not contained in β.

Note that, due to (A1) - (A4), (B1) - (B5), and to other properties of algebras of processes, the relation *pref* is a partial order on the subset $A_{bounded}$ of bounded elements of A.

Given a directed subset D of bounded elements of A with the partial order *pref*, by (B6) we can assign to each $\alpha \in D$ an identity c_α such that $dom(\alpha) + c_\alpha$ equals to the least upper bound with respect to $\sqsubseteq$ of the sources of elements of D. Then the respective $\alpha + c_\alpha$ form a unique direct system D^*. in the category $occ(\mathbf{A})$. This system has the inductive limit δ that can be regarded as a limit of D. By adding all such limits to the set $A_{bounded}$ we obtain the subset $A_{semibounded}$ of those $\alpha \in A$ which possess sources.

3.28. Proposition. The extention $\alpha \sqsubseteq \beta$ of the relation *pref* defined by

$\alpha \sqsubseteq \beta$ whenever every prefix of α is a prefix of β

is a partial order on $A_{semibounded}$. The inductive limits of directed subsets of $A_{semibounded}$ with this order are their least upper bounds. $\sharp$

Proof. Given a directed subset D of the poset $(A_{semibounded}, \sqsubseteq)$, the pefixes of elements of D form a directed set D'. For every element of D' we choose a concrete instance, and we consider α and $\beta = (\alpha + \gamma)\delta$ such that L is the chosen instance of α, L_1 is the chosen instance of β, L_2 is the chosen instance of $\alpha + \gamma$ and $L_3 = head(L_1, c)$ is an instance of $\alpha + \gamma$. Then there exists a unique isomorphism f from L_2 to L_3 since otherwise there would be another isomorphism g and the correspondence $f(x) \mapsto g(x)$ would be different from identity isomorphism between two initial segments of L_1. On the other hand, f determines a unique isomorphism between L and $first(L_2, s)$ with a splitting s due to the fact that the first part of L_2 is determined uniquely by the set of objects which occur in it. $\sharp$

3.29. Definition. The relation $\sqsubseteq$ on $A_{semibounded}$ is called the *prefix order*. The least upper bound of a directed subset D of the partially ordered set $A_{semibounded}$ is called the *limit* of D. $\sharp$

Note that the least upper bounds of directed subsets of $(A_{semibounded}, \sqsubseteq)$ are limits of the corresponding filters in $A_{semibounded}$ with the Scott topology induced by the partial order $\sqsubseteq$.

4

Behaviours

A formal definition of a behaviour

The behaviour of a concurrent system can be represented by the set of its potential processes. The system may be *reactive* in the sense that it may communicate with the environment, behave depending on the data it receives, and act jointly with the environment (cf. [Pn 86]).

A behaviour is potential rather than actual. What has happened up to a certain stage of its potential process is a prefix of this process. What may happen next depends on the presence of suitable instances of objects taking part in the behaviour. Moreover, it is natural to assume that a behaviour contains the existing least upper bound of its subsets. Consequently, a behaviour is a specific set of processes. It automatically posesses the structure of partial order given by the prefix relation, and is a directed complete poset (a DCPO).

In order to define behaviours formally it is convenient to fix an algebra of processes, and think of this algebra as of a framework for the respective definitions. Let $\mathbf{A} = (A, ;, +)$ be an algebra of processes.

4.1. Definition. A *behaviour* represented in $\mathbf{A}$, or a behaviour in $\mathbf{A}$, or simply a behaviour, if $\mathbf{A}$ is known from the context, is a subset B of the set A of processes of $\mathbf{A}$ such that:

(1) B is downward closed with respect to $\sqsubseteq$,

(2) if α and β are initial segments of runs which are maximal elements of B then $\alpha(\gamma + s) \in B$ iff $\beta(\gamma + t) \in B$ for every γ such that $dom(\gamma) + s = cod(\alpha)$ and $dom(\gamma) + t = cod(\beta)$,

(3) $\bigsqcup D \in B$ for every subset D of B such that $\bigsqcup D$ exists. ♯

4.2. Example. The underlying set of any algebra of processes is a behaviour represented in this algebra. Note that such a behaviour contains all the sources of maximal elements of $\mathbf{A}$ with respect to the prefix order. This reflects the indeterministic choice of the initial state of the behaviour from among all the sources of maximal elements of $\mathbf{A}$. ♯

4.3. Example. Consider the machines M_1 and M_2 and their system M in example 1.1.

The behaviour of the machine M_1 working alone can be represented in $\mathbf{PROC}(\mathbf{U}_1)$ as the set of processes a, b, α, α^2, ... , α^ω, β, $\alpha\beta$, $\alpha^2\beta$, ...

The behaviour of the machine M_2 working alone can be represented in $\mathbf{PROC}(\mathbf{U}_1)$ as the set of processes c, d, δ.

The behaviour of the system M can be represented in $\mathbf{PROC}(\mathbf{U}_1)$ as the set B_1 of processes of the subalgebra $\mathbf{A}_1$ of the algebra $\mathbf{PROC}(\mathbf{U}_1)$ that can be obtained by combining $a, b, c, d, \alpha, \beta, \gamma, \delta$ with the aid of compositions and construction of limits.

It is clear that $\mathbf{A}_1$ is an algebra of processes and that B_1 is also a behaviour in $\mathbf{A}_1$. In this behaviour processes which have not in $\mathbf{A}_1$ a common extension (i.e., a processes of which they are predecessors relative to the prefix order) cannot represent initial segments of the same maximal process of M. Note that the lack of such a common extension can be decided without a reference to maximal processes of M.

An initial part of B_1 is depicted in figure 4.1, where the prefix order is indicated by directed edges. $\sharp$

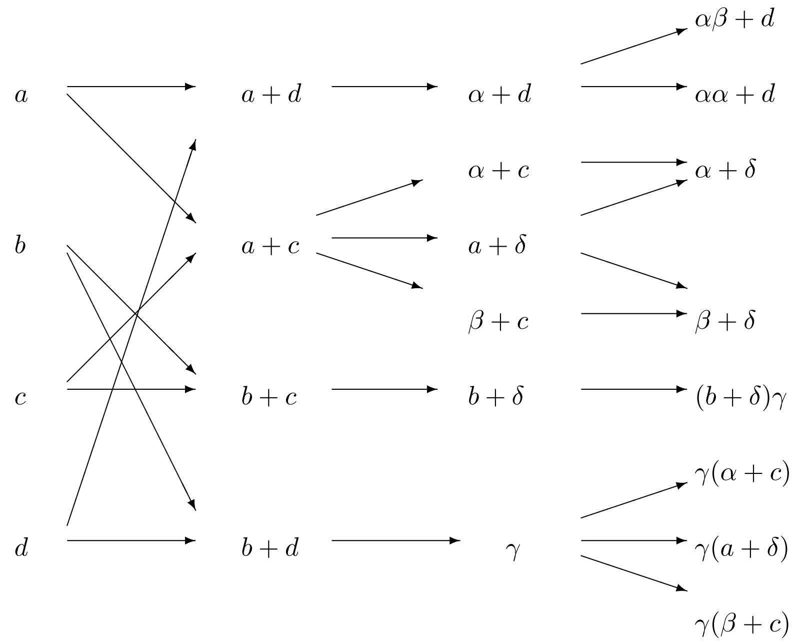


Figure 4.1: An initial part of B_1

4.4. Example. Consider a producer p and a distributor d in example 2.6. By combining the abstract processes corresponding to the possible variants of concrete processes Q and R of the producer and the distributor with the aid of compositions and construction of limits, we obtain a subalgebra $\mathbf{A}_2 = (A_2, ;)$

of $\mathbf{PROC}(\mathbf{U}_2)$. This subalgebra is an algebra of processes. The set B_2 of runs of this algebra is a behaviour represented in $\mathbf{A}_2$. It reflects an independent activity of the producer and the distributor.

By combining the abstract processes corresponding to the possible variants of concrete processes Q, R, S with the aid of compositions and construction of limits, we obtain a subalgebra $\mathbf{A}_3 = (A_3, ;)$ of $\mathbf{PROC}(\mathbf{U}_2)$. This subalgebra is an algebra of processes. The set B_3 of processes of this algebra is a behaviour represented in $\mathbf{A}_3$. It reflects an activity of the producer p and the distributor d that is mainly independent, but from time to time is interrupted by transfer of some material from the producer to the distributor. $\sharp$

4.5. Example. The behaviour of an automaton $\mathcal{A}$ as described in example 2.10 with the initial state $q \in Q$, the sequence $\mu = d_1 d_2 ...$ of input data $d_1 = (v_{11}, i_1, \mathit{input})$, $d_2 = (v_{12}, i_2, \mathit{input})$,... and the sequence $\nu = e_1 e_2 ...$ of output data $e_1 = (v_{21}, j_1, \mathit{output})$, $e_2 = (v_{22}, j_2, \mathit{output})$,... can be defined as the set of prefixes of processes in the universe $\mathbf{U}_{data}$ of data described in example 2.5, namely of the processes whose instances are as P in example 2.10. It will be defined formally in example 4.24.

The behaviour of the same automaton $\mathcal{A}$ for an unspecified initial state and an unspecified sequence of input data can be defined as a closed with respect to the existing least upper bounds of subsets and prefix-closed subset of the algebra of processes in $\mathbf{U}_{data}$, namely the union of the subsets representing the behaviors of $\mathcal{A}$ with all the possible initial states, all the possible sequences of input data, and all the possible sequences of output data. $\sharp$

The following proposition states an important property of behaviours in locally complete partial categories of processes.

4.6. Proposition. If a locally complete partial category of processes $\mathbf{A}$ is a subalgebra of the locally complete partial category $\mathbf{PROC}(\mathbf{U})$ of locally complete processes in a universe $\mathbf{U}$ of objects then every behaviour B in $\mathbf{A}$ with the prefix order is an algebraic domain and thus it is a continuous DCPO. $\sharp$

Proof. Suppose that $\alpha \in B$ is a bounded process with an instance L such that $L = \mathit{head}(L', c)$ for a concrete process L' with $[L'] \in B$ and for c being the least upper bound of cross-sections c' of L' with the underlying sets of $\mathit{head}(L', c')$ containing occurrences $x_1, ..., x_n$ of instances of objects $v_1, ..., v_n$ from a finite subset of V. Then α is a compact element of B. Indeed, suppose that $\alpha \sqsubseteq \bigsqcup S$ for a directed subset S of B. Then all $s \in S$ and $\bigsqcup S$ have instances L_s and L_S that are initial segments of L' such that the underlying set of L_S is the union of the underlying sets of all L_s and it contains the underlying set of L. Consequently, for every $i \in \{1, ..., n\}$ there must be $s_i \in S$ such that the underlying set of L_{s_i} contains x_i. Consequently, $x_1, ..., x_n$ belong to the underlying set of L_s for an upper bound s of $s_1, ..., s_n$ that belongs to S. Consequently, c must be a cross-section of L_s and $\alpha \sqsubseteq s \in S$, as required.

In order to prove that B with the prefix order is algebraic domain, consider any $\alpha \in B$ and its instance L. As every process is an inductive limit of a direct system of its bounded segments, it suffices to consider the case when α is bounded. Then for every finite set $f = \{x_1, ..., x_n\}$ of occurrences of instances of objects $v_1, ..., v_n$ in the underlying set of L there exists the least cross-section c_f of L such that $x_1, ..., x_n$ belong to the underlyin set of $head(L, c_f)$. Then $s_f = [head(L, c_f)]$ is a compact element of B. On the other hand, processes s_f form a directed set S and $\alpha = \bigsqcup S$, as required. $\sharp$

In the next section it will become clear that proposition 4.6 plays an important role in providing random behaviours with suitable probability measures.

Note that from propositions 2.15 and 4.6 it follows that the behaviour B_1 in example 4.3 with the respective prefix order is a continuous DCPO. Note also that behaviour B_2 in example 4.4 with the prefix order is a continuous DCPO if all the variants of Q and R in its processes are complete lattices.

Operations on behaviours

Behaviours can be combined with the aid of operations which can be defined as follows.

First, it is easy to see that the set of behaviours in $\mathbf{A}$ is a complete lattice.

4.7. Proposition. The set $Behaviours(\mathbf{A})$ of behaviours in $\mathbf{A}$ is ordered by inclusion and every family $(B_i : i \in I)$ of its members has the greatest lower bound and the least upper bound. If such a family is nonempty then the intersection $\bigcap(B_i : i \in I)$ is its greatest lower bound and the union $\bigcup(B_i : i \in I)$ augmented by processes whose existence follows from the conditions of definition 4.1 is its least upper bound. The least upper bound of the empty family is the empty behaviour. The greatest upper bound of the empty family is the set of those processes of $\mathbf{A}$ which possess sources. $\sharp$

The operations of forming the greatest lower bound and the least upper bound can be used to define compound behaviours as results of combining their component behaviours.

In order to illustrate this, consider the producer p and the distributor d in as in example 4.4. The behaviour of the producer p is the set $B(p)$ of processes which can be obtained by combining the processes corresponding to the possible variants of the concrete process Q with the aid of compositions and construction of limits. The behaviour of the distributor d is the set $B(d)$ of processes which can be obtained by combining the processes corresponding to the possible variants of the concrete process R with the aid of compositions and construction of limits. The behaviour that consists of independent behaviours of the producer p and the distributor d can be defined as B_1 in example 4.4. On the other hand, this behaviour can be obtained as the least upper bound of the behaviours $B(p)$ and $B(d)$.

The lattice theoretical operations on behaviours are not the only operations we can consider. Now we define also some other operations.

In particular, behaviours can be transformed by preceding them by processes.

4.8. Proposition. For every bounded process α and every behaviour B in $\mathbf{A}$ there exists the least behaviour in $\mathbf{A}$ which contains the set of all processes $\xi \in A$ such that ξ is a prefix of $(\alpha + c)(\beta + d)$ for some $\beta \in B$ and some identities c, d, e such that $cod(\alpha) = d + e$, $dom(\beta) = c + e$, and $c + d + e$ is defined. We write it as $\alpha.B$. $\sharp$

4.9. Definition. The operation $B \mapsto \alpha.B$ is called *prefixing* of α to B. $\sharp$

Next, behaviours can be transformed by replacing some object instances by other object instances.

4.10. Proposition. If $R : \mathbf{A} \to \mathbf{A}$ is an endomorphism of $\mathbf{A}$ then, for every process α and every process β, the congruence $\alpha \sim \beta$ is equivalent to the congruence $R(\alpha) \sim R(\beta)$. Given such an endomorphism, we call it a *replacement*, we call $R(\alpha)$ the *result of applying the replacement* R to α, and write it as $\alpha[R]$. $\sharp$

4.11. Proposition. For every replacement R and every behaviour B in $\mathbf{A}$ the set of all processes $\xi \in A$ such that $\xi = \beta[R]$ for some $\beta \in B$ is a behaviour in $\mathbf{A}$, written as $B[R]$. $\sharp$

Next, every behaviour can be reduced to its subbehaviour that does not absorb or emit some data.

4.12. Definition. A process β of a behaviour B is said to *absorb* (resp.: *strongly absorb*) an object instance m in B iff $m \sqsubseteq dom(\beta)$ (resp.: iff $m \lhd dom(\beta)$ but not $m \lhd \beta$. $\sharp$

Note that β absorbs (resp. strongly absorbs) m in B iff in every instance $L = (X, \leq, ins)$ of β there exists $x \in X$ such that

(1) x is an occurrence of m in L, i.e., $ins(x) = m$,
(2) x is minimal (resp.: minimal but not maximal) element of L with respect to the partial order $\leq$.

4.13. Definition. A process β of a behaviour B is said to *emit* an object instance m in B iff $m \sqsubseteq cod(\beta)$ and for every $\gamma \in B$ such that β is a prefix of γ there exist ρ and δ such that $\gamma = (\beta + \rho)(m + \delta)$ (and thus $m \lhd cod(\gamma)$). ♯

Note that β emits m in B iff in every instance $L' = (X', \leq', ins')$ of every process α of B such that β is a prefix of α there exist a cross-section c, a splitting s of $head(L', c)$, and $x \in X'$ such that

(1) $L = first(head(L', c), s) = (X, \leq, ins)$ is an instance of β
(2) x is an occurrence of m in L and in L', i.e., $ins(x) = ins'(x) = m$,
(3) x is maximal element of L with respect to the partial order $\leq$ and a maximal element of L' with respect to the partial order $\leq'$.

4.14. Proposition. For every subset M of object instances from W and for every behaviour B in $\mathbf{A}$ here exists the least behaviour in $\mathbf{A}$ which contains the set of all processes $\beta \in B$ such that, for every $m \in M$, β does not absorb or emit m in B. We write it as as $B \ddagger M$. ♯

4.15. Definition. The operation $B \mapsto B \ddagger M$ is called an *internalization* of objects from M in B. ♯

Finally, behaviours can be composed in a way which reflects that they exchange data. Following [WiMa 87] the respective composition operation can be defined as follows.

4.16. Definition.
A process α of $\mathbf{A}$ is said to *consist* of processes α_1 and α_2 of $\mathbf{A}$ iff an instance $L = (X, \leq, ins)$ of α has two subsets X_1 and X_2 of its underlying set X such that:

(1) X_1 and X_2 cover X, i.e., $X_1 \cup X_2 = X$,
(2) the restrictions of L to X_1 and X_2 are instances $L_1 = (X_1, \leq_1, ins_1)$ and $L_2 = (X_2, \leq_2, ins_2)$ of α_1 and α_2, respectively,
(3) the partial order $\leq$ is the transitive closure of the union of the partial orders $\leq_1$ and $\leq_2$,
(4) $X_1 \cap X_2$ contains only such elements which are maximal in L_1 and minimal in L_2 or maximal in L_2 and minimal in L_1. ♯

4.17. Example. In figure 4.2 we have processes such that γ consists of α and β. ♯

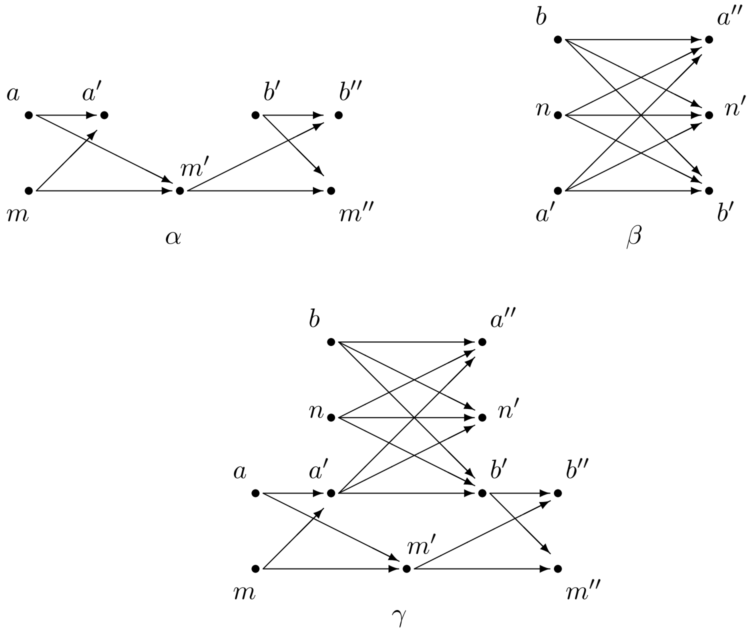


Figure 4.2

Note that every process $\alpha\beta$ consists of α and β, and every process $\delta + \gamma$ consists of δ and γ.

The following proposition are simple consequences of the definition.

4.18. Proposition. If a process γ consists of processes α and β then every prefix of γ consists of some prefixes of α and β. $\sharp$

For example, if $\gamma = \alpha\beta$ and γ' is a prefix of γ such that $\alpha = \gamma'\delta$ with $\delta \neq \mathit{cod}(\alpha)$ then γ' consists of the prefix γ' of α and the prefix 0 of β.

4.19. Proposition. If a process γ consists of processes α and β then it consists of β and α. $\sharp$

4.20. Proposition. If a process φ consists of processes α and δ and δ consists of processes β and γ then there exists a process ε such that ε consists of α and β and φ consists of ε and γ. $\sharp$

Due to proposition 4.18 we obtain the following property.

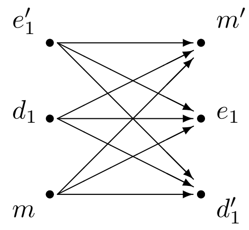
4.21. Proposition. For every two behaviours B and C in $\mathbf{A}$ there exists a unique behaviour D in $\mathbf{A}$, written as $B \parallel C$, such that a process γ is a process of D iff it consists of a process α of B and of a process β of C. $\sharp$

4.22. Definition. The operation $(B, C) \mapsto B \parallel C$ is called a *free composition* or a *merging.* $\sharp$

The lattice operations, prefixing, replacement, internalization, and merging can be used to define behaviours by fixed-point equations. Solutions of such equations exist and can be characterized due to the following theorem which follows easily from the definitions.

4.23. Theorem. The complete lattice of behaviours in $\mathbf{A}$ together with the lattice operations, merging, prefixing, and internalization, as described above, is a continuous algebra, called the *algebra of behaviours* in $\mathbf{A}$, i.e., all the operations preserve the existing least upper bounds. In particular, each derived operation $f : (\mathit{Behaviours}(\mathbf{A}))^n \to (\mathit{Behaviours}(\mathbf{A}))^n$ has the least fixed point B which is given by the least upper bound of the chain $(f^i(\emptyset, ..., \emptyset) : i = 0, 1, 2, ...)$, where $f^0(x) = x$ and $f^{i+1}(x) = f(f^i(x))$. $\sharp$

4.24. Example. Consider an automaton $\mathcal{A}$ as in example 2.10. A move of this automaton can be defined as a behaviour $\mathit{move}(d, m, m', e)$ that consists of the atomic process $\varrho(d, m, m', e)$ shown in figure 4.3 and of its prefixes, where $d = (v', i, \mathit{input})$, $m = (\mathcal{A}, q, \mathit{memory})$,
$m' = (\mathcal{A}, f(i, q), \mathit{memory})$, $e = (v'', g(i, q), \mathit{output})$,
$d' = (v', i, \mathit{sink})$, $e' = (v'', \mathit{none}, \mathit{source})$.



$\varrho(d, m, m', e)$

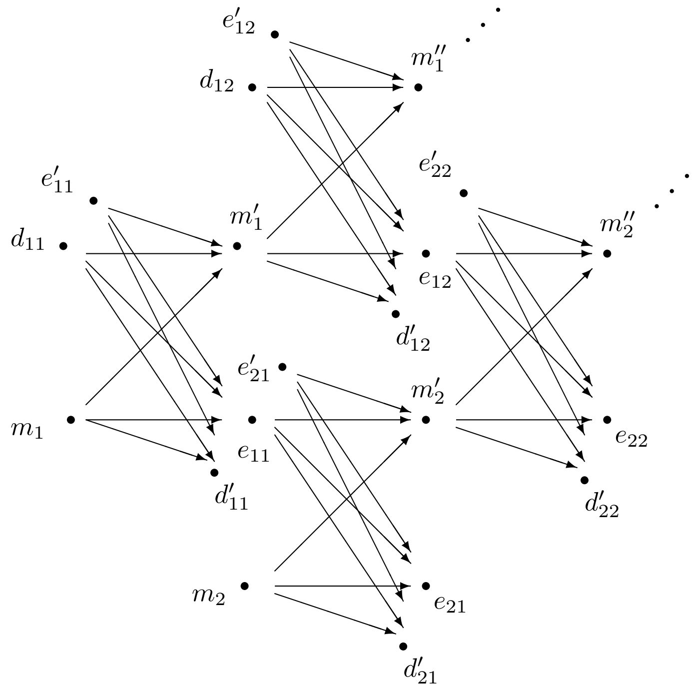
Figure 4.3

The run of $\mathcal{A}$ as described in examples 2.10 and 4.5 can be defined as the component $B(\xi, m, \eta)$ with $\xi = \mu$, $m = (\mathcal{A}, q, \mathit{memory})$, $\eta = \nu$, of the least solution of the following system of equations:
$B(d\xi, m, e\eta) = (((\mathit{move}(d, m, m', e).B(\xi, m', \eta))[k/m]) \ddagger T)[m/k]$,
where $[k/m]$ is the substitution of $k = (\mathcal{A}, q, \mathit{outside})$
for $m = (\mathcal{A}, q, \mathit{memory})$, $[m/k]$ is a substitution of m for k, and T is the set of data d with $\mathit{position}(d) = \mathit{memory}$. $\sharp$

4.25. Example. Consider two copies $\mathcal{A}_1$ and $\mathcal{A}_2$ of an automaton $\mathcal{A}$ as in example 2.10, respectively with the copies Q_1 and Q_2 of Q, the copies memory_1 and memory_2 of memory, the copies input_1 and input_2 of input, and the copies output_1 and output_2 of output, where $\mathit{output}_1 = \mathit{input}_2 = k$. The behaviour of the system of these automata with the initial state q_1 of $\mathcal{A}_1$, the initial state q_2 of $\mathcal{A}_2$, the sequence μ of input data, the sequence ν of output data can be defined as

$R(\mu, m_1, m_2, \nu) = (\bigcup(B(\mu, m_1, \eta) \parallel B(\xi', m_2, \nu))) \ddagger M$

where the union extends on the possible η and ξ', $B(\mu, m_1, \eta)$ and $B(\xi', m_2, \eta)$ are behaviours as in example 4.24, and M is the set of all data d with $\mathit{position}(d) = k$. A process of this behaviour is illustrated in figure 4.4. $\sharp$



$R(\xi, m', m'', \eta)$

Figure 4.4

5

Random behaviours

Faulty computer systems, some production systems controlled by automata, some communication systems, and the like, may show random behaviours. In order to characterize such behaviours it is necessary to define for each system an adequate probability space.

The definition of probability spaces characterizing random behaviours is relatively obvious for sequential systems since processes of such systems and segments of processes can be identified with paths of the corresponding transition systems, and branching of paths at states represents always a choice. It is less obvious for concurrent systems since in such systems branching paths may represent segments of the same process and, consequently, branching at states does not necessarily represents a choice. To see this consider two sequential machines as in example 1.1, the first machine executing each of actions α and β with probability 0.5, each machine working independently and synchronizing with the other by executing action γ. These machines form together a system represented by their product shown in figure 1.2. In this system the paths $(a,c) \xrightarrow{\alpha} (a,c) \xrightarrow{\delta} (a,d)$ and $(a,c) \xrightarrow{\delta} (a,d) \xrightarrow{\alpha} (a,d)$ represent the same initial segment of a process of this system. Consequently, branching at (a,c) does not represent a choice. Similarly, the paths $(a,c) \xrightarrow{\beta} (b,c) \xrightarrow{\delta} (b,d)$ and $(a,c) \xrightarrow{\delta} (a,d) \xrightarrow{\beta} (b,d)$ represent the same initial segment of a process. Consequently, branching at (a,c) does not represent a choice. In particular, the probabilities of transitions from this state to other states need not to sum up to 1, as it really happens.

Sometimes the difficulties of this type can be overcome by representing a concurrent system as collection of sequential modules, each module with its own probabilistic choice of transitions, and by identifying each process of entire system as a sequence of interleaved transitions of its modules (see [HSP 83], [Kw 03], [LSV 07], [ML 07]). However, this is possible only for discrete systems.

In the present paper we present a more general approach. Namely, we define probabilities with which processes of a system enjoy given properties.

Set-theoretical models of random behaviours

A way of defining a probability space representing a random behaviour is to define it as a projective limit of a projective system consisting of a directed family of probability spaces characterizing initial parts of the represented behaviour, each such a space obtained by endowing a set of processes with a suitable σ-algebra of subsets and with a suitable probability measure defined on this σ-algebra. It can be done as follows.

Let B be a behaviour in an algebra of processes $\mathbf{A} = (A, ;, +)$ in the sense of definition 4.1, and let $\Omega(B)$ be the set of maximal elements of B with respect to the prefix order $\sqsubseteq$.

Our aim is to show how to provide $\Omega(B)$ with a suitable probability measure μ on a given σ-algebra $\mathcal{F}$ of subsets of $\Omega(B)$. Our idea is to define μ with the aid of probability distributions on the sets of maximal elements of initial parts of the considered behaviour, called *sections*.

First of all, we define a directed partially ordered set of sections of the behaviour. This can be done as follows.

5.1. Definition. Two elements of B are said to be *confluent* iff they are predecessors of an element of B relative to the prefix order. $\sharp$

5.2. Definition. A set I of elements of B is said to be *confluence-free* iff it does not contain different elements that are confluent. $\sharp$

Note that the set of maximal elements of every subset of B which contains all the least upper bounds of its finite subsets is a confluence-free set.

From Kuratowski - Zorn Lemma, which says that in every partially ordered set in which every chain has an upper bound there exists a maximal element, we obtain the following property.

5.3. Proposition. Each confluence-free set of elements of B is contained in a maximal confluence-free set. $\sharp$

Note that the set of all sources of maximal elements of the behaviour B is a maximal confluence-free set.

5.4. Definition. Each maximal confluence-free set of bounded initial segments of maximal elements of the behaviour B is said to be a *section* of B. $\sharp$

5.5. Example. The following sets of processes of the behaviour B_1 defined in example 4.3 are sections of this behaviour (see figure 5.1):

$I = \{a + c, a + d, b + c, b + d\}$

$J = \{a + d, b + c, b + d, a + \delta\}$

$K = \{a + d, b + c, b + d, \alpha + c, \beta + c\}$

$L = \{a + d, b + c, b + d, \alpha + \delta, \beta + c\}$ $\sharp$

5.6. Example. Let B_2 be the behaviour of a producer p and a distributor d as in example 4.4. For every real $s \geq 0$ there exists a variant Q' of the process Q of the producer that has the length s. Similarly, for every real $t \geq 0$ there exists a variant R' of the process R of the distributor that has the length t. Consequently, for every real $s \geq 0$ and $t \geq 0$, the set of processes of B_2 of the form $\varphi + \psi$ such that φ is a run of the producer of the length s and ψ is a process of the distributor of the length t is a non-empty set $I(s,t)$. As two different members of $I(s,t)$ cannot be prefixes of a process in B_2, the set $I(s,t)$ is a section of B_2.

Let B_3 be the behaviour of a producer p and a distributor d as in example 5.4. For every integer $n \geq 1$, let $J(n)$ be the set of processes of B_3 of the form

$(\varphi_1 + \psi_1)\sigma_1 ... (\varphi_n + \psi_n)\sigma_n$

where φ_i, ψ_i, σ_i represent variants of abstract processes $[Q]$, $[R]$, $[S]$, respectively. As two different members of $J(n)$ cannot be prefixes of a process of B_3, the set $J(n)$ is a section of B_3. $\sharp$

5.7. Definition. We say that a section I of B *precedes* another such a section J, and we write $I \ll J$, iff each element of J has a prefix in I. $\sharp$

5.8. Proposition. The set of all sections of B with the partial order $\ll$ is a directed set $\mathcal{T}(B)$. $\sharp$

For a proof it suffices to consider two arbitrary sections of B, say I and J, and to notice that the set K of maximal elements of the union of the downward closures of I and J is a section of B.

Now, taking into account the directed set $\mathcal{T}(B)$, we may think of defining the required probability space as a limit in the category **PSPACES** of a projective system of simpler probability spaces (see Appendix D for the concept of a projective system and its limit).

For $I \in \mathcal{T}(B)$, let $\mathbf{\Gamma}_I = (\mathit{\Gamma}_I, \mathcal{F}_I, \mu_I)$ be probability spaces such that

(1) $\mathit{\Gamma}_I = I$,
(2) $\mathcal{F}_I$ is a σ-algebra of subsets of I.

For $I, J \in \mathcal{T}(B)$ such that $I \ll J$, let $\pi_{IJ} : \mathit{\Gamma}_J \to \mathit{\Gamma}_I$ be the mappings assigning to each $j \in J$ its predecessor $i \in I$. Due to $I \ll J$ there exists such a predecessor and due to the fact that I is confluence-free it is unique.

The following facts follow easily from definitions.

5.9. Proposition. If $\pi_{IJ}(F) \in \mathcal{F}_I$ for all $F \in \mathcal{F}_J$ and $\mu_J(\pi_{IJ}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{F}_I$ then $\pi_{IJ} : \mathit{\Gamma}_I \leftarrow \mathit{\Gamma}_J$ is a morphism $\pi_{IJ} : \mathbf{\Gamma}_I \leftarrow \mathbf{\Gamma}_J$ $\sharp$

5.10. Proposition. If $\pi_{IJ}(F) \in \mathcal{F}_I$ for all $F \in \mathcal{F}_J$ and $\mu_J(\pi_{IJ}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{F}_I$ then $(\mathbf{\Gamma}_I \stackrel{\pi_{IJ}}{\leftarrow} \mathbf{\Gamma}_J : I, J \in \mathcal{T}(B), I \ll J)$ is a projective system in **PSPACES**. $\sharp$

Let $\mathbf{\Gamma} = (\Omega(B), \mathcal{F}, \mu)$ be a probability space such that $\mathcal{F}$ is the σ-algebra of subsets of $\Omega(B)$ generated by the σ-algebras $\mathcal{G}_I$, $I \in \mathcal{T}(B)$, where every $G \in \mathcal{G}_I$ is an I-cylinder in the sense that together with an element with a prefix belonging to I it contains also all the elements with this prefix, and where $\mathcal{G}_I \subseteq \mathcal{G}_J$ for $I \ll J$. Let π_{I*} be the mapping that assigns to each element of $\Omega(B)$ its unique prefix in I.

5.11. Theorem. The probability space $\mathbf{\Gamma} = (\Omega(B), \mathcal{F}, \mu)$ is a limit of the projective system $(\mathbf{\Gamma}_I \stackrel{\pi_{IJ}}{\leftarrow} \mathbf{\Gamma}_J : I, J \in \mathcal{T}(B), I \ll J)$, where each
$\mathbf{\Gamma}_I = (\Gamma_I, \mathcal{F}_I, \mu_I)$ is the probability space such that

(1) $\Gamma_I = I$,
(2) $\mathcal{F}_I$ is the σ-algebra of those subsets of I whose inverse-images under π_{I*} belong to $\mathcal{G}_I$,
(3) $\mu(\pi_{I*}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{F}_I$,

and every $\pi_{IJ} : \Gamma_I \leftarrow \Gamma_J$ is the morphism assigning to each $j \in J$ its unique predecessor $i \in I$. $\sharp$

5.12. Example. Consider the following probability measures on the sections I, J, K, L defined in example 6.5 of the behaviour B_1 of the system M of machines M_1 and M_2 in example 4.3:

$\mu_I(\{a+c\}) = 1,\ \mu_I(\{a+d\}) = \mu_I(\{b+c\}) = \mu_I(\{b+d\}) = 0$
$\mu_J(\{a+\delta\}) = 1,\ \mu_J(\{a+d\}) = \mu_J(\{b+c\}) = \mu_J(\{b+d\}) = 0$
$\mu_K(\{\alpha+c\}) = \mu_K(\{\beta+c\}) = 0.5$
$\mu_K(\{a+d\}) = \mu_K(\{b+c\}) = \mu_K(\{b+d\}) = 0$
$\mu_L(\{\alpha+\delta\}) = \mu_L(\{\beta+c\}) = 0.5$
$\mu_L(\{a+d\}) = \mu_L(\{b+c\}) = \mu_L(\{b+d\}) = 0.$

Then $I \ll J \ll L$, $I \ll K \ll L$, and it is easy to verify that the probability spaces corresponding to these measures satisfy the conditions of Proposition 5.10. For example, we have

$$\mu_K(\{\alpha+c\}) = \mu_L(\pi_{KL}^{-1}(\{\alpha+c\})) = \mu_L(\{\alpha+\delta\}) = 0.5$$
$$\mu_I(\{a+c\}) = \mu_K(\pi_{IK}^{-1}(\{\alpha+c\})) = \mu_K(\{\alpha+c, \beta+c\}) =$$
$$= \mu_K(\{\alpha+c\}) + \mu_K(\{\beta+c\}) = 0.5 + 0.5 = 1. \ \sharp$$

Random behaviours as described in this paper are similar to classical stochastic processes as defined in [F 66], [Mey 66], and [Par 80]. In order to define them we have to solve the problem of defining the respective projective systems of probability spaces and the problem of the defining for such systems the respective limits.

In the case of the second problem the main point is to guarantee the existence of the required extension of given probability measures. For some behaviours the spaces of their runs are simple enough to exploit the known results on the existence of stochastic processes. For instance, with such a situation we have to do in the case of the behaviour of the system in example 4.3 where the space of processes is contained in the product of finite sets. However, in general we need universal results on the existence of limits of projective systems of probability spaces. One of them can be the result that the respective limit exists if the probability measures of system components are regular in the sense that they can be approximated by their values on members of a compact family of measurable subsets, where compactness means that every subfamily with nonempty intersections of all finite subfamilies has a nonempty intersection (see [Mey 66] for detailed notions and results which can easily be adapted).

In the case of defining for the considered behaviour B a projective system of probability spaces representing initial segments of this behaviour it is sometimes possible to assume a limited dependence of processes of this behaviour on the past, as in Markov processes.

To see this let us consider a random behaviour
$\mathbf{\Gamma} = (\Omega(B), \mathcal{F}, \mu)$ which is a limit of a projective system $(\mathbf{\Gamma}_I \overset{\pi_{IJ}}{\leftarrow} \mathbf{\Gamma}_J : I, J \in \mathcal{T}(B), I \ll J)$ of probability spaces $\mathbf{\Gamma}_I = (\Gamma_I, \mathcal{F}_I, \mu_I)$, and sections I and J such that $I \ll J$.

For every $\beta \in J$ there exists in I a unique prefix $\alpha = \pi_{IJ}(\beta)$, and a unique ξ, written as $link_{IJ}(\beta)$, such that $\alpha\xi = \beta$. We say that the set of ξ such that $\xi = link_{IJ}(\beta)$ for some $\beta \in J$, written as $[I, J]$, is a *segment* of B.

It is clear that the mapping $\pi_{IJ} : J \to I$ is surjective. We call it the *projection* of J on I.

Similarly, it is clear that the mapping $link_{IJ} : J \to [I, J]$ is bijective. We call it the *reduction* of J to $[I, J]$.

Moreover, for every $\xi \in [I, J]$ there exists a unique $\alpha \in I$ such that $\alpha\xi \in J$, written as $pred_{IJ}(\xi)$, and that $\pi_{IJ}(\beta) = pred_{IJ}(link_{IJ}(\beta))$.

Finally, by $\mathcal{F}_{[IJ]}$ we denote the σ-algebra of those $F \subseteq [I, J]$ for which $link_{IJ}^{-1}(F) \in \mathcal{F}_J$.

For every $E \in \mathcal{F}_I$ we have $pred_{IJ}^{-1}(E) \in \mathcal{F}_{IJ}$.

For every $E \in \mathcal{F}_I$ and for $\mu_J \pi_{IJ}^{-1}(E)$ defined as $\mu_J(\pi_{IJ}^{-1}(E))$ we have $\mu_J \pi_{IJ}^{-1}(E) = \mu_I(E)$.

For every $\xi \in \Gamma_I$ and every $F \in \mathcal{F}_J$ we have a conditional probability $\mu_{IJ}(F|\xi)$, where

$$\mu_J(F \cap \pi_{IJ}^{-1}(E)) = \int_E \mu_{IJ}(F|\xi) d\mu_J \pi_{IJ}^{-1}(\xi) \text{ for every } E \in \mathcal{F}_I$$

or, equivalently,

$$\mu_J(F \cap \pi_{IJ}^{-1}(E)) = \int_E \mu_{IJ}(F|\xi) d\mu_I(\xi) \text{ for every } E \in \mathcal{F}_I.$$

Now suppose that the choice of a run in a state does not depend on the past in the sense that $\mu_{IJ}(F|\xi) = \mu_{IJ}(F|\xi')$ whenever $cod(\xi) = cod(\xi')$ and

$\mu_{IJ}(F|\xi) = \mu_{IJ}(F'|\xi)$ whenever $link_{IJ}(F) = link_{IJ}(F')$. Then the conditional probabilities $\mu_{IJ}(F|\xi)$ can be regarded as values $P_{IJ}(G|x)$ of a function P_{IJ} for $G = link_{IJ}(F)$ and $x = cod(\xi)$, where

(*) $P_{IJ}(G|x) = \int_{G'} P_{KJ}(G''|u)dP_{IK}(u|x)$

for $G = G'G''$ with $G' \in \mathcal{F}_{IK}$ and $G'' \in \mathcal{F}_{KJ}$.

Consequently, knowing μ_I for some I and the functions P_{IJ} we can find μ_J using the formula

(**) $\mu_J(F) = \int_{\Gamma_I} P_{IJ}(link_{IJ}(F)|cod(\xi))d\mu_I(\xi)$.

5.13. Example. For the sections
$I = \{a+c, a+d, b+c, b+d\}$,
$K = \{a+d, b+c, b+d, \alpha+c, \beta+c\}$,
$L = \{a+d, b+c, b+d, \alpha+\delta, \beta+c\}$
of the behaviour B_1 in example 4.3 we have
$I \ll K \ll L$,
$[I, K] = \{a+d, b+c, b+d, \alpha+c, \beta+c\}$,
$\pi_{IK}(\alpha+c) = a+c$,
$link_{IK}(\alpha+c) = \alpha+c$,
$[K, L] = \{a+d, b+c, b+d, \alpha+\delta, \beta+c\}$,
$\pi_{KL}(\alpha+\delta) = \alpha+c$,
$link_{KL}(\alpha+\delta) = a+\delta$.
Consequently, for
$\mu_I(\{a+c\}) = 1$,
$P_{IK}(\{\alpha+c\}|a+c) = P_{IK}(\{\beta+c\}|a+c) = 0.5$,
$P_{KL}(\{a+\delta\}|a+c) = P_{KL}(\{b+c\}|b+c) = 1$,
we obtain
$\mu_K(\{\alpha+c\}) = \int_{\Gamma_I} P_{IK}(\{\alpha+c\}|cod(\xi))d\mu_I(\xi)$
$= P_{IK}(\{\alpha+c\}|a+c)\mu_I(\{a+c\}) = 0.5$,

$\mu_K(\{\beta+c\}) = \int_{\Gamma_I} P_{IK}(\{\beta+c\}|cod(\xi))d\mu_I(\xi)$
$= P_{IK}(\{\beta+c\}|a+c)\mu_I(\{a+c\}) = 0.5$,

$\mu_L(\{\alpha+\delta\}) = \int_{\Gamma_K} P_{KL}(\{a+\delta\}|cod(\xi))d\mu_K(\xi)$
$= P_{KL}(\{a+\delta\}|a+c)\mu_K(\{\xi \in K : cod(\xi) = a+c\})$
$= P_{KL}(\{b+c\}|a+c)\mu_K(\{\alpha+c\}) = 0.5$,

$\mu_L(\{\beta+c\}) = \int_{\Gamma_K} P_{KL}(\{\beta+c\}|cod(\xi))d\mu_K(\xi)$
$= P_{KL}(\{b+c\}|b+c)\mu_K(\{\xi \in K : cod(\xi) = b+c\})$
$= P_{KL}(\{b+c\}|b+c)\mu_K(\{\beta+c\}) = 0.5$.

Similarly for other initial segments. ♯

5.14. Example. Consider the behaviour B_2 in example 4.4.

Let Φ and Ψ be respectively the set of processes of the producer and the set of processes of the distributor.

Let Σ be the set of variants of the process $[S]$ of transfer of material from the producer to the distributor.

Let Π be the set of processes of the form $\varphi + \psi$, where $\varphi \in \Phi$ and $\psi \in \Psi$ are respectively the component of the producer and the component of the distributor.

Let $f_s : \Pi \to [0, +\infty)$ be the function with $f_s(\pi)$ defined for every process $\pi \in \Pi$ as the amount of material at disposal of the producer participating in π at the moment s of its local time.

Let $g_s : \Pi \to [0, +\infty)$ be the function with $g_t(\pi)$ defined for every process $\pi \in \Pi$ as the amount of material at disposal of the distributor participating in π at the moment t of its local time.

Given real $b \geq a \geq 0$, $q \geq 0$, and a Borel subset X of the interval $[0, +\infty)$, suppose that $P'_{ab}(X|q)$ is the probability that the producer, which has at the moment a of its local time the amount q of material and acts, gets at the moment b of its local time an amount x of material such that $x \in X$. Suppose that

$$P'_{ac}(X|q) = \int_{[0,+\infty)} P'_{bc}(X|\xi) dP'_{ab}(\xi|q)$$

for all $c \geq b \geq a \geq 0$ and $q \geq 0$.

Given real $b \geq a \geq 0$, $r \geq 0$, and a Borel subset Y of the interval $[0, +\infty)$, suppose that $P''_{ab}(Y|r)$ is the probability that the distributor, which has at the moment a of its local time the amount r of material and acts, gets at the moment b of its local time an amount y of material such that $y \in Y$. Suppose that

$$P''_{ac}(Y|r) = \int_{[0,+\infty)} P''_{bc}(Y|\eta) dP''_{ab}(\eta|r)$$

for all $c \geq b \geq a \geq 0$ and $r \geq 0$.

Given a section $I(s,t)$ of B_2, let $\mathcal{F}_{I(s,t)}$ be the least σ-algebra of subsets of $I(s,t)$ that contains all the inverse-images of Borel subsets of the product $[0, +\infty) \times [0, +\infty)$ under the mappings $h_{s',t'} : \pi \mapsto (f_{s'}(\pi), g_{t'}(\pi))$ with $0 \leq s' \leq s$ and $0 \leq t' \leq t$.

For $0 \leq s' \leq s''$ and $0 \leq t' \leq t''$ we have the σ-algebra $\mathcal{F}_{I(s',t')I(s'',t'')}$ of those $F \subseteq [I(s',t'), I(s'',t'')]$ for which $link^{-1}_{I(s',t')I(s'',t'')}(F) \in \mathcal{F}_{I(s'',t'')}$.

For $q \geq 0$, $r \geq 0$, and Borel subsets X and Y of the interval $[0, +\infty)$, we define

$$P_{I(s',t')I(s'',t'')}(link^{-1}_{I(s',t')I(s'',t'')}\ (f^{-1}_{s''}(X) \cap g^{-1}_{t''}(Y))|\{(p,q),(d,r)\}) =$$
$$= P'_{s's''}(X|q) P''_{t't''}(Y|r)$$

Then for every $q \geq 0$ and $r \geq 0$ the function thus defined extends to a unique probability measure $P_{I(s',t')I(s'',t'')}(.|\{(p,q),(d,r)\})$ on the σ-algebra $\mathcal{F}_{I(s',t')I(s'',t'')}$ of subsets of $[I(s',t'), I(s'',t'')]$ such that the rule (*) is satisfied. Consequently, given a probability measure $\mu_{I(0,0)}$ on the σ-algebra $\mathcal{F}_{I(0,0)}$

of subsets of $I(0,0)$, by applying the rule (**) it is possible to define the probability measures $\mu_{I(s,t)}$ on $\mathcal{F}_{I(s,t)}$ for all $s \geq 0$ and $t \geq 0$, and construct the respective projective system and its limit. As every section of B_2 is dominated by some $I(s,t)$, the result gives the required probability space.

Consider the behaviour B_3 in example 4.4.

Let Φ, Ψ, Π, f_s, g_t, P'_{ab}, P''_{ab}, $h_{s,t}$, $\mathcal{F}_{I(s',t')I(s'',t'')}$, $P_{I(s',t')I(s'',t'')}$, $\mu_{I(s,t)}$ be as before, and let Δ' and Δ'' be given positive real numbers.

Suppose that the producer and the distributor act in steps, the producer Δ' units of its local time in each step, the distributor Δ'' units of its local time in each step, and that each step ends with a transfer of an amount m of material from the producer to the distributor, where $m = \lambda(q', r')$ for the producer with an amount q' of material and the distributor with an amount r' of material.

Then the probability of the system consisting of the producer and the distributor to pass from a state $\xi = \{(p,q),(d,r)\}$ to a state in a Borel subset Z of the product $[0,+\infty) \times [0,+\infty)$ is

$$P_{I(0,0)I(\Delta',\Delta'')}(\Lambda^{-1}_{\Delta'\Delta''}(Z|\xi)$$

where $\Lambda_{\Delta'\Delta''} : \pi \mapsto (f_{\Delta'}(\pi) - \lambda(f_{\Delta'}(\pi), g_{\Delta''}(\pi)), g_{\Delta''}(\pi) - \lambda(f_{\Delta'}(\pi), g_{\Delta''}(\pi)))$.

On the other hand, $\mathcal{F}_{J(n)J(n+1)}$ is the σ-algebra of sets $G(F)$, where $F \in \mathcal{F}_{I(0,0)I(\Delta',\Delta'')}$ and $\gamma \in G(F)$ iff $\gamma = \pi\sigma_\pi$ with $\pi \in F$ and σ_π being the transfer of the amount $\lambda(q', r')$ of material for $\{(p,q'),(d,r')\}$ being the final state of π.

Consequently, for every $n = 1, 2, ...$, every state $\xi = \{(p,q),(d,r)\}$, and every $G(F) \in \mathcal{F}_{J(n)J(n+1)}$ we can define

$$P_{J(n)J(n+1)}(G(F)|\xi) = P_{I(0,0)I(\Delta',\Delta'')}(F|\xi)$$

and then combine $P_{J(n)J(n+1)}$ to define $P_{J(n)J(m)}$ for arbitrary $1 \leq n \leq m$ such that the rule (*) is satisfied. Hence, given a probability measure $\mu_{I(0,0)}$, we can define $\mu_{J(0)} = \mu_{I(0.0)}$ and $\mu_{J(n)}$ for $n = 0, 1, \ldots$, and construct the respective projective system and its limit. As every section of B_3 is dominated by some $J(n)$, the result gives the required probability space. $\sharp$

Models related to Scott topology

The idea described in [VVW 04] can be applied to provide with probability measures behaviours which are continuous directed complete posets. Every such a behaviour B together with its Scott open subsets is a topological space with the Borel σ-algebra $\mathcal{B}$ of subsets generated by Scott open subsets. Every normalized continuous valuation ν of Scott open subsets of B extends uniquely to a probability measure ν' on $\mathcal{B}$. Then the probability measure ν' can be transported to the restriction of B to the subspace $\Omega(B)$ formed by the maximal elements of B. To this end, it suffices to define $\mathcal{B}' = \{f \cap \Omega(B) : F \in \mathcal{B}$ and to assign the value $\nu'(F)$ to every $F \cap \Omega(B)$

with $F \in \mathcal{B}$. Consequently, we obtain a probability space $(\Omega(B), \mathcal{B}', \mu)$, as required.

However, in the present paper we try to develop a basis as universal as possible for describing and studying random behaviours of concurrent systems, a basis that would allow us to describe in a uniform way behaviours of systems of various kinds, including behaviours that need not to be continuous directed complete posets. To this end, we shall describe again how the required measure μ on the σ-algebra $\mathcal{B}$' of subsets of the set $\Omega(B)$ of maximal elements of a behaviour B can be obtained from probability distributions on the sets of maximal elements of initial parts of B. The idea is similar to that for set theoretical models, but now it exploits the topological properties of behaviours.

First of all, we define a directed partially ordered set of subspaces of a behaviour B representing initial parts of B and a directed partially ordered set of subspaces of these subspaces consisting of their maximal elements. This can be done as follows.

5.15. Definition. Each subspace of a behaviour B that is downward closed and contains all the exising least upper bounds of its subsets and all the sources of initial segments of maximal elements of B is called an *initial fragment* of B. The subspace $I = \Omega(P)$ of an initial fragment P of B that consists of the maximal elements of P is called a *topological section* (or briefly a *section*) of B. The set of subsets of $I = \Omega(P)$ of the form $F \cap I$, where F belongs to the Borel σ-algebra $\mathcal{B}$ of subsets of B, is a σ-algebra $\mathcal{B}_I$, called the *natural* σ-algebra of subsets of I. $\sharp$

It follows from this definition that every initial fragment of a behaviour is Scott closed, that it is a directed complete poset, and that every topological section consisting of bounded processes is a section in the sense of definition 5.4.

5.16. Example. Each downward closed subspace of the behaviour B_1 in example 4.3 that contains the existing least upper bounds of its subsets of B_1 and contains the subset $I = \{a+c, a+d, b+c, b+d\}$ of B_1 is an *initial fragment* of B_1. In particular, the following subsets I, E, E', E'', F, G of B_1 are initial fragments of B_1 and the following I, J, J', J'', K, L of B_1 are the corresponding sections of B_1:

$I = \{a+c, a+d, b+c, b+d\}$
$E = \{a+c, a+d, b+c, b+d, a+\delta\}$
$E' = \{a+c, a+d, b+c, b+d, \alpha+c, a+\delta\}$
$E'' = \{a+c, a+d, b+c, b+d, \beta+c, a+\delta\}$
$F = \{a+c, a+d, b+c, b+d, \alpha+c, \beta+c\}$
$G = \{a+c, a+d, b+c, b+d, \alpha+c, a+\delta, \alpha+\delta, \beta+c\}$

and the following subsets I, J, J', J'', K, L of B_1 are the corresponding sections of B_1 (see figure 4.1):

$I = \Omega(I) = \{a + c, a + d, b + c, b + d\}$
$J = \Omega(E) = \{a + d, b + c, b + d, a + \delta\}$
$J' = \Omega(E') = \{a + d, b + c, b + d, \alpha + c, a + \delta\}$
$J'' = \Omega(E'') = \{a + d, b + c, b + d, \beta + c, a + \delta\}$
$K = \Omega(F) = \{a + d, b + c, b + d, \alpha + c, \beta + c\}$
$L = \Omega(G) = \{a + d, b + c, b + d, \alpha + \delta, \beta + c\}$ ♯

5.17. Example. Each set of elements of the behaviour B_2 in example 4.4 that are dominated with respect to the prefix order by elements of a section $I(s,t)$ of this behaviour as in example 5.6 is an initial fragment of B_2. Each section $I(s,t)$ as in example 5.6 is a topological section of B_2 in the sense of definition 5.15.

The σ-algebra $\mathcal{F}_{I(s,t)}$ of subsets of $I(s,t)$ that was defined in example 5.14 consists of intersections of $I(s,t)$ with members of the least σ-algebra cotaining sets $\{\pi \in B_2 : f_{s'}(\pi) \leq x\}$ with $0 \leq s' \leq s$ and sets $\{\pi \in B_2 : g_{t'}(\pi) \leq y\}$ with $0 \leq t' \leq t$. On the other hand, such sets are Scott closed if processes of the producer and distributors consist of continuous segments. Consequently, the σ-algebra $\mathcal{F}_{I(s,t)}$ is then a subalgebra of the natural σ-algebra $\mathcal{B}_{I(s,t)}$.

Each set of elements of the behaviour B_3 in example 4.4 that are dominated by elements of a section $J(n)$ of this behaviour as in example 5.6 is an initial fragment of B_3 and $J(n)$ itself is a topological section of B_3. ♯

A projective system consisting of a directed family of probability spaces characterizing initial parts of a behaviour can be constructed due to the existence of a directed set of topological sections of this behaviour and due to the existence of projections of topological sections on dominated topological sections.

5.18. Proposition. Let P and Q be two initial fragments of a behaviour B such that $P \subseteq Q$, and let $I = \Omega(P)$ and $J = \Omega(Q)$. For every $j \in J$ there exists a unique $i \in I$, written as $\rho_{IJ}(j)$, such that $i \sqsubseteq j$. ♯

Proof. Let X_j be the set of $k \in P$ such that $k \sqsubseteq j$. The set X_j is nonempty since it contains $dom(j)$. It is directed since every two elements of X_j consist of prefixes of j and have the least upper bound that belongs to X_j. Consequently, there exists the least upper bound m of X_j and $m \sqsubseteq j$. As P is Scott closed, we have $m \in P$. As m is the least upper bound of X_j, it must belong to $I = \Omega(P)$, and we can define $\rho_{IJ}(j)$ as m. ♯

From the fact that an initial fragment of a behaviour is downward closed and contains the existing least upper bounds of its subsets we obtain the following proposition.

5.19. Proposition. A subset X of an initial fragment P of a behaviour B is Scott closed iff it is Scott closed in the directed complete poset P. $\sharp$

It follows from proposition 5.18 that for every $U \cap I$ with Scott open U the set $U \cap J$ is the inverse image of $U \cap I$ under $\rho_{IJ}(j)$. Consequently, we obtain the following proposition.

5.20. Proposition. The correspondence $\rho_{IJ} : J \to I$ is a measurable mapping from J equipped with the σ-algebra $\mathcal{B}_J$ to I equipped with the σ-algebra $\mathcal{B}_I$. $\sharp$

The set of initial fragments of a behaviour B is ordered by inclusion. According to proposition 5.18 the set of topological sections of B can be defined as follows.

5.21. Definition. We say that a topological section I of B *precedes* another such a section J, and we write $I \ll J$, iff each element of J has a predecessor in I. $\sharp$

5.22. Proposition. The set of all topological sections of B with the partial order $\ll$ is a directed set $\mathcal{R}(B)$. $\sharp$

For a proof it suffices to consider two arbitrary sections of B, say I and J, and to notice that the set K of maximal elements of the union of the downward closures of I and J is a section of B.

Now we may use the directed set $\mathcal{R}(B)$ to construct the required probability space as a projective limit of a projective system of probability spaces.

A projective system consisting of a directed family of probability spaces characterizing initial fragments of a behaviour can be defined as follows.

For $I \in \mathcal{R}(B)$, let $\mathbf{\Xi}_I = (\Xi_I, \mathcal{X}_I, \mu_I)$ be probability spaces such that

(1) $\Xi_I = I$,
(2) $\mathcal{X}_I$ is the σ-algebra $\mathcal{B}_I$ of subsets of I.

For $I, J \in \mathcal{R}(B)$ such that $I \ll J$, let $\rho_{IJ} : \Xi_J \to \Xi_I$ be the mappings as in proposition 5.18.

The following facts follow easily from definitions.

5.23. Proposition. Every mapping $\rho_{IJ} : \Xi_J \to \Xi_I$ is measurable and the induced mapping $F \mapsto \rho_{IJ}^{-1}(F)$ maps $\mathcal{X}_I$ into $\mathcal{X}_J$. $\sharp$

5.24. Proposition. If $\mu_I(\rho_{IJ}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{X}_I$ then $\rho_{IJ} : \Xi_J \to \Xi_I$ is a morphism $\rho_{IJ} : \mathbf{\Xi}_J \to \mathbf{\Xi}_I$ in **PSPACES**. $\sharp$

5.25. Theorem. If $\mu_J(\rho_{IJ}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{X}_I$ then $(\mathbf{\Xi}_I \overset{\rho_{IJ}}{\leftarrow} \mathbf{\Xi}_J : I, J \in \mathcal{R}(B), I \ll J)$ is a projective system in **PSPACES**. $\sharp$

Let $\mathbf{\Xi} = (\Omega(B), \mathcal{F}, \mu)$ be a probability space such that $\mathcal{F}$ is the σ-algebra $\mathcal{B}_B$ of subsets of $\Omega(B)$.

5.26. Theorem. The probability space $\mathbf{\Xi} = (\Omega(B), \mathcal{F}, \mu)$ is the projective limit of the projective system $(\mathbf{\Xi}_I \overset{\rho_{IJ}}{\leftarrow} \mathbf{\Xi}_J : I, J \in \mathcal{R}(B), I \ll J)$, where each $\mathbf{\Xi}_I = (\Xi_I, \mathcal{X}_I, \mu_I)$ is the probability space such that

(1) $\Xi_I = I$,
(2) $\mathcal{X}_I$ is the σ-algebra $\mathcal{B}_I$,
(3) $\mu(\rho_{IB}^{-1}(F)) = \mu_I(F)$ for all $F \in \mathcal{X}_I$. $\sharp$

The fact that the probability space characterizing a random behaviour of a concurrent system is a projective limit of probability spaces characterizing initial fragments of this behaviour can be exploited in an effective way because referring only to initial fragments of this behaviour we are able to decide which subsets of topological sections belong to the respective σ-algebras. Consequently, we can try approximate the required probability space by simpler probability spaces.

Another approach can be to try to characterize the required probability distribution on the set $\Omega(B)$ with the aid of a probability space $(B, \mathcal{B}, \mu)$ and try to approximate the space $(B, \mathcal{B}, \mu)$ by simpler probability spaces. To this end, we can exploit simple theorems of measure theory.

Given an initial fragment P of a behaviour B, let $\mathcal{B}(P)$ be the σ-algebra of those Borel subsets of B whose inverse images under ρ_{PB} are Borel subsets of P.

5.27. Theorem. For every initial fragments P and Q of B such that $P \subseteq Q$ there exists a conditional probability distribution $\mu_{PQ} : \mathcal{B}(Q) \times \Omega_P \to [0,1]$ on $\mathcal{B}(Q)$ with respect to $\mathcal{B}(P)$ and we have

$$\int_E \mu_{PQ}(F|x) d\mu_P(x) = \mu_Q(F \cap E)$$

for all $F \in \mathcal{B}(Q)$ and $E \in \mathcal{B}(P)$. $\sharp$

A proof follows from the definition of the conditional probability.

5.28. Theorem. For every initial fragments P, Q, R of B such that $P \subseteq Q \subseteq R$, every $G \in \mathcal{B}(R)$, and every $x \in B$, it holds

$\mu_{PR}(G|x) = \int_B \mu_{QR}(G|y) d\mu_{PQ}(y|x)$ ♯

For a proof it suffices to notice that

$\mu_R(E \cap G) = \int_E \mu_{QR}(G|y) d\mu_Q(y) = \int_E \int_B \mu_{QR}(G|y) d\mu_Q(y|x) d\mu_P(x)$
and
$\mu_R(E \cap G) = \int_E \mu_{PR}(G|x) d\mu_P(x)$

Once a probability space $(B, \mathcal{B}, \mu)$ as described is found, it is possible to use it to transport the required probability measure μ to the set $\Omega(B)$. It suffices to define $\mu'(F \cap \Omega(B))$ as $\mu(F)$ for every $F \cap \Omega(B)$ with $F \in \mathcal{B}$.

6

Behaviour-oriented algebras

Basic notions

In chapter 3 it has been shown that every algebra of processes enjoys the properties (A), (B), (C) of proposition 3.22.

In this chapter we introduce abstract algebras in which (A), (B), (C) hold, called in *behaviour-oriented algebras*, and we prove that some of such algebras can be represented as algebras of processes. Such algebras are different from algebras of processes characterized in definition 3.24 in the sense that their elements should be considered as abstract objects without any interenal structure rather than as processes in a universe of objects.

Behaviour-oriented algebras are defined as follows.

6.1. Definition. A *behaviour-oriented algebra* is a partial algebra $\mathbf{A} = (A, ;, +)$, where A is a set, $(\alpha_1, \alpha_2) \mapsto \alpha_1; \alpha_2$ is a partial operation in A, and $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ is a partial operation in A, such that the axioms (A), (B), (C) hold. We say that such a behaviour-oriented algebra is of type K if also (C9) holds. $\sharp$

The composite $\alpha_1; \alpha_2$ is written as $\alpha_1 \alpha_2$.

The reduct $(A, ;)$ of $\mathbf{A}$ is a partial category $\mathbf{pcat}(\mathbf{A})$ satisfying (A1) - (A10), called the *underlying partial category* of $\mathbf{A}$. In this partial category two partial unary operations $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable that assign to an element a source and a target, if they exist. The reduct $(A, +)$ of $\mathbf{A}$ is a partial commutative monoid $\mathbf{pmon}(\mathbf{A})$ satisfying (C1) - (C8) and containing a zero element 0 such that $\alpha + 0 = \alpha$ for every α.

An element of A is said to be *bounded* if it has a source and a target. An element $\alpha \neq 0$ of A is said to be a $(+)$-*atom* of $\mathbf{A}$ provided that for every $\alpha_1 \in A$ and $\alpha_2 \in A$ the equality $\alpha = \alpha_1 + \alpha_2$ implies that either $\alpha_1 = 0$ and $\alpha_2 = \alpha$ or $\alpha_1 = \alpha$ and $\alpha_2 = 0$. An identity of $\mathbf{pcat}(\mathbf{A})$ that is also a $(+)$-atom is said to be an *atomic identity*.

An element α of A is said to be a $(;)$-*atom* of $\mathbf{A}$ provided that it is not an identity of $pcat(\mathbf{A})$ and for every $\alpha_1 \in A$ and $\alpha_2 \in A$ the equality $\alpha = \alpha_1 \alpha_2$ implies that either α_1 is an identity and $\alpha_2 = \alpha$ or $\alpha_1 = \alpha$ and α_2 is an identity. An element α of A which is both a $(+)$-atom and $(;)$-atom is said to be a $(+, ;)$-*atom*. In particular, atomic identities are $(+, ;)$-atoms.

We say that **A** is *discrete* if every $\alpha \in A$ that is not an identity can be represented in the form $\alpha = \alpha_1 \ldots \alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are $(;)$-atoms.

Let $\mathbf{A} = (A, ;, +)$ be a behaviour-oriented algebra.

6.2. Definition. Given $\alpha \in A$, by a *cut* of α we mean a pair (α_1, α_2) such that $\alpha_1\alpha_2 = \alpha$. $\sharp$

Due to the property (A5) the algebra $\mathbf{A}$ has the properties of partial algebras of processes described in propositions 3.17 and 3.18. Consequently, cuts of every $\alpha \in A$ are partially ordered by the relation $\sqsubseteq_\alpha$, where $x \sqsubseteq_\alpha y$ with $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ means that $\eta_1 = \xi_1\delta$ with some δ. Due to (A1) and (A2) for $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \sqsubseteq_\alpha y$ there exists a unique δ such that $\eta_1 = \xi_1\delta$, written as $x \to y$. As in proposition 3.18 the partial order $\sqsubseteq_\alpha$ makes the set of cuts of α a lattice LT_α. Given two cuts x and y, by $x \sqcup_\alpha y$ and $x \sqcap_\alpha y$ we denote respectively the least upper bound and the greatest lower bound of x and y. From (A5) it follows that $(x \leftarrow x \sqcap_\alpha y \to y, x \to x \sqcup_\alpha y \leftarrow y)$ is a bicartesian square.

6.3. Definition. Given $\alpha \in A$ and its cuts $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \sqsubseteq_\alpha y$, by a *segment* of α from x to y we mean $\beta \in A$ such that $\xi_2 = \beta\eta_2$ and $\eta_1 = \xi_1\beta$, written as $\alpha|[x, y]$. A segment $\alpha|[x', y']$ of α such that $x \sqsubseteq_\alpha x' \sqsubseteq_\alpha y' \sqsubseteq_\alpha y$ is called a *subsegment* of $\alpha|[x, y]$. If $x = x'$ (resp. if $y = y'$) then we call it an *initial* (resp. a *final*) subsegment of $\alpha|[x, y]$. An initial segment of α is called also a *full prefix* of α. $\sharp$

In the sequel elements of A are called *hypothetical processes* (or briefly, *processes*) of $\mathbf{A}$. Processes of $\mathbf{A}$ which are identities of the underlying partial category $\mathbf{pcat}(\mathbf{A})$ are called *hypothetical states* (or briefly *states*) of $\mathbf{A}$. Processes which are atomic identities are called *atomic states*. A process α is said to be *global* if $\alpha + \beta$ is defined only for $\beta = 0$. A process α is said to be *bounded* if it has the source $dom(\alpha)$ and the target $cod(\alpha)$. For every process α, the existing states $u = dom(\alpha)$ and $v = cod(\alpha)$ are called respectively the *initial state* and the *final state* of α and we write α as $u \xrightarrow{\alpha} v$. The operations $(\alpha_1, \alpha_2) \mapsto \alpha_1\alpha_2$ and $(\alpha_1, \alpha_2) \mapsto \alpha_1 + \alpha_2$ are called respectively the *sequential composition* and the *parallel composition*.

6.4. Definition. If processes α_1 and α_1 are such that $\alpha_1 + \alpha_2$ is defined then we say that they are *concurrent* and write α_1 *co* α_2. The relation *co* thus defined is called the *concurrency relation* of $\mathbf{A}$. $\sharp$

For example, processes α and δ in figure 3.2 are concurrent.

With the aid of concurrency relation we can generalize the introduced in [Wink 03] notions of parallel and sequential independence of processes of Condition/Event Petri nets (cf. also [EK 76]).

6.5. Definition. Processes α_1 and α_2 such that $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$ and $\alpha_2 = c + dom(\varphi_1) + \varphi_2$ for a state c and actions φ_1 and φ_2 such that $c + \varphi_1 + \varphi_2$ is defined are said to be *parallel independent.* $\sharp$

In particular, processes $\alpha_1 = \varphi_1 + dom(\varphi_2)$ and $\alpha_2 = dom(\varphi_1) + \varphi_2$, where φ_1 and φ_2 are concurrent, are parallel independent.

6.6. Definition. Bounded processes α_1 and α_2 such that $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$ and $\alpha_2 = c + cod(\varphi_1) + \varphi_2$ for a state c and actions φ_1 and φ_2 such that $c + \varphi_1 + \varphi_2$ is defined are said to be *sequential independent.* $\sharp$

In particular, bounded processes $\alpha_1 = c + \varphi_1 + dom(\varphi_2)$ and $\alpha_2 = c + cod(\varphi_1) + \varphi_2$, where φ_1 and φ_2 are concurrent, are sequential independent.

An important feature of behaviour-oriented algebras is that in such algebras concurrency of processes implies their independence. This is a direct consequence of (C8).

From (C8) we obtain the following characterization of the parallel and the sequential independence of processes.

6.7. Theorem. Processes of the pair $v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w$ $(= (v \stackrel{\alpha_1}{\leftarrow} u, u \stackrel{\alpha_2}{\rightarrow} w))$ are parallel independent iff there exists a unique pair $v \stackrel{\alpha_2'}{\rightarrow} u' \stackrel{\alpha_1'}{\leftarrow} w$ such that $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w, v \stackrel{\alpha_2'}{\rightarrow} u' \stackrel{\alpha_1'}{\leftarrow} w)$ is a bicartesian square. $\sharp$

6.8. Theorem. Processes of the pair $u \stackrel{\alpha_1}{\rightarrow} v \stackrel{\alpha_2'}{\rightarrow} u'$ are sequential independent iff there exists a unique pair $u \stackrel{\alpha_2}{\rightarrow} w \stackrel{\alpha_1'}{\rightarrow} u'$ such that $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w, v \stackrel{\alpha_2'}{\rightarrow} u' \stackrel{\alpha_1'}{\leftarrow} w)$ is a bicartesian square. $\sharp$

Note that independence of any finite set of bounded processes can be defined as independence of every two different processes from this set. Due to (A7) the independence thus defined is equivalent to the existence of the corresponding bicartesian n-cube.

Underlying partial monoids

Let $\mathbf{A} = (A, ;, +)$ be a behaviour-oriented algebra with the underlying partial category $\mathbf{pcat}(\mathbf{A})$, with the underlying partial monoid $\mathbf{pmon}(\mathbf{A})$, with the operation $\triangle$ of taking the greatest lower bound with respect to the partial order $\triangleleft$, where $\alpha_1 \triangleleft \alpha_2$ iff $\alpha_2 = \alpha_1 + \rho$ for some ρ, and with the function $\alpha \mapsto h(\alpha)$ that assigns to each α the set of $(+)$-atoms less than or equal α with respect to the partial order $\triangleleft$.

Let A_+ denote the set of (+)-atoms of $\mathbf{A}$. Let A_0 denote the set of identities of the underlying partial category $pcat(\mathbf{A})$, and $A_{+0} = A_+ \cap A_0$ the subset of atomic identities.

6.9. Lemma. If $\alpha_1 + \alpha_2$ is defined then the greatest lower bound $\alpha_1 \triangle \alpha_2$ of α_1 and α_2 is 0. $\sharp$

Proof. Let $\alpha_1 = (\alpha_1 \triangle \alpha_2) + \xi$ and $\alpha_2 = (\alpha_1 \triangle \alpha_2) + \eta$. Since $\alpha_1 + \alpha_2$ is defined, we have $\alpha_1 + \alpha_2 = (\alpha_1 \triangle \alpha_2) + (\alpha_1 \triangle \alpha_2) + \xi + \eta$. Thus $(\alpha_1 \triangle \alpha_2) + (\alpha_1 \triangle \alpha_2)$ is defined and, by (B2), $\alpha_1 \triangle \alpha_2 = 0$. $\sharp$

6.10. Lemma. If $\alpha_1 + \alpha_2$ is defined then there exists the least upper bound of α_1 and α_2, written as $\alpha_1 \bigtriangledown \alpha_2$, and $\alpha_1 \bigtriangledown \alpha_2 = \alpha_1 + \alpha_2$. $\sharp$

Proof. $\alpha_1 + \alpha_2$ is an upper bound of α_1 and α_2. If ζ is another upper bound of α_1 and α_2 then for $\theta = \zeta \triangle (\alpha_1 + \alpha_2)$ we have $\alpha_1 \triangleleft \theta$ and $\alpha_2 \triangleleft \theta$, $\theta + \gamma = \alpha_1 + \alpha_2$, $\alpha_2 + \delta = \theta$, and $\alpha_2 + \epsilon = \theta$. Hence $\alpha_1 + \delta + \gamma = \alpha_1 + \alpha_2$ and $\alpha_2 + \epsilon + \gamma = \alpha_1 + \alpha_2$. Thus $\delta + \gamma = \alpha_2$ and $\epsilon + \gamma = \alpha_1$. Hence $\gamma \triangleleft \alpha_1$ and $\gamma \triangleleft \alpha_2$, i.e., $\gamma = 0$ by lemma 6.9. Consequently, $\theta = \zeta \triangle (\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2$. Finally, $\alpha_1 + \alpha_2 \triangleleft \zeta$, i.e., $\alpha_1 + \alpha_2 = \alpha_1 \bigtriangledown \alpha_2$. $\sharp$

6.11. Lemma. The correspondence $\alpha \mapsto h(\alpha)$ enjoys the following properties:

(1) if $\alpha_1 \neq \alpha_2$ then $h(\alpha_1) \neq h(\alpha_2)$,
(2) $h(\alpha_1 \triangle \alpha_2) = h(\alpha_1) \cap h(\alpha_2)$,
(3) if $\alpha_1 + \alpha_2$ is defined then $h(\alpha_1) \triangle h(\alpha_2) = \emptyset$,
(4) if $\alpha_1 + \alpha_2$ is defined then $h(\alpha_1 + \alpha_2) = h(\alpha_1) \cup h(\alpha_2)$. $\sharp$

Proof. For (1) refer to (B11). For (2) notice that $\xi \triangleleft \alpha_1 \triangle \alpha_2$ iff $\xi \triangleleft \alpha_1$ and $\xi \triangleleft \alpha_2$. For (3) notice that if $\alpha_1 + \alpha_2$ is defined then by proposition 2.11 we have $\alpha_1 \triangle \alpha_2 = 0$. Consequently, $h(\alpha_1 \cap \alpha_2) = \emptyset$ and it suffices to apply (2). For (4) notice that if $\xi \in h(\alpha_1 + \alpha_2)$ then $\xi \triangleleft \alpha_1 + \alpha_2$ and thus $\xi \triangleleft \alpha_1$ or $\xi \triangleleft \alpha_2$ since ξ is a (+)-atom. Consequently, $\xi \in h(\alpha_1)$ or $\xi \in h(\alpha_2)$. Conversely, if $\xi \in h(\alpha_1)$ or $\xi \in h(\alpha_2)$ then $\xi \in \alpha_1$ or $\xi \in \alpha_2$, i.e., $\xi \in h(\alpha_1 + \alpha_2)$. $\sharp$

We recall that a tolerance relation in a set is a reflexive and symmetric binary relation in this set, that for such a relation a tolerance preclass is a set whose every two elements are in this relation, and that a tolerance class is a maximal tolerance preclass.

The relation $\overline{co}$, where $\alpha_1 \ \overline{co}\ \alpha_2$ iff α_1 and α_2 are concurrent or $\alpha_1 = \alpha_2$, is a tolerance relation. We call it *the tolerance relation of* $\mathbf{A}$ and say about actions α_1 and α_2 such that $\alpha_1 \ \overline{co}\ \alpha_2$ that they *tolerate* each other.

By *tol* we denote the restriction of $\overline{co}$ to the set A_+ of (+)-atoms of $\mathbf{A}$.

The following fact is a consequence of (B7) and (B8).

6.12. Lemma. For each process α the set $h(\alpha)$ of (+)-atoms contained in α is a tolerance preclass of the relation *tol*. $\sharp$

The following fact is a consequence of (B4).

6.13. Lemma. For every tolerance preclass C of the relation *tol* there exists a process α such that $h(\alpha) = C$. $\sharp$

From lemmas 6.11 - 6.13 we obtain that elements of the partial monoid $\mathbf{pmon}(\mathbf{A})$ can be represented as tolerance preclasses of the relation *tol* and combined with the aid of set theoretical operations. More precisely, we obtain the following theorem.

6.14. Theorem. The underlying partial monoid $\mathbf{pmon}(\mathbf{A}) = (A, +)$ of $\mathbf{A}$ is isomorphic to a partial commutative monoid $\mathbf{M} = (A', +')$ with the neutral element $0'$ of tolerance preclasses of the tolerance relation *tol*, where

(1) A' is the set of tolerance preclasses of *tol* that contains all finite preclasses and is closed with respect to intersections and unions of families with an upper bound in A',
(2) the operation $+'$ is defined for pairs of disjoint preclasses from A' as the set theoretical union provided that its results belong to A',
(3) $0'$ is the empty set.

The isomorphism is given by the correspondence $\alpha \mapsto h(\alpha)$. $\sharp$

Let $\sim$ be the least congruence whose existence is guaranteed by (C7). Let *nat* be the natural homomorphism from $\mathbf{A}$ to the quotient algebra $\mathbf{A}/\sim$.

6.15. Definition. Given an atomic identity $p \in A_{+0}$, the image $nat(p)$ of p under the natural homomorphism *nat* is called an *object* corresponding to p, and p is called an *instance* of this object. $\sharp$

By $\mathbf{A}_{ob}$ we denote the set of objects corresponding to atomic identities of $\mathbf{A}$ and we call elements of $\mathbf{A}_{ob}$ *objects definable in* $\mathbf{A}$. We show that the identities of $\mathbf{pcat}(\mathbf{A})$ can be viewed as partial functions from $\mathbf{A}_{ob}$ to A_{+0}.

6.16. Theorem. The restriction of $\mathbf{pmon}(\mathbf{A})$ to the subset A_0 of identities is isomorphic to a partial commutative monoid $\mathbf{N} = (A'', +'')$ with the neutral element $0''$ of partial functions, where A'' is a set of partial functions from $\mathbf{A}_{ob}$ to A_{+0}, $u +'' v$ denotes the set theoretical union of partial functions u and

v provided that such functions have disjoint domains and their union belongs to A'', and $0''$ is the empty partial function. $\sharp$

Proof. Given an identity u, we define H_u as the set of pairs $(nat(p), p)$ with $p \in h(u)$. From the fact that $\sim$ is a strong congruence on $\mathbf{A}$ it follows that $nat(p_1) = nat(p_2)$ implies $p_1 = p_2$ since otherwise $p_1 + p_2$ would be defined and, consequently, $nat(p_1) + nat(p_2)$ would also be defined, and (B2) could not hold. Hence H_u is a partial function. The fact that $u \mapsto H_u$ defines an isomorphism follows from theorem 6.14. $\sharp$

Given an identity $u \in A_0$, each pair $(nat(p), p) \in H_u$ can be interpreted as a representant of an instance p of the object $nat(p) \in \mathbf{A}_{ob}$. Consequently, H_u can be interpreted as a partial function defined on a set of objects definable in $\mathbf{A}$ that assigns an instance to each object from its domain. For example, conditions of a Condition/Event Petri net are objects definable in the algebra of finite processes of this net and a function that for each condition from a subset of conditions of the net assigns to this condition its logical value is a state of the net.

Elements of behaviour-oriented algebras as processes

Let $\mathbf{A} = (A, ;, +)$ be a behaviour-oriented algebra of type K. With the characterization just described of identities of $\mathbf{pcat}(\mathbf{A})$ we can characterize arbitrary elements of $\mathbf{A}$.

We shall represent each such element α by a partially ordered labelled set $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$. Each element $x \in X_\alpha$ will play the role of an occurrence of the instance $l_\alpha(x)$ of the object $nat(l_\alpha(x))$. The partial order $\leq_\alpha$ will reflect how occurrences of instances of objects arise from other instances.

This way of representing elements of A will allow us to extend the correspondence $u \mapsto H_u$ by assigning to each $\alpha \in A$ the isomorphism class of partially ordered labelled sets that contains L_α.

The elements of X_α will be defined as packets of cuts of α, where a cut is a decomposition of α into two components the sequential composition of which yields α (see definition 6.2).

We start with some notions and observations.

Given a cut $x = (\xi_1, \xi_2)$ of α and an atomic identity p, we say that p *occurs* in x and call (x, p) an *occurrence* of p in x if p is contained in $cod(\xi_1) = dom(\xi_2)$.

Given an occurrence (x, p) of an atomic identity p in a cut $x = (\xi_1, \xi_2)$ of α, and an occurrence (y, q) of an atomic identity q in a cut $y = (\eta_1, \eta_2)$ of α, we say that these occurrences are *adjacent* and write $(x, p) \sim_\alpha (y, q)$ if $p = q$ and $p \sqsubseteq (x \sqcap_\alpha y \to x \sqcup_\alpha y)$, that is if $p = q$ and $(x \sqcap_\alpha y \to x \sqcup_\alpha y) = c + \varphi_1 + \varphi_2$ with an identity c that contains p and with $(x \sqcap_\alpha y \to x) = c + \varphi_1 + dom(\varphi_2)$, $(x \sqcap_\alpha y \to y) = c + dom(\varphi_1) + \varphi_2$, $(y \to x \sqcup_\alpha y) = c + \varphi_1 + cod(\varphi_2)$, $(x \to x \sqcup_\alpha y) = c + cod(\varphi_1) + \varphi_2$.

Given a cut x of α, by *atomicid*(x) we denote the set of atomic identities that occur in x. From (C7) we obtain that the cardinality of the set *atomicid*(x) is the same for all cuts of α. We call it the *width* of α and write as *width*(α). Taking into account also (C7) we obtain that the set of objects definable in $\mathcal{A}$ and having instances in *atomicid*(x) is also the same for all cuts of α. We call it the *range* of α and write as *range*(α).

6.17. Lemma. For each $\alpha \in A$ the adjacency relation $\sim_\alpha$ is an equivalence relation. $\sharp$

Proof. It suffices to prove that $\sim_\alpha$ is transitive. Suppose that $(x,p) \sim_\alpha (y,q)$ with $p = q$ and $p \sqsubseteq (x \sqcap_\alpha y \to x \sqcup_\alpha y)$, and that $(y,q) \sim_\alpha (z,r)$ with $p = q = r$ and $p \lhd (y \sqcap_\alpha z \to y \sqcup_\alpha z)$. Hence by (C6) we have $p \lhd \sigma$ for every σ that is a segment of $(x \sqcap_\alpha y \to x \sqcup_\alpha y)$ or $(y \sqcap_\alpha z \to y \sqcup_\alpha z)$. On the other hand, taking into account the fact that the set of cuts of α is a lattice, we obtain that $(x \sqcap_\alpha z \to x \sqcup_\alpha z)$ can be represented as the result of composing sequentially such segments. Consequently, $p \lhd (x \sqcap_\alpha z \to x \sqcup_\alpha z)$. Hence $(x,p) \sim_\alpha (z,r)$. Thus $\sim_\alpha$ is transitive. $\sharp$

6.18. Definition. Given $\alpha \in A$ and an atomic identity p, by an *occurrence* of p in α we mean an equivalence class of occurrences of p in cuts of α. $\sharp$

6.19. Definition. Given $\alpha \in A$, the set of occurrences of atomic identities in α, written as X_α, is called the *canonical underlying set* of α. $\sharp$

6.20. Definition. Given $\alpha \in A$, the correspondence $[(x,p)] \mapsto p$ between occurrences of atomic identities in α and the atomic identities themselves, written as l_α, is called the *canonical labelling* of (occurrences of atomic identities in) α. $\sharp$

The partial order $\leq_\alpha$ on X_α can be defined as follows.

Given an occurrence (x,p) of an atomic identity p in a cut $x = (\xi_1, \xi_2)$ of α and an occurrence (y,q) of an atomic identity q in a cut $y = (\eta_1, \eta_2)$ of α, we say that (x,p) *precedes* (y,q) and write $(x,p) <_\alpha (y,q)$ if $x \sqsubseteq_\alpha y$, p occurs in x, q occurs in y, and there is no cut v of $x \to y$ such that $(x,p) \sim_\alpha (v,p)$ and $(y,q) \sim_\alpha (v,q)$.

6.21. Lemma. For each element α of A the relation $<_\alpha$ is irreflexive and transitive. $\sharp$

Proof. The irreflexivity of $<_\alpha$ follows directly from the definition. For the transitivity suppose that $(x,p) <_\alpha (y,q)$ and $(y,q) <_\alpha (z,r)$. Then from $x \sqsubseteq_\alpha y$ and $y \sqsubseteq_\alpha z$ we obtain $x \sqsubseteq_\alpha z$. On the other hand, p occurs in x and

r occurs in z. So, it remains to prove that there is no cut v of $x \to z$ such that $(x,p) \sim_\alpha (v,p)$ and $(z,r) \sim_\alpha (v,r)$. To this end suppose the contrary and consider $y \sqcap_\alpha v \to y \sqcup_\alpha v = c + \varphi_1 + \varphi_2$, where c is an identity and $cod(\eta_1) = c + cod(\varphi_1) + dom(\varphi_2)$ for $(\eta_1, \eta_2) = y$. It cannot be $q \lhd c + cod(\varphi_1)$ since it would imply $(y \sqcap_\alpha v, p) \sim_\alpha (x,p)$ and $(y \sqcap_\alpha v, q) \sim_\alpha (y,q)$. Similarly, it cannot be $q \lhd c + dom(\varphi_2)$ since it would imply $(y \sqcup_\alpha v, r) \sim_\alpha (z,r)$ and $(y \sqcup_\alpha v, q) \sim_\alpha (z,q)$. Consequently, q cannot occur in y as it follows from $(x,p) <_\alpha (y,q)$ and $(y,q) <_\alpha (z,r)$. $\sharp$

6.22. Lemma. For each element α of A the relation $\leq_\alpha$ on X_α, where $u \leq_\alpha v$ iff $u \sim_\alpha v$ or $(x,p) <_\alpha (y,q)$ for some $(x,p) \in u$ and $(y,q) \in v$, is a partial order. $\sharp$

Proof. It suffices to prove that $(x,p) <_\alpha (y,q)$ excludes $(y,q) <_\alpha (x,p)$. To this end it suffices to notice that otherwise the identity $x \to x$ would be the result of composing sequentially $x \to y$ and $y \to x$, what is impossible according to (A3). $\sharp$

6.23. Definition. Given $\alpha \in A$, the partial order $\leq_\alpha$ is called the *canonical partial order* of (occurrences of atomic identities in) α, and $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ is called the *canonical instance* of α. $\sharp$

6.24. Lemma. Given an $\alpha \in A$, if $nat(l_\alpha(u)) = nat(l_\alpha(v))$ for some $u, v \in X_\alpha$ then $u \leq_\alpha v$ or $v \leq_\alpha u$. $\sharp$

Proof. It suffices to consider the case $u \neq v$. From $nat(l_\alpha(u)) = nat(l_\alpha(v))$ it follows that in this case $p = l_\alpha(u)$ and $q = l_\alpha(v)$ cannot occur in the same cut. Consequently, $(x,p) \in u$ and $(y,q) \in v$ for some cuts x and y such that $x \neq y$. Moreover, x and y can be chosen such that $x \sqsubseteq_\alpha y$ or $y \sqsubseteq_\alpha x$ and then we obtain respectively $(x,p) \leq_\alpha (y,q)$ or $(y,q) \leq_\alpha (x,p)$. $\sharp$

6.25. Lemma. For each $\alpha \in A$ and each object $s \in \mathbf{A}_{ob}$ the set $Z_\alpha(s)$ of $u \in X_\alpha$ such that $l_\alpha(u) = p$ for an instance p of s is a maximal chain with respect to the partial order $\leq_\alpha$ or it is empty. $\sharp$

Proof. Let $Z_\alpha(s) = \{u \in X_\alpha : l_\alpha(u) = p$ for some p with $nat(p) = s\}$. Suppose that $u_1 <_\alpha u <_\alpha u_2$ for some $u_1, u_2 \in Z_\alpha(s)$ and u with $l_\alpha(u)$ not being an instance of s. Then there exists $(x,q) \in u$ with q being an instance of some $s' \in \mathbf{A}_{ob}$ that is different from s and has an occurrence in a cut that does not contain an occurrence of s. But this is impossible since every cut of α contains an occurrence of s. $\sharp$

6.26. Lemma. For each $\alpha \in A$ of finite width a subset $Y \subseteq X_\alpha$ is a maximal antichain of the partially ordered set $(X_\alpha, \leq_\alpha)$ iff it corresponds to the set of occurrences of atomic identities in a cut y of α. $\sharp$

Proof. Let y be a cut of α. From the definition of the partial order $\leq_\alpha$ we obtain that equivalence classes of occurrences of atomic identities in y are pairwise incomparable. Thus they form an antichain $Y = H'(y)$. According to (C7) for each $u \in X_\alpha$ that does not belong to Y there exists $v \in Y$ such that $nat(l_\alpha(u)) = nat(l_\alpha(v))$ and, by lemma 6.24, v is comparable with u. Consequently, Y is a maximal antichain.

Let Y be a maximal antichain of $(X_\alpha, \leq_\alpha)$. Then all different $u, v \in Y$ are incomparable with respect to $\leq_\alpha$ and it follows from the definition of $\leq_\alpha$ that there exists a cut x of α such that for some atomic identities p and q (x, p) is an instance of u and (x, q) is an instance of v. As α is of finite width, it is possible to construct step by step a cut y such that each element of Y has an instance in y. Namely, given a cut y_n such that (y_n, p_1),...,(y_n, p_n) are instances of elements u_1,...,u_n of Y, and an element u of Y that is incomparable with u_1,...,u_n and has instances (x_1, p_{n+1}),...,(x_n, p_{n+1}) such that $(x_1, p_1) \sim_\alpha (y_n, p_1)$,..., $(x_n, p_n) \sim_\alpha (y_n, p_n)$, we define y_{n+1} as $(x_1 \sqcup_\alpha y_n) \sqcap_\alpha ... \sqcap_\alpha (x_n \sqcup_\alpha y_n)$ if $(y_n, q) <_\alpha (x_1, p_{n+1})$ for some q, or as $(x_1 \sqcap_\alpha y_n) \sqcup_\alpha ... \sqcup_\alpha (x_n \sqcap_\alpha y_n)$ if $(x_1, p_{n+1}) <_\alpha (y_n, q)$ for some q. In the first case $(x_i \sqcap_\alpha y_n \to x_i \sqcup_\alpha y_n) = c_i + \varphi_{i1} + \varphi_{i2}$ with an identity c_i containing p_i and $cod(\varphi_{i2})$ containing p_{n+1}, and we obtain $(x_i \to x_i \sqcup_\alpha y_n) = c_i + \varphi_{i1} + cod(\varphi_{i2})$ with p_{n+1} contained in $c_i + cod(\varphi_{i2})$ and $(y_n \to x_i \sqcup_\alpha y_n) = c_i + cod(\varphi_{i1}) + \varphi_{i2}$ with p_i contained in $c_i + cod(\varphi_{i1})$. Hence $(x_i, p_i) \sim_\alpha (x_i \sqcup_\alpha y_n, p_i)$ and $(x_i \sqcup_\alpha y_n, p_{n+1}) \sim_\alpha (x_i, p_{n+1})$. From $(y_n \to x_i \sqcup_\alpha y_n) = c_i + cod(\varphi_{i1}) + \varphi_{i2}$ and $y_n \to y_{n+1} \to x_i \sqcup_\alpha y_n$ we obtain by (B4) $(y_n \to y_{n+1}) = c_i + cod(\varphi_{i1}) + \gamma_i$ and $(y_{n+1} \to x_i \sqcup_\alpha y_n) = c_i + cod(\varphi_{i1}) + \delta_i$. Hence $(x_i, p_i) \sim_\alpha (y_{n+1}, p_i)$. From $(x_i \sqcup_\alpha y_n, p_{n+1}) \sim_\alpha (x_i, p_{n+1})$ and $(x_1, p_{n+1}) \sim_\alpha ... \sim_\alpha (x_n, p_{n+1})$ we obtain $(x_i \sqcup_\alpha y_n, p_{n+1}) \sim_\alpha (x_1, p_{n+1})$ for all $i \in \{1, ..., n\}$. Hence $(x_1 \sqcap (x_i \vee_\alpha y_n) \to x_1 \sqcup (x_i \sqcup_\alpha y_n)) = d_i + \psi_{i1} + \psi_{i2}$ with identities d_i containing p_{n+1} for all $i \in \{1, ..., n\}$ and, finally, $(x_1 \sqcap y_{n+1} \to x_1 \sqcup y_{n+1}) = d + \psi_1 + \psi_2$ with an identity d containing p_{n+1}. Thus $(y_{n+1}, p_1) \sim_\alpha (y_n, p_1)$,..., $(y_{n+1}, p_n) \sim_\alpha (y_n, p_n)$, $(y_{n+1}, p_{n+1}) \sim_\alpha (x_1, p_{n+1})$. Similarly, in the second case $(y_{n+1}, p_1) \sim_\alpha (y_n, p_1)$,..., $(y_{n+1}, p_n) \sim_\alpha (y_n, p_n)$, $(y_{n+1}, p_{n+1}) \sim_\alpha (x_1, p_{n+1})$. $\sharp$

6.27. Corollary. If the set $\mathbf{A}_{ob}$ of objects definable in $\mathbf{A}$ is finite then for every $\alpha \in A$ a subset $Y \subseteq X_\alpha$ is a maximal antichain of the partially ordered set $(X_\alpha, \leq_\alpha)$ iff it corresponds to the set of occurrences of atomic identities in a cut y of α. $\sharp$

6.28. Lemma. If $\alpha \in A$ is of finite width then the canonical partial order $\leq_\alpha$ is strongly K-dense. $\sharp$

Proof. Suppose that Y is a maximal antichain of $(X_\alpha, \leq_\alpha)$ that consists of the equivalence classes of occurrences of atomic identities in a cut y of α. Suppose that Z is a maximal chain of $(X_\alpha, \leq_\alpha)$. If all elements of Z are not above Y then for each $z \in Z$ the set $f(z, Y)$ of successors of z in Y is non-empty and it can at most decrease with the increase of z. As α is of finite width and thus $f(z, Y)$ is finite, there exists at least one element of Z that belongs to Y. Similarly when all elements of Z are not below Y. Finally, if Z has elements both below and above Y, then the set $g(z_1, z_2, Y)$ of elements of Y that are between an element z_1 of Z that is below Y and an element z_2 of Z that is above Y is non-empty due to (C9) and it can at most decrease when z_1 and z_2 approach Y. As α is of finite width and thus such a set is finite, Z has an element in Y. $\sharp$

It is straighforward that if $\mathbf{A}$ is of type K, as supposed, and the set $\mathbf{A}_{ob}$ of objects definable in $\mathbf{A}$ is finite then the correspondence $\alpha \mapsto L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ just described between elements of $\mathbf{A}$ and lposets enjoys the following properties.

6.29. Lemma. Let $\mathbf{A}$ is a behaviour-oriented algebra of type K, as supposed, and let the set $\mathbf{A}_{ob}$ of objects definable in $\mathbf{A}$ be finite. If $\gamma = \alpha + \beta$ then L_γ is a coproduct object in **LPOSETS** of L_α and L_β with the canonical morphisms given by the correspondences

$$i_{\alpha,\alpha+\beta} : [((\xi_1, \xi_2), p)] \mapsto [((\xi_1 + dom(\beta), \xi_2 + \beta), p)]$$

$$i_{\beta,\alpha+\beta} : [((\eta_1, \eta_2), p)] \mapsto [((dom(\alpha) + \eta_1, \alpha + \eta_2), p)] \quad \sharp$$

6.30. Lemma. Let $\mathbf{A}$ is a behaviour-oriented algebra of type K, as supposed, and let the set $\mathbf{A}_{ob}$ of objects definable in $\mathbf{A}$ be finite. If $\gamma = \alpha\beta$ with $cod(\alpha) = dom(\beta) = c$ then L_γ is the pushout object in **LPOSETS** of the injections of L_c in L_α and in L_β given by

$$k_{c,\alpha} : [((c, c), p)] \mapsto [((\alpha, c), p)]$$

$$k_{c,\beta} : [((c, c), p)] \mapsto [((c, \beta), p)]$$

with the canonical morphisms given by the correspondences

$$j_{\alpha,\alpha\beta} : [((\xi_1, \xi_2), p)] \mapsto [((\xi_1, \xi_2\beta), p)]$$

$$j_{\beta,\alpha\beta} : [((\eta_1, \eta_2), p)] \mapsto [((\alpha\eta_1, \eta_2), p)] \quad \sharp$$

Existence of a representing homomorphism

In the case of a discrete behaviour-oriented algebra $\mathbf{A}$ of type K, i.e. a discrete behaviour-oriented algebra in which (C9) holds, where the set $\mathbf{A}_{ob}$ of objects definable in $\mathbf{A}$ is finite, all the lposets L_α are finite and thus they do not contain segments with isomorphic proper subsegments. Consequently, all L_α are strongly K-dense processes in the universe $U(\mathbf{A}) = (\mathbf{A}_{ob}, A_{+0}, nat|A_{+0})$ and they can be composed as it is described in section 3. Thus we come to the following representation of behaviour-oriented algebras.

6.31. Theorem. If $\mathbf{A}$ is is a discrete behaviour-oriented algebra of type K such that the set $\mathbf{A}_{ob}$ of objects definable in $\mathbf{A}$ is finite then the correspondence $\alpha \mapsto [L_\alpha]$ is a homomorphism from $\mathbf{A}$ to the algebra $\mathbf{KPROC}(U(\mathbf{A}))$ of weakly K-dense processes in the universe $U(\mathbf{A})$ of objects which are definable in $\mathbf{A}$. $\sharp$

In the case of a behaviour-oriented algebra $\mathbf{A}$ in which (C9) holds and $\mathbf{A}_{ob}$ is finite but not discrete it is not obvious that the lposets L_α are processes because in order to be processes they must satisfy the condition (3.3) of definition 2.6 that is trivial only for discrete lposets. However, the fact that the lposets L_α satisfy this condition is a consequence of the strong property (A4). Thus we come to the following result.

6.32. Theorem. If $\mathbf{A}$ is a behaviour-oriented algebra of type K such that the set $\mathbf{A}_{ob}$ of objects definable in $\mathbf{A}$ is finite then the correspondence $\alpha \mapsto [L_\alpha]$ is a homomorphism from $\mathbf{A}$ to the algebra $\mathbf{KPROC}(U(\mathbf{A}))$ of weakly K-dense processes in the universe $U(\mathbf{A})$ of objects which are definable in $\mathbf{A}$. $\sharp$

The representation for algebras of processes

In the case of behaviour-oriented algebras which are algebras of processes the lposets consisting of canonical underlying sets, canonical partial orders, and canonical labellings of their elements are instances of processes being these elements.

In order to demonstrate this suppose that $\mathbf{A} = (A, ;, +)$ is an algebra of weakly K-dense processes in a universe $\mathbf{U}$ of objects. Let α be a process from $\mathbf{A}$ and let $L = (X, \leq, ins)$ be an instance of α.

6.33. Lemma. There exists an isomorphic correspondence $\lambda_{\alpha,L}$ between the partially ordered set of cuts of α and the partially ordered set of cross-sections of L. $\sharp$

For a proof it suffices to apply proposition 2.12.

6.34. Lemma. To every occurrence (x, p) of an object instance p there corresponds a unique element $\mu_{\alpha,L}(x, p)$ of the cross-section $\lambda_{\alpha,L}(x)$ such that $ins(\mu_{\alpha,L}(x, p)) = p$. $\sharp$

A proof is immediate.

6.35. Lemma. Occurrences (x, p) and (y, q) of object instances are adjacent iff $\mu_{\alpha,L}(x, p) = \mu_{\alpha,L}(y, q)$. $\sharp$

A proof follows due to lemmas 6.33, 6.34, (A5) and (C8).

6.36. Corollary. The adjacency relation $\sim_\alpha$ is an equivalence relation. $\sharp$

The elements of the underlying set X_α of the canonical instance of a process α can be defined as equivalence classes of $\sim_\alpha$.

6.37. Definition. Given an atomic identity p, by an *occurrence* of p in α we mean an equivalence class of occurrences of p in cuts of α. $\sharp$

6.38. Definition. The set of occurrences of atomic identities in α, written as X_α, is called the *canonical underlying set* of α. $\sharp$

6.39. Definition. The correspondence $[(x, p)] \mapsto p$ between occurrences of atomic identities in α and the atomic identities themselves, written as ins_α, is called the *canonical labelling* of (occurrences of atomic identities in) the element α. $\sharp$

The partial order $\leq_\alpha$ on X_α can be defined as follows.

Given an occurrence (x, p) of an atomic identity p in a cut $x = (\xi_1, \xi_2)$ of α and an occurrence (y, q) of an atomic identity q in a cut $y = (\eta_1, \eta_2)$ of α, we say that (x, p) *precedes* (y, q) and write $(x, p) <_\alpha (y, q)$ if $x \sqsubseteq_\alpha y$, p occurs in x, q occurs in y, and there is no cut v of $x \to y$ such that $(x, p) \sim_\alpha (v, p)$ and $(y, q) \sim_\alpha (v, q)$.

6.40. Lemma. The relation $(x, p) <_\alpha (y, q)$ holds if and only if $\mu_{\alpha,L}(x, p) < \mu_{\alpha,L}(y, q)$. $\sharp$

A proof follows from the definition of $(x, p) <_\alpha (y, q)$ due to the weak K-density of L.

6.41. Corollary. For each $\alpha \in A$ the relation $\leq_\alpha$ on X_α, where $u \leq_\alpha v$ iff $u \sim_\alpha v$ or $(x, p) <_\alpha (y, q)$ for some $(x, p) \in u$ and $(y, q) \in v$, is a partial order. $\sharp$

6.42. Definition. The partial order $\leq_\alpha$ is called the *canonical partial order* of (occurrences of atomic identities in) α. The triple $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ is called the *canonical instance* of α. $\sharp$

It is straightforward that the correspondence $\alpha \mapsto L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ just described between actions of **KPROC**(**U**) and their canonical instances enjoys the following properties.

6.43. Lemma. If $\gamma = \alpha + \beta$ then L_γ is a coproduct object in **KPROC**(**U**) of L_α and L_β with the canonical morphisms given by the correspondences $i_{\alpha,\alpha+\beta}$ and $i_{\beta,\alpha+\beta}$ as in lemma 6.29. $\sharp$

6.44. Lemma. If $\gamma = \alpha\beta$ with $cod(\alpha) = dom(\beta) = c$ then L_γ is the pushout object in **KPROC**)**U**) of the injections of L_c in L_α and in L_β given by $k_{c,\alpha}$ and $k_{c,\beta}$ as in lemma 6.30 with the canonical morphisms given by the correspondences $j_{\alpha,\alpha\beta}$ and $j_{\beta,\alpha\beta}$ as in lemma 6.30. $\sharp$

7

Providing processes with structures

The idea

We have shown that every element of a behaviour-oriented algebra defines a unique set (the canonical underlying set) and a unique structure on this set (the structure that consists of the canonical partial order and the canonical labelling), and a unique lposet (the canonical instance). Now we want to show how some elements of such an algebra or, more precisely, their canonical underlying sets, can be provid ed with some additional structures.

Lemmas 6.29 and 6.30 of the previous chapter suggest that structures for the canonical instances of elements should be related to the structures for the canonical instances of the components of these elementss.

Let $T = (B, mor)$ be a structure type as defined in Appendix E.

Let $\mathbf{A} = (A, ;, +)$ be an algebra of weakly K-dense processes in a universe $\mathbf{U} = (V, W, ob)$ of objects.

The canonical instance of each element of $\mathbf{A}$ can be provided with a structure of type T on its underlying set. However, the choice of such a structure cannot be arbitrary since elements of the algebra $\mathbf{A}$ and their instances can be related and then we expect also the corresponding structures to be related in a similar way. Consequently, we propose to formalize such a choice by assigning to each $\alpha \in A$ the canonical instance $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$, by providing the assigned instances with a suitable structures str_α in a way consistent with the operations on processes, and by transporting the structures thus introduced from the canonical instances of processes to arbitrary isomorphic lposets with the aid of the respective isomorphisms. This can be done as follows (cf. [Wink 07b]).

The structures for the canonical instances of elements of $\mathbf{A}$ should be related as follows to the structures for the canonical instances of the components of these elements.

7.1. Definition. Elements of the algebra $\mathbf{A}$ are said to be *consistently provided* with structures of type T if there exists a correspondence $\alpha \mapsto str_\alpha$ such that, for every $\alpha \in A$, str_α is a structure of type T on the canonical underlying set X_α of α and the following conditions are fulfilled:

(1) if $\alpha + \beta$ is defined then $str_{\alpha+\beta}$ is the coproduct object in

STRUCT(T) of str_α and str_β with the canonical injections $i_{\alpha,\alpha+\beta}$ and $i_{\beta,\alpha+\beta}$ as in lemma 6.29,

(2) if $\alpha\beta$ is defined and $cod(\alpha) = dom(\beta) = c$ then $str_{\alpha\beta}$ is the pushout object in **STRUCT**(T) of the injections $k_{c,\alpha}$ and $k_{c,\beta}$ of str_c in str_α and in str_β as in lemma 6.30 with the canonical injections $j_{\alpha,\alpha\beta}$ and $j_{\beta,\alpha\beta}$ as in lemma 6.30. $\sharp$

Examples

Examples that follow illustrate the idea.

7.2. Example. Let LPO be the structure type of labelled partial orders with order and labelling preserving morphisms. To each element α of $\mathbf{A}$ we can assign the structure $lpo_\alpha = (\leq_\alpha, l_\alpha)$ on the canonical underlying set X_α. If the set $\mathbf{A}_{ob}$ of objects occurring in $\mathbf{A}$ is finite then 6.29 and 6.30 imply that the propositions correspondence $\alpha \mapsto lpo_\alpha$ fulfils the conditions (1) and (2) of definition 7.1 for the structure type LPO. $\sharp$

7.3. Example. Let WPO be the structure type of weighted partial orders $wpo = (\leq, d)$, where $\leq$ is a partial order on a set X
and $d : X \times X \to Real \cup \{-\infty, +\infty\}$ is a function such that

(a) $d(x, x) = 0$,
(b) $d(x, y) = -\infty$ if x and y are incomparable with respect to $\leq$,
(c) $d(x, y) = sup\{d(x, z) + d(z, y) : z \neq x, z \neq y, x \leq z \leq y\}$ if there exists z such that $z \neq x$, $z \neq y$, $x \leq z \leq y$,

and where morphisms are order and weight preserving mappings. If the algebra $\mathbf{A}$ is generated by a set of $(+, ;)$-atomic processes and if the set $\mathbf{A}_{ob}$ of objects occurring in $\mathbf{A}$ is finite then to each process α of A we can assign structure $wpo_\alpha = (\leq_\alpha, d_\alpha)$ To this end it suffices to define d_α on $(+, ;)$-atoms generating $\mathbf{A}$ and then extend it on entire $\mathbf{A}$ such that the conditions (1) and (2) of definition 7.1 are fulfilled for the structure type WPO. Values of functions d_α can be interpreted as delays between elements of the canonical underlying set X_α of α. Together with data about occurrence times of minimal elements of X_α they determine occurrence times of all elements of X_α. For instance, in the case of an action α with a linear flow order the occurrence time of each $x \in X_\alpha$ is $t' + d_\alpha(x', x)$, where x' is the minimal element of X_α and t' is the occurrence time of x'. $\sharp$

7.4. Example. Suppose that the set $\mathbf{A}_{ob}$ of objects occurring in $\mathbf{A}$ is finite. Suppose that B is a subset of $(;)$-atoms of $\mathbf{A}$ such that to each $\beta \in B$ there corresponds a structure gr_β of a graph on the canonical set X_β of β. Suppose that $\mathbf{A}'$ is the subalgebra of $\mathbf{A}$ generated by B. Then $gr_{dom(\beta)}$ and $gr_{cod(\beta)}$

must be graphs and the correspondence $\beta \mapsto gr_\beta$ has a unique extension on entire subalgebra $\mathbf{A}'$ and this extension fulfils the conditions (1) and (2) of definition 7.1 for the structure type *GRAPHS*. Notice that elements of $\mathbf{A}'$ thus provided can be interpreted as derivations of graphs from graphs by applying graph grammar productions in the sense of the so called double pushout approach (cf. [CMR 96]). $\sharp$

Providing processes with context relations

Applications of graph grammar productions to graphs in the sense of double pushout approach are examples of processes in which some subgraphs of transformed graphs are involved but remain unchanged. Put in another way, some object occurrences in processes play the role of a context for other object occurrences. Situations of this kind can be reflected by providing processes with the respective acyclic binary relations of context dependence. This can be done as follows.

7.5. Proposition. If the algebra $\mathbf{A}$ is generated by a set A' of not necessarily atomic processes and if it is possible to assign to each $\alpha \in A'$ an acyclic binary relation cxt_α on X_α, called after [Wink 05] a *context relation*, such that:

(1) for all elements of X_α, $(x, y) \in cxt_\alpha$ excludes both $x \leq_\alpha y$ and $y \leq_\alpha x$, and the reflexive and transitive closure of the following relation R, where cxt^+_α denotes the transitive closure of cxt_α, is a partial order with the same minimal and maximal elements as for $\leq_\alpha$:
$(x, y) \in R$ iff
$x \leq_\alpha y$ or
$(x <_\alpha z$ and $(z, y) \in cxt^+_\alpha$ for some $z)$ or
$(x \leq_\alpha t$ and $z <_\alpha y$ and $(z, t) \in cxt_\alpha$ for some z and $t)$,
(2) the conditions (1) and (2) of definition 7.1 are fulfilled for the structure type *ABREL* of acyclic binary relations,

then it is possible to extend the correspondence $\alpha \mapsto cxt_\alpha$ on A such that the conditions (1) and (2) of definition 7.1 are fulfilled for the structure type *ABREL*. $\sharp$

Proof. It suffices to prove that $cxt_{\alpha\beta}$ is an acyclic binary relation in $X_{\alpha\beta}$. To this end suppose the contrary and suppose that Z is a cycle in $cxt_{\alpha\beta}$. Suppose that c is the cross-section of $L_{\alpha\beta}$ such that $head(L_{\alpha\beta}, c)$ and L_α are isomorphic and $tail_($$L_{\alpha\beta}, c)$ and L_β are isomorphic. As cxt_α and cxt_β are acyclic, Z must consist of a part Z_- in $head_($$L_{\alpha\beta}, c)$ and a part Z_+ in $tail_($$L_{\alpha\beta}, c)$. However, this is impossible since otherwise there would be x, y, z such that x and z are in c, they are different, $x \leq_\beta y$, and $(y, z) \in cxt_\beta$, and it would imply that the partial order defined by $\leq_\beta$ and cxt_β could not have the same minimal elements as for $\leq_\beta$. $\sharp$

7.6. Example. Suppose that a machine m produces some material for users exploiting it in an unspecified manner. Suppose that the machine m is equipped with a switch S to resume production (the position *on*) and to break it (the position *off*). Define an instance of m to be a pair (m, a), where $a \geq 0$ is the available amount of material. Define an instance of S to be a pair (S, s), where s is *on* or *off*. Define $V' = \{m, S\}$, $W' = W_m \cup W_S$, where $W_m = \{(m, a) : a \geq 0\}$, and $W_S = \{(S, on), (S, \mathit{off})\}$. Define $ob'(w) = m$ for $w = (m, a) \in W_m$ and $ob'(w) = S$ for $w = (S, s) \in W_S$. Then $\mathbf{U}' = (V', W', ob')$ is a universe of objects.

The work of the machine m in an interval $[t', t'']$ of global time is a concrete process in $\mathbf{U}'$ that, when considered without taking into account the switch, can be defined as $WORK = (X_{WORK}, \leq_{WORK}, ins_{WORK})$, where
X_{WORK} is the set $\{q(t) : t \in [t', t'']\}$ of values of the real-valued function $t \mapsto q(t)$ that specifies the amount of material that has been produced until $t \in [t', t'']$,
$\leq_{WORK}$ is the restriction of the usual order of numbers to X_{WORK},
$ins_{WORK}(x) = (m, a(t))$ for $x = q(t)$, where $a(t)$ is the amount of material available at $t \in [t', t'']$.

Switching on the machine m in a state $s_0 = (m, a_0)$ is a concrete process that can be defined as $ON = (X_{ON}, \leq_{ON}, ins_{ON})$, where
$X_{ON} = \{x_1, x_2, x_3, x_4\}$,
$x_1 <_{ON} x_3, \quad x_1 <_{ON} x_4, \quad x_2 <_{ON} x_3, \quad x_2 <_{ON} x_4$,
$ins_{ON}(x_1) = ins_{ON}(x_3) = s_0$, $ins(x_2) = (S, \mathit{off})$,
$ins_{ON}(x_4) = (S, on)$.

Switching off the machine m in a state $s_1 = (m, a_1)$ is a concrete process that can be defined as $OFF = (X_{OFF}, \leq_{OFF}, ins_{OFF})$, where
$X_{OFF} = \{x_1, x_2, x_3, x_4\}$,
$x_1 <_{OFF} x_3, \quad x_1 <_{OFF} x_4, \quad x_2 <_{OFF} x_3, \quad x_2 <_{OFF} x_4$,
$ins_{OFF}(x_1) = ins_{OFF}(x_3) = s_1$,
$ins_{OFF}(x_2) = (S, on)$, $ins_{OFF}(x_4) = (S, \mathit{off})$.

Switching on the machine m in a state s_0 followed by a work of m and by switching off m in a state s_1 is a concrete process that can be defined as $RUN = (X_{RUN}, \leq_{RUN}, ins_{RUN})$, where
$X_{RUN} = X_{WORK'} \cup X_{ON'} \cup X_{WORK'}$,
$\leq_{RUN}$ is the transitive closure of $\leq_{WORK'} \cup \leq_{ON'} \cup \leq_{OFF'}$,
$ins_{RUN} = ins_{WORK'} \cup ins_{ON'} \cup ins_{OFF'}$,
for a variant $WORK'$ of $WORK$, a variant ON' of ON, and a variant OFF' of OFF, such that the maximal element of $X_{ON'}$ with the label (S, on) coincides with the minimal element of $X_{OFF'}$ with the label (S, on), the maximal element of $X_{ON'}$ with the label s_0 coincides with the minimal element of $X_{WORK'}$ with the label s_0, the maximal element of $X_{WORK'}$ with the label s_1 coincides with the minimal element of $X_{OFF'}$ with the label s_1, and these are the only common elements of pairs of sets from among $X_{WORK'}$, $X_{ON'}$, $X_{OFF'}$.

The abstract processes $[WORK]$, $[ON]$, $[OFF]$, and $[RUN]$, are represented graphically in figure 7.1.

Consider the processes $[WORK]$, $[ON]$, $[OFF]$, $[RUN]$. In the case of such processes and their combinations, we can consider the subalgebra of the respective algebra of processes generated by variants of $([WORK] + \{(S, on)\})$, $[ON]$, $[OFF]$, and endow $([WORK] + \{(S, on)\})$ with a context relation as it is illustrated in figure 7.2 with a dotted arrow. $\sharp$

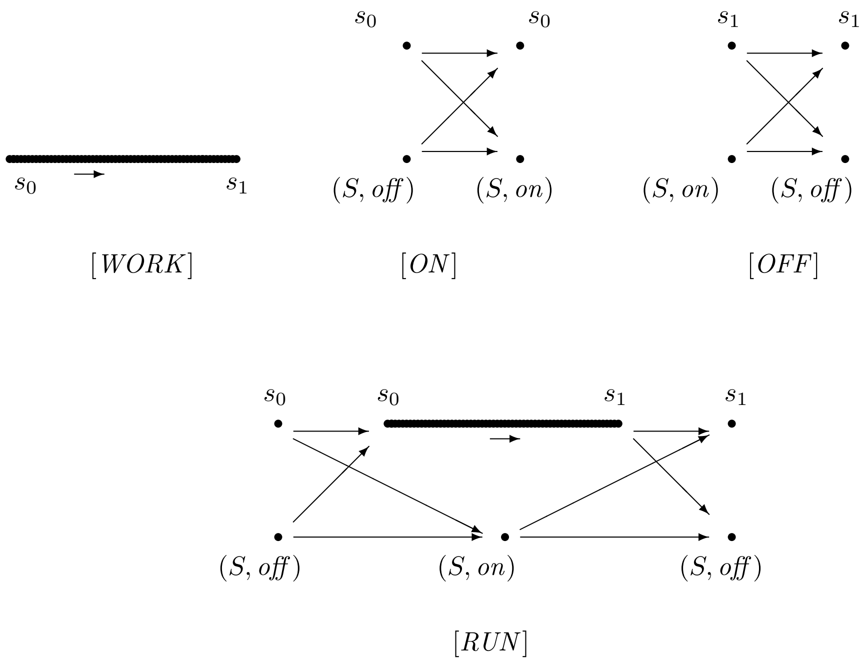


Figure 7.1: $[WORK]$, $[ON]$, $[OFF]$, $[RUN]$

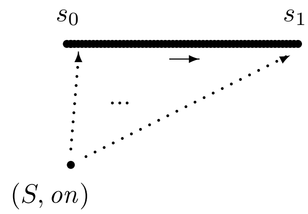


Figure 7.2: $[WORK] + \{(S, on)\}$ with a context relation

7.7. Proposition. If the algebra $\mathbf{A}$ is generated by a set A_0 of $(;)$-atoms such that the elements of A_0 that are not $(+)$-atoms cannot be obtained by composing in parallel other elements of A_0 and if the elements of A_0 can be provided with context relations cxt^+_α such that the condition (1) of proposition 7.5 is fulfilled then:

(1) it is possible to extend the correspondence $\alpha \mapsto cxt_\alpha$ on A such that the conditions (1) and (2) of definition 7.1 are fulfilled for the structure type *ABREL*,

(2) a diagram $D = (v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ in $\mathbf{pcat}(\mathbf{A})$ is a bicartesian square in $\mathbf{pcat}(\mathbf{A})$ if and only if there exist c, φ_1, φ_2 such that
c is an identity,
there is no identity $d \neq 0$ such that $d \sqsubseteq \varphi_1$ or $d \sqsubseteq \varphi_1$,
$c + \varphi_1 + \varphi_2$ is defined with a partition of $X_{c+\varphi_1+\varphi_2}$ into three mutually disjoint subsets X'_c, X'_{φ_1}, X'_{φ_2} such that the restrictions of $L_{c+\varphi_1+\varphi_2}$ to these subsets are respectively instances of c, φ_1, φ_2,
$\alpha_1 = c + \varphi_1 + dom(\varphi_2)$, $\alpha_2 = c + dom(\varphi_1) + \varphi_2$,
$\alpha_1' = c + \varphi_1 + cod(\varphi_2)$, $\alpha_2' = c + cod(\varphi_1) + \varphi_2$,
$(x, y) \in cxt_{c+\varphi_1+\varphi_2}$ only if both x and y belong to X'_{φ_1} or to X'_{φ_2}, or x belongs to X'_c. $\sharp$

Proof. The first part of the proposition is immediate. The fact that the existence of the respective c, φ_1, φ_2 implies that the diagram D is a bicartesian square in $\mathbf{pcat}(\mathbf{A})$ follows from (C8) and from the fact that the conditions (1) and (2) of definition 7.1 are satisfied for the correspondence $\alpha \mapsto cxt_\alpha$. To prove the converse take into account the fact that, due to the assumed properties of $\mathbf{A}$, every diagram in $\mathbf{pcat}(\mathbf{A})$ that is a bicartesian square in the algebra of processes that contains $\mathbf{A}$ is a bicartesian square in $\mathbf{pcat}(\mathbf{A})$ as well. Consequently, it suffices to prove that cxt_α enjoys the expected properties for the respective c, φ_1, φ_2, X'_c, X'_{φ_1}, X'_{φ_2}. To this end suppose the contrary. Then in one of the sets X'_{φ_1}, X'_{φ_2}, say in X'_{φ_1}, there exists x that is not minimal and such that $(x, y) \in cxt_{c+\varphi_1+\varphi_2}$ for some $y \in X'_{\varphi_2}$ and, consequently, $x' <_{c+\varphi_1+\varphi_2} y$ for some $x' \in X'_{\varphi_1}$. However this is impossible because then $cxt_{c+\varphi_1+\varphi_2}$ could not be a context relation for $\alpha_2\alpha_1$. $\sharp$

8

Behaviour-oriented partial categories

Basic notions

In chapter 3, proposition 3.13, it has been shown that every partial category of processes enjoys the properties (A1) - (A10).

In this chapter we introduce abstract algebras in which (A1) - (A10) hold, called in *behaviour-oriented partial categories*, and we prove that such partial categories can be represented as partial categories of processes.

Behaviour-oriented partial categories are essentially specific *multiplicative transition systems* in the sense of [Wink 11]. They are defined as follows.

8.1. Definition. A *behaviour-oriented partial category*, or briefly a BOPC, is a partial category $\mathbf{A} = (A, ;)$, where A is a set and $(\alpha_1, \alpha_2) \mapsto \alpha_1; \alpha_2$ is a partial operation in A such that the axioms (A1) - (A10) hold. $\sharp$

In $\mathbf{A}$ two partial unary operations $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable that assign to an element a source and a target, if they exist.

An element α of A is said to be a *atom* of $\mathbf{A}$ provided that it is not an identity, has a source and a target, and for every $\alpha_1 \in A$ and $\alpha_2 \in A$ the equality $\alpha = \alpha_1\alpha_2$ implies that either α_1 is an identity and $\alpha_2 = \alpha$ or α_2 is an identity and $\alpha_1 = \alpha$.

We say that $\mathbf{A}$ is *discrete* if every $\alpha \in A$ that is not an identity can be represented in the form $\alpha = \alpha_1 ... \alpha_n$, where $\alpha_1, ..., \alpha_n$ are atoms.

Note that if $\mathbf{A}$ is discrete then its every element has a source and a target and thus $\mathbf{A}$ is a category.

As in the case of behaviour-oriented algebras, by a *cut* of $\alpha \in A$ we mean a pair (α_1, α_2) such that $\alpha_1\alpha_2 = \alpha$.

The partial category $\mathbf{A}$ has the properties of partial categories of processes described in propositions 3.17 and 3.18. Consequently, cuts of every $\alpha \in A$ are partially ordered by the relation $\sqsubseteq_\alpha$, where $x \sqsubseteq_\alpha y$ with $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ means that $\eta_1 = \xi_1\delta$ with some δ. Due to (A1) and (A2) for $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \sqsubseteq_\alpha y$ there exists a unique δ such that $\eta_1 = \xi_1\delta$, written as $x \to y$. As in proposition 3.18 the partial order $\sqsubseteq_\alpha$ makes the set of cuts of α a lattice LT_α. The lattice LT_α is obviously a behaviour-oriented partial category. Given two cuts x and y, by $x \sqcup_\alpha y$ and $x \sqcap_\alpha y$ we denote respectively the least upper bound and the greatest lower bound of x and y. From (A5) it follows that $(x \leftarrow x \sqcap_\alpha y \to y, x \to x \sqcup_\alpha y \leftarrow y)$ is a bicartesian square.

Given $\alpha \in A$ and its cuts $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \sqsubseteq_\alpha y$, by a *segment* of α from x to y we mean $\beta \in A$ such that $\xi_2 = \beta\eta_2$ and $\eta_1 = \xi_1\beta$, written as $\alpha|[x, y]$. A segment $\alpha|[x', y']$ of α such that $x \sqsubseteq_\alpha x' \sqsubseteq_\alpha y' \sqsubseteq_\alpha y$ is called a *subsegment* of $\alpha|[x, y]$. If $x = x'$ (resp. if $y = y'$) then we call it an *initial* (resp. a *final*) subsegment of $\alpha|[x, y]$. An initial segment ι of α is called also a *prefix* of α, written as $\iota \, pref \, \alpha$.

As in the case of partial categories of processes, in the set $A_{semibounded}$ of those $\alpha \in A$ which are semibounded in the sense that their source $dom(\alpha)$ one can define as follows a relation $\sqsubseteq$, where

$\alpha \sqsubseteq \beta$ whenever every prefix of α is a prefix of β

and this relation is a partial order, i.e. $(A_{semibounded}, \sqsubseteq)$ is a poset.

As in the case of behaviour-oriented algebras, elements of A are called *hypothetical processes* (or briefly, *processes*) of $\mathbf{A}$. Processes of $\mathbf{A}$ which are identities of $\mathbf{A}$ are called *hypothetical states* (or briefly *states*) of $\mathbf{A}$. Processes which are atomic identities are called *atomic states.* A process α is said to be *bounded* if it has the source $dom(\alpha)$ and the target $cod(\alpha)$. For every process α, the existing states $u = dom(\alpha)$ and $v = cod(\alpha)$ are called respectively the *initial state* and the *final state* of α and we write α as $u \overset{\alpha}{\to} v$. The operation $(\alpha_1, \alpha_2) \mapsto \alpha_1\alpha_2$ is called the *composition.* The independence of bounded processes can be defined exploiting the chracterization of parallel and sequential independence of processe in theorems 6.7 and 6.8.

8.2. Definition. Processes $u \overset{\alpha_1}{\to} v$ and $u \overset{\alpha_2}{\to} w$ are said to be parallel independent iff there exist unique processes $v \overset{\alpha_2'}{\to} u'$ and $w \overset{\alpha_1'}{\to} u'$ such that $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\to} w, v \overset{\alpha_2'}{\to} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square. $\sharp$

8.3. Definition. Processes $u \overset{\alpha_1}{\to} v$ and $v \overset{\alpha_2'}{\to} u'$ are said to be sequential independent iff there exist unique processes $u \overset{\alpha_2}{\to} w$ and $w \overset{\alpha_1'}{\to} u'$ such that $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\to} w, v \overset{\alpha_2'}{\to} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square. $\sharp$

These definitions are adequate in subalgebras of behaviour-oriented partial categories provided that bicartesian squares in such subalgebras are bicartesian squares in the original behaviour-oriented partial categories. This appears to be true if the respective subalgebras are inheriting in the following sense.

8.4. Definition. A subalgebra $\mathbf{A}'$ of a behaviour-oriented partial category $\mathbf{A}$ is said to be *inheriting* if it is closed with respect to components of its elements in the sense that arrows α and β of $\mathbf{A}$ are also arrows of $\mathbf{A}'$ whenever $\alpha\beta$ is an arrow of $\mathbf{A}'$. $\sharp$

This following proposition reflects the crucial property of inheriting subalgebras of behaviour-oriented partial categories.

8.5. Proposition. If $\mathbf{A}'$ is an inheriting subalgebra of a behaviour-oriented partial category $\mathbf{A}$ then:

(1) each bicartesian square of $\mathbf{A}$ whose arrows are in $\mathbf{A}'$ is a bicartesian square in $\mathbf{A}'$,
(2) each bicartesian square in $\mathbf{A}'$ is a bicartesian square in $\mathbf{A}$. $\sharp$

Proof. The first part of this proposition is immediate. For the second part it suffices to exploit the property (A6) of $\mathbf{A}$ and the fact that $\mathbf{A}'$ is an inheriting subalgebra of $\mathbf{A}$.

Behaviour-oriented partial categories are models of concurrent system richer than transition systems in the sense that they specify not only states, transitions, and independence of transitions of the modelled systems, but also their processes (runs) and how processes compose. Moreover, independence becomes a definable notion, and it can be defined not only for transitions, but also for compound processes.

8.6. Example. Consider the universe $\mathbf{U}_2$ of a producer and a distributor and the conrete processes Q, R, S in $\mathbf{U}_2$ described in example 2.8. By combining the abstract processes corresponding to the possible variants of concrete processes Q, R, S we obtain a subalgebra $\mathbf{A}_2 = (A_2, ;)$ of the partial category $\mathbf{pcatgPROC}(\mathbf{U}_2)$ of global processes in $\mathbf{U}_2$. This subalgebra is a BOPC in the sense of definition 8.1. $\sharp$

8.7. Example. Define a transition system without a distinguished initial state as $M = (S, E, T)$ such that S is a set of states, E is a set of events, and $T \subseteq S \times E \times S$ is a set of transitions, where $(s, e, s') \in T$ stands for the transition from the state s to the state s' due to the event e. Assume that E contains a distinguished element $*$ standing for "no event", and assume that for every state $s \in S$ the set T contains an idle transition $(s, *, s)$ standing for "stay in s". Then M can be represented by the graph $G(M) = (T, dom, cod)$, where $dom(s, e, s') = (s, *, s)$ and $cod(s, e, s') = (s', *, s')$ for every $(s, e, s') \in T$.

Write $s \stackrel{e}{\to} s'$ to indicate that $(s, e, s') \in T$. Denote by Lts the set of triples of the form $\alpha = s \stackrel{x}{\to} s'$ where x is any finite word over the alphabet $E - \{*\}$ such that $x = e_1 ... e_m$ for $\alpha = s_0 \stackrel{e_1}{\to} s_1 \stackrel{e_2}{\to} s_2 ... s_{m-1} \stackrel{e_m}{\to} s_m$ with $s_0 = s$ and $s_m = s'$, or x is the empty word represented by $*$ and $s' = s$.
Define $dom(s \stackrel{x}{\to} s') = s \stackrel{*}{\to} s$ and $cod(s \stackrel{e_1}{\to} s') = s' \stackrel{*}{\to} s'$.
For triples $\alpha_1 = s_1 \stackrel{x_1}{\to} s'_1$ and $\alpha_2 = s_2 \stackrel{x_2}{\to} s'_2$ such that $s'_1 = s_2$ define the result of composing α_1 and α_2 as $\alpha_1 \alpha_2 = s_1 \stackrel{x_1 x_2}{\to} s'_2$.
It is easy to verify that the set Lts with the composition thus defined is a BOPC $LTS(M)$ in the sense of definition 8.1. In this BOPC each ordering $\sqsubseteq_\alpha$ is linear and $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\to} w, v \stackrel{\alpha'_2}{\to} u' \stackrel{\alpha'_1}{\leftarrow} w)$ is a bicartesian square iff α_1 and α'_1 are identities or α_2 and α'_2 are identities. $\sharp$

8.8. Example. Consider the transition system M from example 8.7. Consider a symmetric irreflexive relation $I \subseteq (E - \{*\})^2$, called an independence relation, and the least equivalence relation $\|_I$ between words over the alphabet $E - \{*\}$ such that words $uabv$ and $ubav$ are equivalent whenever $(a, b) \in I$. The equivalence classes of such a relation are known in the literature as Mazurkiewicz traces with respect to I (see [Maz 88]). Denote by Ts the set of triples as in example 8.7 but with words over the alphabet $E - \{*\}$ replaced by traces with respect to I. Define *dom* and *cod* and the composition as in example 8.7, but with the concatenation of words replaced by the induced concatenation of traces.

It is easy to verify that the set Ts with the composition thus defined is a BOPC $TS(M, I)$ in the sense of definition 8.1, and that this BOPC is a homomorphic image of the BOPC from example 8.7. However, in this system there exist nontrivial bicartesian squares, namely, the squares ($v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w$) such that $\alpha_1 = u \overset{x_1}{\rightarrow} v$, $\alpha_2 = u \overset{x_2}{\rightarrow} w$,
$\alpha_1' = w \overset{x_1}{\rightarrow} u'$, $\alpha_2' = v \overset{x_2}{\rightarrow} u'$ with $(a, b) \in I$ for all (a, b) such that a occurs in x_1 and b occurs in x_2. $\sharp$

Independence and equivalence of transitions

In the definitions 8.2 and 8.3 we have characterized the natural concepts of sequential and parallel independence of processes similar to the concepts introduced in [EK 76] as the existence in the respective BOPC of appropriate bicartesian squares. Now we shall use this characterization to define independence and a natural equivalence of elements of behaviour-oriented partial categories similar to the considered in [WN 95] independence and equivalence of transitions in transition systems with independence. This will allow us to adapt and study the concept of a region similar to that introduced in [ER 90].

8.9. Examples. In the BOPC $\mathbf{A}_2$ in example 8.6 processes $\pi + dom(\rho)$ and $dom(\pi) + \rho$ are parallel independent, processes $\pi + dom(\rho)$ and $cod(\pi) + \rho$ are sequential independent, and transitions
$dom(\pi) + \rho$ and $\pi + cod(\rho)$ are sequential independent. In the BOPC $LTS(M)$ in example 8.7 processes $u \overset{\alpha_1}{\rightarrow} v$ and $u \overset{\alpha_2}{\rightarrow} w$ are *parallel independent* only if one of them is an identity. Similarly, processes $u \overset{\alpha_1}{\rightarrow} v$ and $v \overset{\alpha_2'}{\rightarrow} u'$ are sequential independent only if one of them is an identity. In the BOPC $TS(M)$ in example 8.8 processes $u \overset{\alpha_1}{\rightarrow} v$ and $u \overset{\alpha_2}{\rightarrow} w$ are *parallel independent* iff $(a, b) \in I$ for all a occurring in α_1 and all b occurring in α_2. Similarly, processes $u \overset{\alpha_1}{\rightarrow} v$ and $v \overset{\alpha_2'}{\rightarrow} u'$ are sequential independent iff $(a, b) \in I$ for all (a, b) such that a occurs in α_1 and b occurs in α_2'. $\sharp$

8.10. Definition. By the *natural equivalence* of elements of a BOPC $\mathbf{A} = (A, ;)$ we mean the least equivalence relation $\equiv$ in A such that $\alpha_1 \equiv \alpha'_1$ whenever in this BOPC there exists a bicartesian square $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w, v \stackrel{\alpha'_2}{\rightarrow} u' \stackrel{\alpha'_1}{\leftarrow} w)$. $\sharp$

8.11. Examples. In the BOPC $\mathbf{A}_2$ in example 8.6 processes $\pi + dom(\rho)$ and $cod(\rho) + \pi$ are equivalent in the sense of definition 8.10. In the BOPC $LTS(M)$ in example 8.7 the natural equivalence coincides with the identity relation. In the BOPC $TS(M)$ in example 8.8 we have $\alpha_1 \equiv \alpha'_1$ whenever $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w, v \stackrel{\alpha'_2}{\rightarrow} u' \stackrel{\alpha'_1}{\leftarrow} w)$ with α_1 and α'_1 representing the same trace t_1, and α_2 and α'_2 representing the same trace t_2, for $(a, b) \in I$ for all (a, b) such that a occurs in t_1 and b occurs in t_2. $\sharp$

Regions

The existence in behaviour-oriented partial categories of the natural equivalence of processes allows us to adapt and exploit the concept of a region similar to that introduced in [ER 90].

8.12. Definition. By a *region* of a BOPC $\mathbf{A} = (A, ;)$ we mean a nonempty subset r of the set of states of $\mathbf{A}$ such that:

$dom(\alpha) \in r$ and $cod(\alpha) \notin r$ and $\alpha' \equiv \alpha$
implies $dom(\alpha') \in r$ and $cod(\alpha') \notin r$,
$dom(\alpha) \notin r$ and $cod(\alpha) \in r$ and $\alpha' \equiv \alpha$
implies $dom(\alpha') \notin r$ and $cod(\alpha') \in r$. $\sharp$

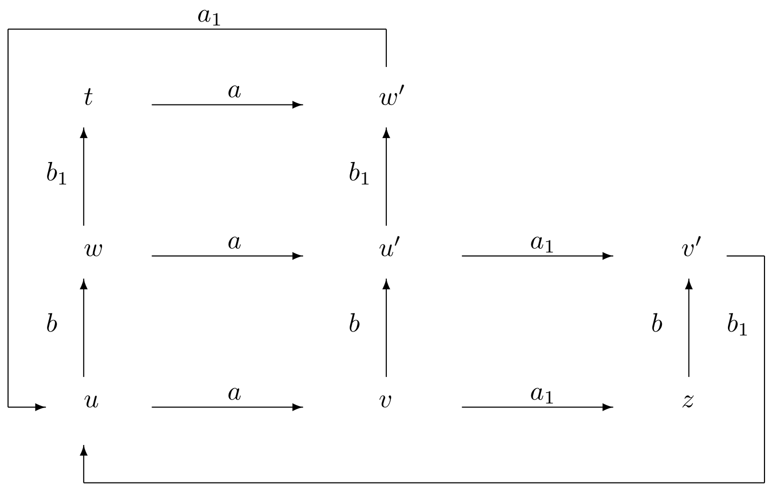
8.13. Example. Consider the BOPC $\mathbf{A}_2$ in example 8.6. In this BOPC the sets $[(p, q)] = \{(p, q)\} \times (\{d\} \times [0, +\infty))$ with $q \geq 0$, the sets $[(d, r)] = \{(d, r)\} \times (\{p\} \times [0, +\infty))$ with $r \geq 0$, and disjoint unions of such sets are regions. $\sharp$

8.14. Example . Consider the transition system M' in figure 8.1. Consider the independence relation

$I' = \{(a, b), (a, b_1), (a_1, b), (a_1, b_1)\}$

and the BOPC $TS(M', I')$. In this BOPC we have processes $\alpha = u \stackrel{[a]}{\rightarrow} v$, $\beta = u \stackrel{[b]}{\rightarrow} w$, $\alpha' = w \stackrel{[a]}{\rightarrow} u'$, $\beta' = v \stackrel{[b]}{\rightarrow} u'$ $\alpha'' = t \stackrel{[a]}{\rightarrow} w'$, $\beta'' = z \stackrel{[b]}{\rightarrow} v'$, $\alpha_1 = u' \stackrel{[a_1]}{\rightarrow} v'$, $\beta_1 = u' \stackrel{[b_1]}{\rightarrow} w'$, $\alpha'_1 = w' \stackrel{[a_1]}{\rightarrow} u$, $\beta'_1 = v' \stackrel{[b_1]}{\rightarrow} u$ $\alpha''_1 = v \stackrel{[a_1]}{\rightarrow} z$, $\beta''_1 = w \stackrel{[b_1]}{\rightarrow} t$, where $[a]$,$[a_1]$,$[b]$,$[b_1]$ are traces correspondig to a, a_1, b, b_1, and compositions of these processes. For example, $\alpha\beta' = \beta\alpha' = \gamma = u \stackrel{[ab]}{\rightarrow} u'$, $\alpha_1\beta'_1 = \beta_1\alpha'_1 = \gamma_1 = u' \stackrel{[a_1b_1]}{\rightarrow} u$, processes α,α' are equivalent, processes β,β'

are equivalent, and we have regions $E = \{u, w, t, v', z\}$, $F = \{u, v, z, t, w'\}$, $G = \{v, u', w'\}$, $H = \{w, u', v'\}$, $E \cup G$, $F \cup H$, and $\{u, v, w, z, t, u', v', w'\}$. $\sharp$



M'

Figure 8.1

From the definition of a region we obtain the following proposition.

8.15. Proposition. If $\mathbf{A} = (A, ;)$ is a BOPC, r is a region of $\mathbf{A}$, and $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in $\mathbf{A}$, then $v \in r$ implies that $u \in r$ or $u' \in r$. $\sharp$

Due to the property (A7) of behaviour-oriented partial categories we obtain the following proposition.

8.16. Proposition. If $\mathbf{A} = (A, ;)$ is a BOPC, r is a region of $\mathbf{A}$, and $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\rightarrow} w, v \overset{\alpha_2'}{\rightarrow} u' \overset{\alpha_1'}{\leftarrow} w)$ is a bicartesian square in $\mathbf{A}$ with morphisms which are not identities, then for every decomposition $u \overset{\alpha_1}{\to} v = u \overset{\alpha_{11}}{\to} v_1 \overset{\alpha_{12}}{\to} v$ such that $u, v \in r$ we have $v_1 \in r$, and for every decomposition $w \overset{\alpha_1'}{\to} u' = w \overset{\alpha_{11}'}{\to} w_1 \overset{\alpha_{12}'}{\to} u'$ such that $w, u' \in r$ we have $w_1 \in r$. $\sharp$

The following three propositions follow from the definition of a region.

8.17. Proposition. The set of all states of **A** is a region of **A**. $\sharp$

8.18. Proposition. If p and q are disjoint regions of **A** then $p \cup q$ is a region of **A**. $\sharp$

8.19. Proposition. If p and q are different regions of **A** such that $p \subseteq q$ then $q - p$ is a region of **A**. $\sharp$

Moreover, we are also able to prove the following proposition.

8.20. Proposition. Every region of **A** contains a minimal region. $\sharp$

Proof. Let r be a region of **A** and let x be an element of r. Given a chain $(r_i : i \in I)$ of regions of **A** that are contained in r and contain and element x, for $r' = \bigcap(r_i : i \in I)$ and a transition α such that $dom(\alpha) \in r'$ and $cod(\alpha) \notin r'$, there exists $i_0 \in I$ such that $dom(\alpha) \in r_i$ and $cod(\alpha) \notin r_i$ for $i > i_0$. Consequently, for every transition α' such that $\alpha' \equiv \alpha$ we have $dom(\alpha') \in r_i$ and $cod(\alpha') \notin r_i$ for $i > i_0$, and thus $dom(\alpha') \in r'$ and $cod(\alpha') \notin r'$. Similarly, for α such that $dom(\alpha) \notin r'$ and $cod(\alpha) \in r'$ and for $\alpha' \equiv \alpha$. So, r' is a region. Consequently, in the set of regions that are contained in r and contain x there exists a minimal region. $\sharp$

The propositions 8.19 and 8.20 imply the following properties.

8.21. Proposition. Every state of **A** belongs to a minimal region. $\sharp$

8.22. Proposition. If a state s of **A** does not belong to a region r then there exists a minimal region r' such that $r \cap r' = \emptyset$ and s belongs to r'. $\sharp$

8.23. Proposition. Every region of **A** can be represented as a disjoint union of minimal regions. $\sharp$

Proof. Let m be the disjoint union of a family M of minimal regions of **A**. Then m is a region of **A** and if it does not cover A then $A - m$ is a region of **A** and the family M can be extended by a minimal region of **A** that contains a given element of $A - m$ as in the proof of Proposition 8.20. Consequently, a family of disjoint minimal regions of **A** can be defined such that its union covers A. ♯

Processes as labelled posets

Now we shall concentrate on behaviour-oriented partial categories which enjoy a specific but still very natural property. We shall call them *clean behaviour-oriented partial categories*, and we shall show that their elements can be interpreted as processes in a universe of objects.

We start with suitable notions and observations.

Let $\mathbf{A} = (A, ;)$ be a BOPC.

8.24. Definition. Given $\alpha \in A$ and a cut $x = (\xi_1, \xi_2)$ of α, by a *state* corresponding to such a cut x we mean $cod(\xi_1)$, and we write such a state as $state_\alpha(x)$. ♯

It is easy to see that the lattice LT_α of cuts of α viewed as a category is a BOPC and that the obvious extension of the correspondence $x \mapsto state_\alpha(x)$ to the mapping mp_α from LT_α to **A** preserves the composition. Given two cuts x and y, by $x \sqcup_\alpha y$ and $x \sqcap_\alpha y$ we denote respectively the least upper bound and the greatest lower bound of x and y. The diagram $(x \leftarrow x \sqcap_\alpha y \rightarrow y, x \rightarrow x \sqcup_\alpha y \leftarrow y)$ is a bicartesian square in LT_α. From (A5) it follows that the image under the mapping mp_α of such a diagram is a bicartesian square in **A**.

8.25. Example. Consider the BOPC $\mathbf{A}_2$ in example 8.6. For the process $\tau = [T] = \sigma'(\pi + \rho)\sigma''$ of this BOPC described in example 2.8 we have the BOPC LT_τ shown in figure 8.2 and its minimal regions

$i = \{(u, \tau)\}$,
$j = \{(\sigma', (\pi + \rho)\sigma''), ..., (\sigma'(\pi + dom(\rho)), (cod(\pi) + \rho)\sigma'')\}$,...,
$j' = \{(\sigma'(dom(\pi) + \rho), (\pi + cod(\rho))\sigma''), ..., (\sigma'(\pi + \rho), \sigma'')\}$,...,
$k = \{(\sigma', (\pi + \rho)\sigma''), ..., (\sigma'(dom(\pi) + \rho), (\pi + cod(\rho))\sigma'')\}$,...,
$k' = \{(\sigma'(\pi + dom(\rho)), (cod(\pi) + \rho)\sigma''), ..., (\sigma'(\pi + \rho), \sigma'')\}$,
$l = \{(\tau, u)\}$. ♯

8.26. Example. Consider the BOPC $TS(M', I')$ in example 8.14. For the process $\delta = \gamma\gamma_1 = \alpha\beta'\alpha_1\beta'_1$ of this system we have the BOPC LT_δ shown in figure 8.3 and its minimal regions
$e = \{(u, \delta), (\beta, \alpha'\gamma_1), (\beta\beta''_1, \alpha''\alpha'_1)\}$, $g = \{(\alpha, \beta'\gamma_1), (\gamma, \gamma_1), (\gamma\beta_1, \alpha'_1)\}$,
$e' = \{(\alpha\alpha''_1, \beta''\beta'_1), (\gamma\alpha_1, \beta'_1), (\delta, u)\}$, $f = \{(u, \delta), (\alpha, \beta'\gamma_1), (\alpha\alpha''_1, \beta''\beta'_1)\}$,
$h = \{(\beta, \alpha'\gamma_1), (\gamma, \gamma_1), (\gamma\alpha_1, \beta'_1)\}$, $f' = \{(\beta\beta''_1, \alpha''\alpha'_1), (\gamma\beta_1, \alpha'_1), (\delta, u)\}$. ♯

$$\begin{tikzcd}
 & (\sigma'(dom(\pi)+\rho),(\pi+cod(\rho))\sigma'') \arrow[r] & \cdots \arrow[r] & (\sigma'(\pi+\rho),\sigma'') \arrow[r] & (\tau,u) \\
(u,\tau) \arrow[r] & (\sigma',(\pi+\rho)\sigma'') \arrow[r] \arrow[u] & \cdots \arrow[r] & (\sigma'(\pi+dom(\rho)),(cod(\pi)+\rho)\sigma'') \arrow[u] &
\end{tikzcd}$$

LT_τ

Figure 8.2

$$\begin{tikzcd}
(\beta\beta_1'',\alpha''\alpha_1') \arrow[r, "\alpha''"] & (\gamma\beta_1,\alpha_1') \arrow[r, "\alpha_1'"] & (\delta,u) \\
(\beta,\alpha'\gamma_1) \arrow[r, "\alpha'"] \arrow[u, "\beta_1''"] & (\gamma,\gamma_1) \arrow[r, "\alpha_1"] \arrow[u, "\beta_1"] & (\gamma\alpha_1,\beta_1') \arrow[u, "\beta_1'"] \\
(u,\delta) \arrow[r, "\alpha"] \arrow[u, "\beta"] & (\alpha,\beta'\gamma_1) \arrow[r, "\alpha_1''"] \arrow[u, "\beta'"] & (\alpha\alpha_1'',\beta''\beta_1') \arrow[u, "\beta''"]
\end{tikzcd}$$

LT_δ

Figure 8.3

Let $\mathbf{A} = (A, ;)$ be an arbitrary BOPC. system.

Given an element α of $\mathbf{A}$, by R_α we denote the set of minimal regions of the BOPC LT_α.

Using regions of $\mathbf{A}$ we want to assign to each process α of $\mathbf{A}$ a labelled partially ordered set (an lposet)
$L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$. Each element $x \in X_\alpha$ is supposed to play the role of an occurrence in α of a minimal region $l_\alpha(x)$ of $\mathbf{A}$. The partial order $\leq_\alpha$ is supposed to reflect how occurrences of minimal regions arise from other minimal occurrences.

The underlying set X_α of L_α is supposed to be defined referring to the set R_α of minimal regions of the BOPC LT_α and to a relation $\vdash_\alpha$ between minimal regions of LT_α and minimal regions of $\mathbf{A}$.

We are going to show how to define the respective lposet L_α for every element of $\mathbf{A}$.

8.27. Proposition. Every minimal region $r \in R_\alpha$ is *convex* in the sense that $w \in r$ for every w such that $u \sqsubseteq_\alpha w \sqsubseteq_\alpha v$ for some $u \in r$ and $v \in r$. $\sharp$

Proof. Suppose that $r \in R_\alpha$ and $a \sqsubseteq_\alpha c \sqsubseteq_\alpha b$ for $a, b \in r$ and $c \notin r$. Define r^- to be the set of $u \in r$ such that $u \sqsubseteq_\alpha c$ or $u' \sqsubseteq_\alpha c$ for some u' that can be connected with u by a side of a bicartesian square with the nodes of the opposite side not in r. Define r^+ to be the set of $u \in r$ such that $c \sqsubseteq_\alpha u$ or $c \sqsubseteq_\alpha u'$ for some u' that can be connected with u by a side of a bicartesian square with the nodes of the opposite side not in r. There is no bicartesian square with a side connecting some $u \in r$ and $v \in r$ such that $u \sqsubseteq_\alpha c \sqsubseteq_\alpha v$ and with the nodes of the opposite side not in r because by (A6) it would imply $c \in r$. By (A8) there are no bicartesian squares with sides connecting some u' with $u \in r$ and $v \in r$ such that $u \sqsubseteq_\alpha c \sqsubseteq_\alpha v$ and with the nodes of the opposite sides not in r. Consequently, the sets r^- and r^+ are disjoint. On the other hand, r is a minimal region of LT_α and thus $r \subseteq r^- \cup r^+$. Moreover, there is no bicartesian square connecting an element of r^- with an element of r^+ and with the nodes of the opposite side not in r. Consequently, r cannot be a minimal region of LT_α as supposed. $\sharp$

In the set R_α there exists a partial order that can be defined as follows.

8.28. Definition. Given $x, y \in R_\alpha$, we write $x \preceq_\alpha y$ iff for every $v \in y$ there exists $u \in x$ such that $u \sqsubseteq_\alpha v$, for every $u \in x$ there exists $v \in y$ such that $u \sqsubseteq_\alpha v$, and the following conditions are satisfied:

(1) $t \in x$ iff $w \in y$, for every bicartesian square $(u \leftarrow t \rightarrow w, u \rightarrow v \leftarrow w)$ with $u \in x$ and $v \in y$,

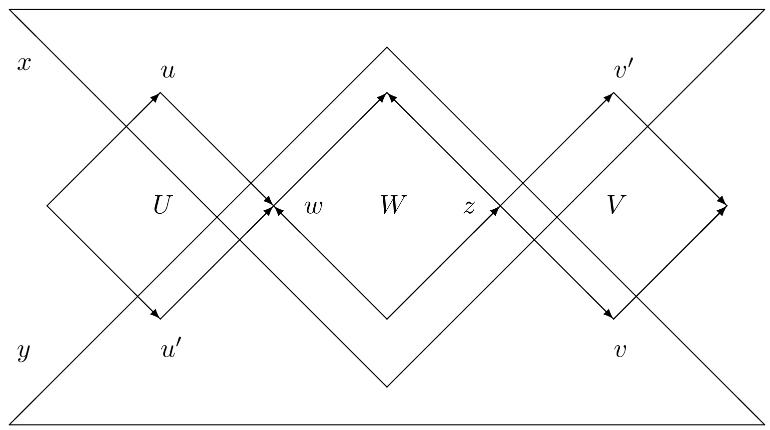
(2) $t' \in x$ iff $w' \in y$, for every bicartesian square $(t' \leftarrow u \rightarrow v, t' \rightarrow w' \leftarrow v)$ with $u \in x$ and $v \in y$. $\sharp$

8.29. Proposition. If minimal regions $x, y \in R_\alpha$ are not disjoint and different then neither $x \preceq_\alpha y$ nor $y \preceq_\alpha x$. $\sharp$

Proof. Suppose that x and y are different minimal regions of LT_α such that $x \cap y \neq \emptyset$. Then $x - y$ and $y - x$ are nonempty and there exist $u \in x - y$, $v \in y - x$, and $w, z \in x \cap y$ such that u and w are adjacent nodes of a bicartesian square U, z and v are adjacent nodes of a bicartesian square V, and the nodes of the bicartesian square $W = (w \leftarrow w \sqcap_\alpha z \rightarrow z, w \rightarrow w \sqcup_\alpha z \leftarrow z)$ are in $x \cap y$.

Consider the case in which $w = u \sqcup_\alpha u'$ for some u' not in x and $z = v \sqcap_\alpha v'$ for some v' not in y, as it is depicted in figure 8.4. Then $u' \in y$, $v' \in x$, and the condition (1) is not satisfied for $z \sqsubseteq_\alpha v$ and the bicartesian square $(v \leftarrow z \rightarrow v', v \rightarrow v \sqcup_\alpha v' \leftarrow v')$. Consequently, $x \preceq_\alpha y$ does not hold.

Similarly, in the other possible cases we come to the conclusion that neither $x \preceq_\alpha y$ nor $y \preceq_\alpha x$. $\sharp$



$x, y \in R_\alpha$

Figure 8.4

8.30. Proposition. If minimal regions $x, y \in R_\alpha$ are disjoint then either $x \preceq_\alpha y$ or $y \preceq_\alpha x$. $\sharp$

Proof. It is impossible that u and v are incomparable for all $u \in x$ and $v \in y$ since one of the regions x or y contains $u \sqcup_\alpha v$ or $u \sqcap_\alpha v$.

Suppose that $u \sqsubseteq_\alpha v$ for $u \in x$ and $v \in y$. As x and y are disjoint and convex, it suffices to prove that every element of y has a predecessor in x.

Consider $w \in y$. If $v \sqsubseteq_\alpha w$ then $u \sqsubseteq_\alpha w$. If $w \sqsubseteq_\alpha v$ then $u' \sqsubseteq_\alpha w$ for $u' = u \sqcap_\alpha w$ and by considering the bicartesian square $(u \leftarrow u' \rightarrow w, u \rightarrow w' \leftarrow w)$ we obtain that $w' \in y$ because y is convex. Hence $u' \in x$. If w and v are incomparable then either $v \sqcap_\alpha w \in y$ and we may replace w by $v \sqcap_\alpha w$

and proceed as in the previous case, or $v \sqcup_\alpha w \in y$ and we may replace v by $v \sqcup_\alpha w \in y$ and proceed as in the previous case. On the other hand, $u \sqsubseteq_\alpha v$ for $u \in x$ and $v \in y$ excludes $v' \sqsubseteq_\alpha u'$ for $u' \in x$ and $v' \in y$ since x and y are convex. Hence $x \preceq_\alpha y$.

Similarly, in the case $v \sqsubseteq_\alpha u$ we obtain $y \preceq_\alpha x$. $\sharp$

8.31. Proposition. The relation $\preceq_\alpha$ is a partial order on R_α.

Proof. The transitivity of the relation $\preceq_\alpha$ follows from the definition of this relation. The antisymmetry follows from the transitivity and from the propositions 8.29 and 8.30. $\sharp$

The relation $\vdash_\alpha$ between minimal regions of LT_α and minimal regions of **A** can be defined as follows.

8.32. Proposition. For every minimal region m of LT_α there exists a minimal region r of **A** such that the set $state_\alpha(m) = \{state_\alpha(u) : u \in m\}$ is contained in r, and we write $m \vdash_\alpha r$. $\sharp$

Proof. Given a minimal region m of LT_α, let r be a minimal element of the set of regions of **A** containing the set $state_\alpha(m)$. As the image of every bicartesian square of LT_α under the mapping mp_α from LT_α to **A** is a bicartesian square in **A**, and for every partition of m into two disjoint nonempty subsets m' and m'' there exists in LT_α a bicartesian square connecting m' and m'', the same holds true for r. Consequently, r is a minimal region of **A**. $\sharp$

Finally, the lposet $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ can be defined by defining X_α as the set of pairs (m, r) such that $m \in R_\alpha$ and $m \vdash_\alpha r$, the relation $\leq_\alpha$ as the partial order on X_α such that $x \leq_\alpha x'$ for $x = (m, r)$ and $x' = (m', r')$ whenever $m \preceq_\alpha m'$, and $l_\alpha(x)$ as r for $x = (m, r) \in X_\alpha$.

8.33. Example. Consider the BOPC $\mathbf{A}_2$ described in example 8.6, its minimal regions $[(p, q)]$, $[(d, r)]$ described in example 8.13, and the minimal regions i, j,...,j', k,...,k', l of LT_τ for $\tau = [T] = \sigma'(\pi + \rho)\sigma''$ as in example 8.25. We obtain $L_\tau = (X_\tau, \leq_\tau, l_\tau)$, where

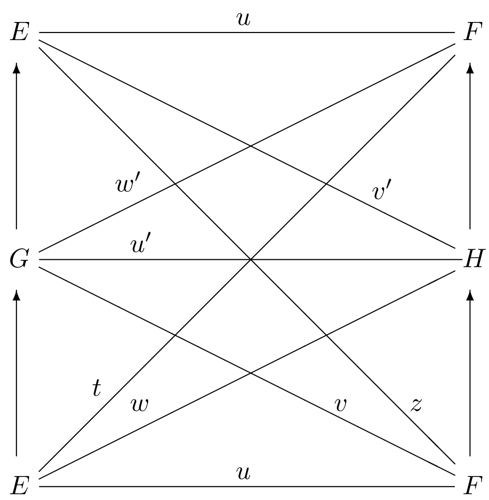
$$X_\tau = \{(i, [(p, q_0 + m)]), (i, [(d, r_0 - m)]),\ (j, [(p, q_0)]), ..., (j', [(p, q_1)]),$$
$$(k, [(d, r_0)]), ..., (k', [(d, r_1)]),\ (l, [(p, q_1 - m')]), (l, [(d, r_1 + m')])\},$$

$$(i, [(p, q_0 + m)]), (i, [(d, r_0 - m)]) \leq_\tau$$
$$\{(j, [(p, q_0)]) \leq_\tau ... \leq_\tau (j', [(p, q_1)])\},\ \{(k, [(d, r_0)]) \leq_\tau ... \leq_\tau (k', [(d, r_1)])\}$$
$$\leq_\tau (l, [(p, q_1 - m')]), (l, [(d, r_1 + m')]),$$

$l_\tau((i,[(p,q_0+m)])) = [(p,q_0+m)]$, $l_\tau((j,[(p,q_0)])) = [(p,q_0)]$,
$l_\tau((j',[(p,q_1)])) = [(p,q_1)]$, $l_\tau((k,[(d,r_0)])) = [(d,r_0)]$,...,
$l_\tau((k',[(d,r_1)]) = [(d,r_1)]$, $l_\tau((l,[(p,q_1-m')])) = [(p,q_1-m')]$,
$l_\tau((l,[(d,r_1+m')])) = [(d,r_1+m')]$.

The corresponding $[L_\tau]$ is essentially as that in figure 2.2. $\sharp$

8.34. Example. Consider the BOPC $TS(M',I')$ described in example 8.14, its minimal regions E, F, G, H, and the minimal regions e, g, e', f, h, f' of LT_δ for $\delta = \gamma\gamma_1 = \alpha\beta'\alpha_1\beta'_1$ as in example 8.26. We obtain $L_\delta = (X_\delta, \leq_\delta, l_\delta)$, where $X_\delta = \{(e,E),(g,G),(e',E),(f,F),(h,H),(f',F)\}$, $(e,E) \leq_\delta (g,G) \leq_\delta (e',E)$, $(f,F) \leq_\delta (h,H) \leq_\delta (f',F)$, $l_\delta((e,E)) = l_\delta((e',E) = E$, $l_\delta((g,G)) = G$, $l_\delta((f,F)) = l_\delta((f',F)) = F$, $l_\delta((h,H)) = H$. The corresponding $[L_\delta]$ is presented in figure 8.5. $\sharp$



L_δ

Figure 8.5

8.35. Proposition. For every element u of LT_α, and for every $x, y \in R_\alpha$ such that $x \preceq_\alpha y$, and $x \preceq_\alpha x'$ for some $x' \in X_\alpha$ such that $u \in x'$, and $y' \preceq_\alpha y$ for some $y' \in X_\alpha$ such that $u \in y'$, there exists $z \in X_\alpha$ such that $u \in z$, and $x \preceq_\alpha z$, and $z \preceq_\alpha y$. $\sharp$

Proof. For $x' = x$ it suffices to define z as x. For $y' = y$ it suffices to define z as y. Consider the case in which $x' \neq x$ and $y' \neq y$. By proposition 8.29 in this case x and y are disjoint, x' and x are disjoint, and y' and y are disjoint. Consequently, u does not belong to x, u does not belong to y, and, by proposition 8.22, there exists $z \in X_\alpha$ that is disjoint both with x and with y, as required. $\sharp$

Crucial for a representation of behaviour-oriented partial categories are the properties of $\mathbf{A}$ described in proposition 8.35 and in the following propositions.

8.36. Proposition. Every two different minimal regions x and y of LT_α such that $x \vdash_\alpha r$ and $y \vdash_\alpha r$ for a minimal region r of $\mathbf{A}$ are disjoint. $\sharp$

Proof. The correspondence between $u \xrightarrow{\delta} v$ such that $u = (\xi_1, \xi_2)$, $v = (\eta_1, \eta_2)$, $\eta_1 = \xi_1\delta$, $\xi_2 = \delta\eta_2$ and $mp_\alpha(u) \xrightarrow{\delta} mp_\alpha(v)$ is a functor F_α from LT_α to $\mathbf{A}$. Due to (A5) this functor preserves bicartesian squares and, consequently, $mp_\alpha^{-1}(r)$ is a region in LT_α. Indeed, the image of a bicartesian square
$D = (v \leftarrow t \rightarrow w, v \rightarrow u \leftarrow w)$ of LT_α under F_α is a bicartesian square
$E = (v' \leftarrow t' \rightarrow w', v' \rightarrow u' \leftarrow w')$ of $\mathbf{A}$ since otherwise due to (A6) there would be a bicartesian square
$E' = (v' \leftarrow t'' \rightarrow w', v' \rightarrow u'' \leftarrow w')$ that would be the image of a diagram
$D' = (v \leftarrow \bar{t} \rightarrow w, v \rightarrow \bar{u} \leftarrow w)$ with $\bar{t} \neq t$ or $\bar{u} \neq u$, what is impossible in LT_α.

Say that elements $u, v \in mp_\alpha^{-1}(r)$ are connected if in LT_α there exists a bicartesian square S with one side with the vertices u and v and with the opposite side with the images of vertices under F_α not in r. Divide $mp_\alpha^{-1}(r)$ into parts such that different parts have no connected vertices and consider maximal decreasing chains of parts thus obtained. Each part is a region of LT_α and for every element x of this part the intersection of a chain of regions contained in this part and containing x is a region as in the proof of Proposition 8.20. Consequently, there exists a minimal region of LT_α that is contained in the considered part and contains x. Consequently, $mp_\alpha^{-1}(r)$ can be represented in a unique way as the union of disjoint minimal regions of LT_α. As these are the only minimal regions contained in $mp_\alpha^{-1}(r)$, the required conclusion follows. $\sharp$

8.37. Proposition. For every α in $\mathbf{A}$ and for $x, y \in X_\alpha$, the equality $l_\alpha(x) = l_\alpha(y)$ implies $x \leq_\alpha y$ or $y \leq_\alpha x$. $\sharp$

Proof. It suffices to take into account propositions 8.30 and 8.36. $\sharp$

Towards a representation

The construction of the labelled poset $L_\alpha = (X_\alpha, \leq_\alpha, l_\alpha)$ for every element α of a BOPC $\mathbf{A}$ is such that due to the properties (A1) - (A4) of $\mathbf{A}$ we obtain that no segment of L_α is isomorphic to its subsegment. This suggests that elements of BOPCs represent processes in a universe of objects.

To see this, consider the universe $U(\mathbf{A}) = (V(\mathbf{A}), W(\mathbf{A}), ob(\mathbf{A}))$ of objects, where $V(\mathbf{A})$ is the set of decompositions of the set of states of $\mathbf{A}$ into disjoint unions of minimal regions of $\mathbf{A}$, $W(\mathbf{A})$ is the set of pairs $w = (v, r)$ consisting of a decomposition v of the set of states of $\mathbf{A}$ into a disjoint union of minimal regions of $\mathbf{A}$ and of a minmal region $r \in v$, and $(ob(\mathbf{A}))(w) = v$ for every $w = (v, r) \in W(\mathbf{A})$. Due to proposition 8.23 the sets $V(\mathbf{A})$ and $W(\mathbf{A})$ are nonempty. Given $\alpha \in A$, consider the lposet $L^*_\alpha = (X^*_\alpha, \leq^*_\alpha, l^*_\alpha)$, where X^*_α is the set of triples (m, v, r) such that such that $m \in R_\alpha$ and $m \vdash_\alpha r$ and $(v, r) \in W(\mathbf{A})$, the relation $\leq^*_\alpha$ is the partial order on X^*_α such that $x \leq^*_\alpha x'$ for $x = (m, r, v)$ and $x' = (m', r', v')$ whenever $m \preceq_\alpha m'$ and $r = r'$ implies $v = v'$ and $m = m'$ implies $r = r'$, and $l^*_\alpha(x) = (v, r)$ for $x = (m, r, v) \in X^*_\alpha$. a As the minimal regions of every decomposition $v \in V(\mathbf{A})$ are disjoint, due to proposition 8.30 we obtain easily that the set $X^*_\alpha|v = \{x \in X^*_\alpha : (ob(\mathbf{A}))(l^*_\alpha(x)) = v\}$ is a maximal chain and has an element in every cross-secton of L^*_α. As also every element of X^*_α belongs to a cross-section of L^*_α, we obtain that L^*_α is a concrete process in $U(\mathbf{A})$. Consequently, we obtain the following proposition.

8.38. Proposition. Given a behaviour-oriented partial category $\mathbf{A}$, the correspondence

$$\alpha \mapsto [L^*_\alpha] = [(X^*_\alpha, \leq^*_\alpha, l^*_\alpha)]$$

between elements of $\mathbf{A}$ and pomsets is a mapping from $\mathbf{A}$ to the partial category of processes in the universe $U(\mathbf{A}) = (V(\mathbf{A}), W(\mathbf{A}), ob(\mathbf{A}))$. $\sharp$

8.39. Example. Consider the BOPC represented by the diagram in figure 8.6, where $\alpha\beta' = \beta\alpha'$, $\alpha'\gamma' = \gamma\alpha''$, $\delta\gamma'' = \gamma'\delta'$. In this system the diagrams $(v \overset{\alpha}{\leftarrow} u \overset{\beta}{\rightarrow} w, v \overset{\beta'}{\rightarrow} u' \overset{\alpha'}{\leftarrow} w)$, $(u' \overset{\alpha'}{\leftarrow} w \overset{\gamma}{\rightarrow} \overline{u}, u' \overset{\gamma'}{\rightarrow} z \overset{\alpha''}{\leftarrow} \overline{u})$, $(t \overset{\delta}{\leftarrow} u' \overset{\gamma'}{\rightarrow} z, t \overset{\gamma''}{\rightarrow} u'' \overset{\delta'}{\leftarrow} z)$ are cartesian squares, the sets $uw\overline{u} = \{u, w, \overline{u}\}$, $vu'z = \{v, u', z\}$, $tu'' = \{t, u''\}$, $wu'\overline{u}z = \{w, u', \overline{u}, z\}$, $uv = \{u, v\}$, $wu't = \{w, u', t\}$, $\overline{u}zu'' = \{\overline{u}, z, u''\}$ are minimal regions, and we have the following decompositions of the set of states into disjoint unions of minimal regions
$I = \{uw\overline{u}, vu'z, tu''\}$, $J = \{uv, wu'\overline{u}z, tu''\}$, $K = \{uv, wu't, \overline{u}zu''\}$.
Consequently, the respective universe of objects is
$\mathbf{U}' = (W', V', ob')$, where

$V' = \{I, J, K\}$,
$W' = \{(I, uw\overline{u}), (I, vu'z), (I, tu''), (J, uv), (J, wu'\overline{u}z), (J, tu''),$
$(K, uv), (Kwu't), (K, \overline{u}zu'')\}$,
$ob'(I, uw\overline{u}) = ob'(I, vu'z) = ob'(I, tu'') = I$,
$ob'(J, uv) = ob'(J, wu'\overline{u}z) = ob'(J, tu'') = J$,
$ob'(K, uv) = ob'(K, wu't) = ob'(K, \overline{u}zu'') = K$.

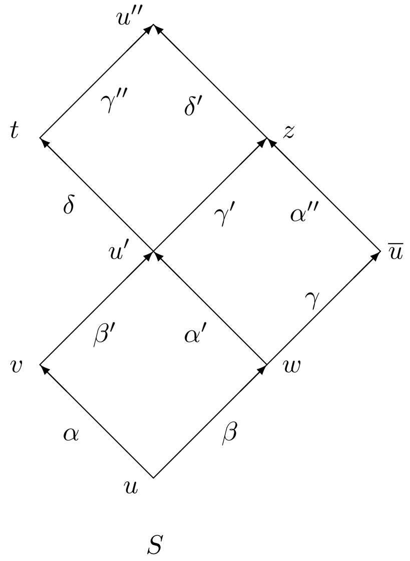


Figure 8.6

Consider the process $\pi = \alpha\beta'\delta\gamma''$ of this system. The lattice LT_π of decompositions of this process is essentially identical with the system itself, and we have the following set of minimal regions of this lattice

$R_\pi = \{uw\overline{u}, vu'z, tu'', uv, wu'\overline{u}z, wu't, \overline{u}zu''\}$,

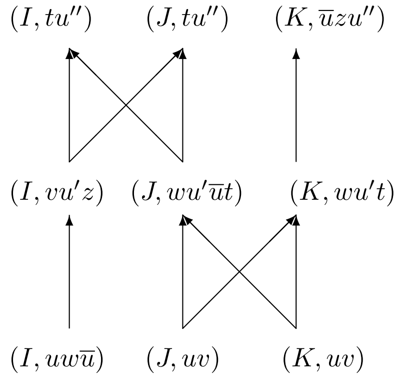
where

$uw\overline{u} \preceq_\pi vu'z \preceq_\pi tu''$, $uv \preceq_\pi wu'\overline{u}z \preceq_\pi tu''$, $uv \preceq_\pi wu't \preceq_\pi \overline{u}zu''$.

Consequently,

$$X^*_\pi = \{(uw\overline{u}, I, uw\overline{u}), (vu'z, I, vu'z)(tu'', I, tu''), (uv, J, uv), (wu'\overline{u}z, J, wu'\overline{u}z), (tu'', J, tu''), (uv, K, uv), (wu't, K, wu't), (\overline{u}zu'', K, \overline{u}zu'')\}$$

with the partial order $\leq_\pi$ induced by $\preceq_\pi$, and we obtain the process in $\mathbf{U}'$ shown in figure 8.7. $\sharp$



$[L^*_\pi]$

Figure 8.7

8.40. Example. Consider the BOPC represented by the diagram in figure 8.8, where $\alpha\beta' = \beta\alpha' \neq \varphi$. In this diagram $(q \xleftarrow{\alpha} p \xrightarrow{\beta} r, q \xrightarrow{\beta'} s \xleftarrow{\alpha'} r)$ is a bicartesian square, the sets $pq = \{p, q\}$, $pr = \{p, r\}$, $qs = \{q, s\}$, $rs = \{r, s\}$ are minimal regions, and $X = \{pq, rs\}$, $Y = \{pr, qs\}$ are decompositions of the set of states into disjoint unions of minimal regions. For the process φ the lattice LT_φ of decompositions of this process consists of the least element $a = (p, \varphi)$ and the greatest element $b = (\varphi, s)$. Consequently, L^*_φ is a process as shown in figure 8.9 and it is identical with L^{**}_φ. $\sharp$

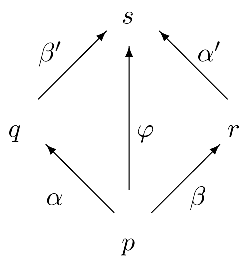


Figure 8.8

$$\begin{tikzcd}
(b,rs,X) & & (b,qs,Y) \\
(a,pq,X) \arrow[u] \arrow[urr] & & (a,pr,Y) \arrow[u] \arrow[ull]
\end{tikzcd}$$

Figure 8.9

Note that the correspondence $\alpha \mapsto [L^*_\alpha] = [(X^*_\alpha, \leq^*_\alpha, l^*_\alpha)]$ need not be a homomorphism. To see this it suffices to consider a BOPC $\mathbf{A}$ that is the reduct of an algebra of processes, and in this BOPC a process $\gamma = \alpha\beta$, where $\alpha = dom(\varphi) + \psi$ and $\beta = \varphi + cod(\psi)$. It is easy to see that $[L^*_\gamma] \neq [L^*_\alpha][L^*_\beta]$.

However, every process L^*_α can be transformed into a process L^{**}_α such that the correspondence $\alpha \mapsto [L^{**}_\alpha]$ is a homomorphism. This can be done as follows.

The fact that all $(m, r, v) \in X^*_\alpha$ with the same r and v form a chain implies the following proposition.

8.41. Proposition. The following relation between elements of X^*_α is an equivalence relation: $(m, r, v) \simeq_\alpha (m', r', v')$ iff $v' = v$, $r' = r$, $m \vdash_\alpha r$, $m' \vdash_\alpha r$, and $m'' \vdash_\alpha r$ for all m'' such that $m \sqsubseteq_\alpha m'' \sqsubseteq_\alpha m'$ or $m' \sqsubseteq_\alpha m'' \sqsubseteq_\alpha m$. $\sharp$

Due to this proposition it is straightforward to prove the following proposition.

8.42. Proposition. The triple $L^{**}_\alpha = (X^{**}_\alpha, \leq^{**}_\alpha, l^{**}_\alpha)$ with $X^{**}_\alpha = X^*_\alpha / \simeq_\alpha$, $x \leq^{**}_\alpha x'$ whenever $(m, r, v) \leq^*_\alpha (m', r', v')$ for all $(m, r, v) \in x$ and $(m', r', v') \in x'$, and $l^{**}_\alpha(x) = l^*_\alpha(m, r, v)$ for $(m, r, v) \in x$, is a concrete process in $U(\mathbf{A})$. $\sharp$

8.43. Example. Consider a system M consisting of machines M_1 and M_2 as in example 2.7. Its global processes form a subalgebra $\mathbf{A_1}$ of the algebra $\mathbf{pcatgPROC}(\mathbf{U_1})$ of global processes in the universe $\mathbf{U_1}$ described in example 2.2. This subalgebra consists of processes that can be obtained by combining the processes $a + c$, $a + d$, $b + c$, $b + d$, $\alpha_c = \alpha + c$, $\alpha_d = \alpha + d$, $\beta_c = \beta + c$, $\beta_d = \beta + d$, γ, $\delta_a = \delta + a$, $\delta_b = \delta + b$ with the aid of composition and construction of limits. It is a BOPC with bicartesian squares $(a + c \overset{\alpha_c^m}{\leftarrow} a + c \overset{\delta_a}{\rightarrow} a + d, a + c \overset{\delta_a}{\rightarrow} a + d \overset{\alpha_d^m}{\leftarrow} a + d)$, $(b + c \overset{\beta_c}{\leftarrow} a + c \overset{\delta_a}{\rightarrow} a + d, b + c \overset{\delta_b}{\rightarrow} b + d \overset{\beta_d}{\leftarrow} a + d)$, minimal regions $A = \{a + c, a + d\}$, $B = \{b + c, b + d\}$,

$C = \{a + c, b + c\}$, $D = \{a + d, b + d\}$, and decompositions $P = \{A, B\}$, $Q = \{C, D\}$ of the set of states into disjoint unions of minimal regions. The respective universe of objects is $\mathbf{U}(\mathbf{A_1}) = (V(\mathbf{A_1}), W(\mathbf{A_1}), ob(\mathbf{A_1}))$, where $W(\mathbf{A_1}) = \{A, B, C, D\}$, $V(\mathbf{A_1}) = \{P, Q\}$, $(ob(\mathbf{A_1}))(A) = (ob(\mathbf{A_1}))(B) = P$, $(ob(\mathbf{A_1}))(C) = (ob(\mathbf{A_1}))(D) = Q$. For every process π of $\mathbf{A_1}$ we have the corresponding lattice LT_π of decompositions of π, the corresponding set R_π of minimal regions of this lattice, the corresponding partial order $\preceq_\pi$ on R_π, and the corresponding process L^*_π in $\mathbf{U_1}$. For example, for $\pi = \alpha_c\beta_c\delta_b\gamma\beta_c$ we have the lattice of decompositions of π shown in figure 8.10, the set $R_\pi = \{x, y, z, p, q, r, s\}$ of minimal regions, where

$x = \{(a + c, \pi)\} \vdash A, C,$
$y = \{(\alpha_c, \beta_c\delta_b\gamma\beta_c), (\alpha_c\delta_a, \beta_d\gamma\beta_c)\} \vdash A$
$z = \{(\alpha_c\beta_c, \delta_b\gamma\beta_c), (\alpha_c\beta_c\delta_b, \gamma\beta_c)\} \vdash B$
$p = \{(\alpha_c, \beta_c\delta_b\gamma\beta_c), (\alpha_c\beta_c, \delta_b\gamma\beta_c)\} \vdash C$
$q = \{(\alpha_c\delta_a, \beta_d\gamma\beta_c), (\alpha_c\beta_c\delta_b, \gamma\beta_c)\} \vdash D$
$r = \{(\alpha_c\beta_c\delta_b\gamma, \beta_c)\} \vdash A, C$
$s = \{(\pi, b + c))\} \vdash B, C$

the process L^*_π in $\mathbf{U_1}$ shown in figure 8.11, and the corresponding process L^{**}_π in $\mathbf{U_1}$ shown in figure 8.12. $\sharp$

$$\begin{tikzcd}
 & (\alpha_c\delta_a, \beta_d\gamma\beta_c) \arrow[r, "\beta_d"] & (\alpha_c\beta_d\delta_b, \gamma\beta_c) \arrow[r, "\gamma"] & (\alpha_c\beta_c\delta_b\gamma, \beta_c) \arrow[r, "\beta_c"] & (\pi, b + c) \\
(a + c, \pi) \arrow[r, "\alpha_c"'] & (\alpha_c, \beta_c\delta_b\gamma\beta_c) \arrow[u, "\delta_a"] \arrow[r, "\beta_c"'] & (\alpha_c\beta_c, \delta_b\gamma\beta_c) \arrow[u, "\delta_b"] & &
\end{tikzcd}$$

Figure 8.10

$$\begin{tikzcd}
(x,C,Q) \arrow[r] \arrow[dr] & (p,C,Q) \arrow[r, "\delta"] & (q,D,Q) \arrow[r] \arrow[dr] & (r,C,Q) \arrow[r] \arrow[dr] & (s,C,Q) \\
(x,A,P) \arrow[r] \arrow[ur] & (y,A,P) \arrow[r, "\beta"'] & (z,B,P) \arrow[r] \arrow[ur] & (r,A,P) \arrow[r] \arrow[ur] & (s,B,P) \\
{} \arrow[r, phantom, "\alpha_c"] & {} & {} \arrow[r, phantom, "\gamma"] & {} \arrow[r, phantom, "\beta_c"] & {}
\end{tikzcd}$$

Figure 8.11

$$\begin{tikzcd}
(x,C,Q) \arrow[r] & (p,C,Q) \arrow[r, "\delta"] & (q,D,Q) \arrow[r] \arrow[dr] & (r,C,Q) \arrow[r] & (s,C,Q) \\
(x,A,P) \arrow[r] & (y,A,P) \arrow[r, "\beta"'] & (z,B,P) \arrow[r] \arrow[ur] & (r,A,P) \arrow[r] & (s,B,P) \\
{} \arrow[r, phantom, "\alpha_c"] & {} & {} \arrow[r, phantom, "\gamma"] & {} & {}
\end{tikzcd}$$

Figure 8.12

Now we want to prove that the correspondence $\alpha \mapsto [L^{**}_\alpha] = [(X^{**}_\alpha, \leq^{**}_\alpha, l^{**}_\alpha)]$ between elements of a BOPC $\mathbf{A}$ and processes in the universe $U(\mathbf{A}) = (V(\mathbf{A}), W(\mathbf{A}), ob(\mathbf{A}))$ of objects enjoys the following property.

8.44. Proposition. If $\gamma = \alpha\beta$ with $cod(\alpha) = dom(\beta) = c$ then L^{**}_γ is the pushout object in the category **LPOSETS** of the injections of L^{**}_c in L^{**}_α and in L^{**}_β. $\sharp$

Proof. Let $d \in LT_\gamma$ be the cut (α, β) of γ. The correspondence $i_\alpha : (\alpha_1, \alpha_2) \mapsto (\alpha_1, \alpha_2\beta)$ is an isomorphism between the lattice LT_α and the sublattice $LT_{\gamma,\alpha}$ of LT_γ consisting of the cuts between $(dom(\gamma), \gamma)$ and (α, β). Similarly, the correspondence $i_\beta : (\beta_1, \beta_2) \mapsto (\alpha\beta_1, \beta_2)$ is an isomorphism between the lattice LT_β and the sublattice $LT_{\gamma,\beta}$ of LT_γ consisting of the cuts between (α, β) and $(\gamma, cod(\gamma))$.

Let r be a region of LT_γ and let r_α and r_β be respectively the part of r in $LT_{\gamma,\alpha}$ and the part of r in $LT_{\gamma,\beta}$. Every bicartesian square that is contained in $LT_{\gamma,\alpha}$ and has a side outside of r_α must be disjoint with r_α or must have the entire opposite side in r_α. Consequently, r_α is a region of $LT_{\gamma,\alpha}$. Similarly, r_β is a region of $LT_{\gamma,\beta}$.

Due to (A6) every bicartesian square that is contained in LT_γ and has a side in r_α and the opposite side disjoint with r can be decomposed into two bicartesian squares of which one has a side in r_α and the opposite side disjoint with r_α. Consequently, r_α is a minimal region of $LT_{\gamma,\alpha}$ whenever r is a minimal region of LT_γ, and $r_\alpha \subseteq m$ for every minimal region of LT_γ that contains m. Similarly, every bicartesian square that is contained in LT_γ and has a side in r_β and the opposite side disjoint with r can be decomposed into two bicartesian squares of which one has a side in r_β and the opposite side disjoint with r_β. Consequently, r_β is a minimal region of $LT_{\gamma,\beta}$ whenever r is a minimal region of LT_γ, and $r_\alpha \subseteq n$ for every minimal region of LT_γ that contains n.

Thus every minimal region r of LT_γ has a part r_α in $LT_{\gamma,\alpha}$ and a part r_β in $LT_{\gamma,\beta}$, these parts are minimal regions of $LT_{\gamma,\alpha}$ and $LT_{\gamma,\beta}$, respectively, and they determine r uniquely. Moreover, if both r_α and r_β are nonempty then, due to the convexity of minimal regions of LT_γ, the cut $d = (\alpha, \beta)$ belongs to r.

Exploiting these facts we can verify that

$(L^{**}_\alpha \overset{k_{\gamma,\alpha}}{\leftarrow} L^{**}_\gamma \overset{k_{\gamma,\beta}}{\rightarrow} L^{**}_\beta)$ is a pushout of $(L^{**}_\alpha \overset{j_{\alpha,c}}{\leftarrow} L^{**}_c \overset{j_{\beta,c}}{\rightarrow} L^{**}_\beta)$

with

$j_{\alpha,c} : [m, r, v] \mapsto [m', r, v]$ for m containing (c, c) and m' containing (α, c)

$j_{\beta,c} : [m, r, v] \mapsto [m', r, v]$ for m containing (c, c) and m' containing (c, β)

$k_{\gamma,\alpha} : [m, r, v] \mapsto [m', r, v]$ for m containing (α_1, α_2) and m' containing $(\alpha_1, \alpha_2\beta)$

$k_{\gamma,\beta} : [m, r, v] \mapsto [m', r, v]$ for m containing (β_1, β_2) and m' containing $(\alpha\beta_1, \beta_2)$ $\sharp$

Consequently, we obtain the following result.

8.45. Proposition. Given a behaviour-oriented partial category $\mathbf{A}$, the correspondence $\alpha \mapsto [L_\alpha^{**}] = [(X_\alpha^{**}, \leq_\alpha^{**}, l_\alpha^{**})]$ between elements of $\mathbf{A}$ and processes in the universe $U(\mathbf{A}) = (V(\mathbf{A}), W(\mathbf{A}), ob(\mathbf{A}))$ of objects is a homomorphism from $\mathbf{A}$ to the partial category of processes in $U(\mathbf{A})$. $\sharp$

9

Discrete BOPCs

As we have observed in the previous chapter, discrete behaviour-oriented partial categories are in fact arrows-only categories. If we reduce such categories to their states and bounded atoms then we obtain transition systems. If we endow the transition systems thus obtained with the existing in the original categories information on independence of atomic bounded processes then we obtain structures close to introduced in [WN 95] transition systems with independence and to other similar models as those in [Sh 85] and [Bedn 88].

Transition systems with independence

For the rest of the paper transition systems with independence are defined as follows.

9.1. Definition. A *transition system with independence* is $\Theta = (S, \mathit{Tran}, \mathit{dom}, \mathit{cod}, I)$, where S is a set of *states*, Tran is a set of *transitions*, $\mathit{dom}, \mathit{cod} : \mathit{Tran} \to S$ are functions assigning to each transition τ a *source*, $\mathit{dom}(\tau)$, and a *target*, $\mathit{cod}(\tau)$, and I is a binary *independence relation* in Tran such that

(1) $(s, \alpha, s')I(u, \beta, u')$ implies $s = u$ or $s' = u$,
(2) $(s, \alpha, s_1)I(s, \beta, s_2)$ implies the existence of unique (s_1, β', u) and (s_2, α', u) such that $(s, \alpha, s_1)I(s_1, \beta', u)$ and $(s, \beta, s_2)I(s_2, \alpha', u)$,
(3) $(s, \alpha, s_1)I(s_1, \beta', u)$ implies the existence of unique (s, β, s_2) and (s_2, α', u) such that $(s, \alpha, s_1)I(s, \beta, s_2)$ and $(s, \beta, s_2)I(s_2, \alpha', u)$,
(4) if $\pi = ((s, \pi_i, s_i) : i \in \{1, ..., n\})$ is a family of transitions such that $(s, \pi_i, s_i)I(s, \pi_j, s_j)$ for all $i, j \in \{1, ..., n\}$ such that $i \neq j$ then in $T(\Pi)$ regarded as a graph there exists a unique n-cube $Q(\pi)$ such that $(u, \alpha, v)I(u, \beta, w)$ and $(u, \beta, w)I(w, \delta, t)$ and $(u, \alpha, v)I(v, \gamma, t)$ for each 2-face of this cube that consists of transitions (u, α, v), (u, β, w), (v, γ, t), (w, δ, t). $\sharp$

Note that the properties (1) - (3) correspond to the basic axioms characterizing transition systems with independence of [WN 95].

The following proposition describes how descrete categories of processes define transition systems with independence.

9.2. Proposition. Let Π be a discrete BOPC with the set S_Π of states and the set A_Π of atomic processes. Let $T(\Pi) = (S, \mathit{Tran}, \mathit{dom}, \mathit{cod}, I)$, where

$S = S_\Pi$, *Tran* is the set of triples (s, α, s') such that $\alpha \in A_\Pi$, $s = dom(\alpha)$, $s' = cod(\alpha)$, *dom* and *cod* are the mappings from *Tran* to S defined by $dom(s, \alpha, s') = s$ and $cod(s, \alpha, s') = s'$, and I is the least binary relation in *Tran* such that $(s, \alpha, s_1)I(s, \beta, s_2)$ whenever α and β are parallel independent and $(s, \alpha, s_1)I(s_1, \beta', u)$ whenever α and β' are sequential independent.
Then $T(\Pi)$ is a transition system with independence. $\sharp$

The properties (1) - (3) formulated in definition 9.1 follow from the definition of independence of processes in behaviour-oriented partial categories as the existence of a suitable bicartesian square. The property (4) follows from (A7). Thus we may call $T(\Pi)$ the transition system with independence corresponding to the category of processes Π.

Generated behaviour-oriented partial categories

By defining $Paths(\Theta)$ as the set of paths of Θ, and by defining in the obvious way the source and the target of each path p and the composition of paths p_1 and p_2 such that p_2 follows p_1, we obtain the category of paths of Θ, written as $PATHS(\Theta)$. By defining $\sim_\Theta$ as the least equivalence relation in $Paths(\Theta)$ such that $p_1 \sim_\Theta p_2$ whenever $p_1 = r\alpha\beta s$ and $p_2 = r\beta'\alpha' s$ with $\alpha I\beta$ and the unique α' and β' such that $\alpha I\beta'$ and $\beta' I\alpha'$, we obtain a congruence in the category $PATHS(\Theta)$, and the respective quotient category, $RUNS(\Theta)$, called the category of runs of Θ.

9.3. Theorem. For each transition system with independence, Θ, the category of its runs, $RUNS(\Theta)$, is a discrete behaviour-oriented partial category. $\sharp$

Proof outline.

A diagram $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ in $RUNS(\Theta)$ is a bicartesian square in iff it consists of independent transitions or by applying decompositions as in (A6) it can be decomposed into bicartesian squares consisting of independent transitions. As among the other required properties only (A5) and (A7) are not obvious, it suffices to verify (A5) and (A7).

For (A5) this can be done as follows.

First, it is convenient to fix some terminology. Given two paths p_1 and p_2 such that $p_1 = r\alpha\beta s$ and $p_2 = r\beta'\alpha' s$ with $\alpha I\beta$ and the unique α' and β' such that $\alpha I\beta'$ and $\beta' I\alpha'$, we call the pair (p_1, p_2) a *derivation step*. Given a sequence $p_1, ...p_n$ of paths such that each pair (p_i, p_{i+1}) of contiguous paths in this sequence is a derivation step, we call such a sequence a *derivation* of p_n from p_1. Given two paths p_1 and p_2, by the *distance* between p_1 and p_2, written as $d(p_1, p_2)$ we mean the length of the shortest derivation of p_2 from p_1, if such a derivation exists, or $+\infty$ otherwise. Finally, given two representations $\xi_1\xi_2$ and $\eta_1\eta_2$ of a run from $RUNS(\Theta)$, i.e., $\xi_1\xi_2 = \eta_1\eta_2$, by the *distance* between such representations, written as $d(\xi_1, \xi_2; \eta_1, \eta_2)$, we mean the least distance between paths p_1 and p_2 such that $p_1 = p_{11}p_{12}$ for some $p_{11} \in \xi_1$ and $p_{12} \in \xi_2$, and $p_2 = p_{21}p_{22}$ for some $p_{21} \in \eta_1$ and $p_{22} \in \eta_2$.

In order to verify that the equality $\xi_1\xi_2 = \eta_1\eta_2$ implies the existence of σ_1, σ_2, π_1, π_2, π_1', π_2' as in (A5) we proceed by induction on the distance between the representations $\xi_1\xi_2$ and $\eta_1\eta_2$.

If the distance between the representations is 0 then the required property is immediate.

Suppose that the property holds true for the distance not exceeding n and consider ξ_1, ξ_2, η_1, η_2 such that $d(\xi_1, \xi_2; \eta_1, \eta_2) = n + 1$.

In $RUNS(\Theta)$ there exist ζ_1 and ζ_2 such that $d(\xi_1, \xi_2; \zeta_1, \zeta_2) = n$ and $d(\zeta_1, \zeta_2; \eta_1, \eta_2) = 1$. Consequently, there exist unique τ_1, τ_2, and a unique bicartesian square $(v \stackrel{\alpha_1}{\leftarrow} u \stackrel{\alpha_2}{\rightarrow} w, v \stackrel{\alpha_2'}{\rightarrow} u' \stackrel{\alpha_1'}{\leftarrow} w)$ such that $\xi_1 = \tau_1\alpha_1$, $\xi_2 = \alpha_2'\tau_2$, $\zeta_1 = \tau_1\alpha_2$, $\zeta_2 = \alpha_1'\sigma_2$.

Now, if one of the equalities $\eta_1 = \zeta_1$, or $\eta_2 = \zeta_2$, holds true then also the other holds true, and we have the required property.

Otherwise, there exist γ_1, γ_2, and indecomposable β_1, β_2, β_1', β_2' such that $\beta_1 I \beta_2$, $\beta_1 I \beta_2'$, $\beta_2 I \beta_1'$, and $\zeta_1 = \gamma_1\beta_1$, $\eta_1 = \gamma_1\beta_2$, $\zeta_2 = \beta_2'\gamma_2$, $\eta_2 = \beta_1'\gamma_2$, as shown in figure 9.1.

As $d(\tau_1, \alpha_2; \gamma_1, \beta_1) \leq n$, $d(\alpha_1', \tau_2; \beta_2', \gamma_2) \leq n$, and β_1, β_2, β_1', β_2' are indecomposable, we obtain one of the diagrams in figure 9.2 with all their rectangles being bicartesian squares and the outermost rectangle determining the respective representation of $\xi_1\xi_2 = \eta_1\eta_2$, as required.

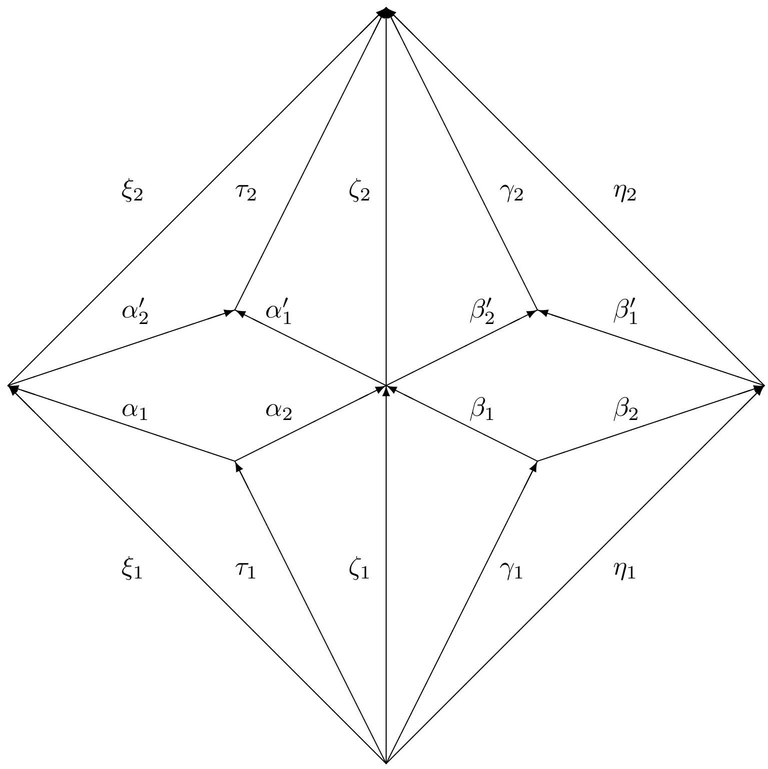


Figure 9.1: A representation of $\xi_1\xi_2 = \eta_1\eta_2$

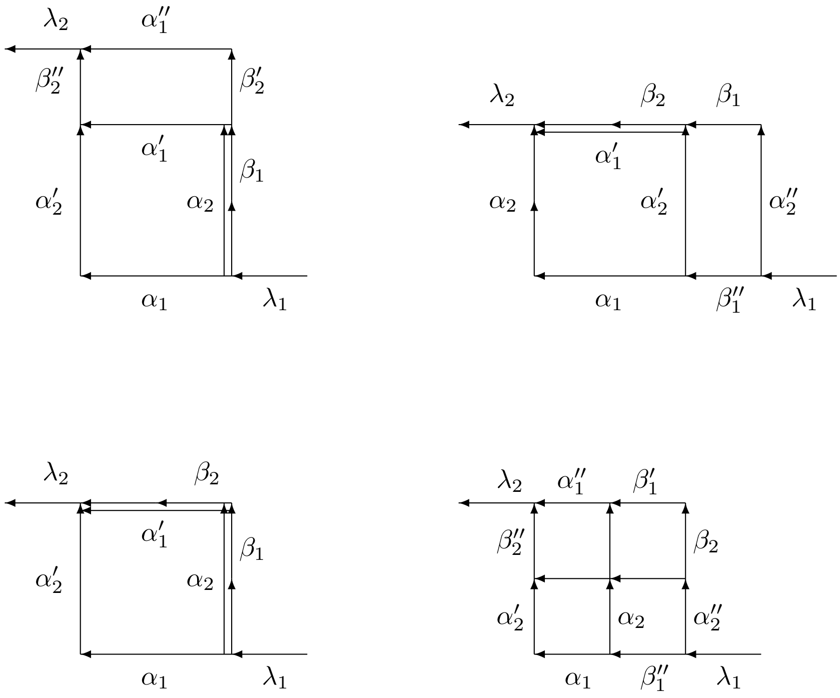


Figure 9.2: More detailed representations of $\xi_1\xi_2 = \eta_1\eta_2$

A proof of (A7) can be carried out by decomposing the bicartesian squares $(v_i \xleftarrow{\pi_i} u \xrightarrow{\pi_j} v_j, v_i \xrightarrow{\pi'_j} u'_{ij} \xleftarrow{\pi'_i} v_j)$ into atomic bicartesian squares which correspond to pairs of independent transitions, by exploiting the properties (1) - (4) of the independence relation of Θ and constructing from the atomic bicartesian squares thus obtained the corresponding atomic bicartesian n-cubes, and by combining these n-cubes along their matching $(n-1)$-faces and thus constructing the required bicartesian n-cube for the original runs. $\sharp$

The relation between transition systems with independence and categories of processes can be described regarding these structures as objects of categories which can be defined as follows.

9.4. Definition. A *morphism* from a transition system with independence $\Theta = (S, \mathit{Tran}, \mathit{dom}, \mathit{cod}, I)$ to another such a system

$\Theta' = (S', \mathit{Tran}', \mathit{dom}', \mathit{cod}', I')$ is a pair (f, g) of mappings $f : S \to S'$ and $g : \mathit{Tran} \to \mathit{Tran}'$ such that $\mathit{dom}'(g(\alpha)) = f(\mathit{dom}(\alpha))$, $\mathit{cod}'(g(\alpha)) = f(\mathit{cod}(\alpha))$, and $\alpha I \beta$ implies $g(\alpha) I' g(\beta)$. $\sharp$

By **TI** we denote the category of transition systems with independence and their morphisms.

9.5. Definition. A *morphism* from a discrete behaviour-oriented partial category Π to a discrete behaviour-oriented partial category Π' is a functor from Π to Π' that preserves bicartesian squares. $\sharp$

By **P** we denote the category of discrete behaviour-oriented partial categories and their morphisms.

Due to theorem 9.3 we obtain the following result.

9.6. Theorem. Each transition system with independence Θ generates freely the discrete behaviour-oriented partial category $\mathit{RUNS}(\Theta)$ in the sense that each morphism from Θ to the transition system with independence $T(\Pi)$ that corresponds to a discrete behaviour-oriented partial category Π has a unique extension to a morphism from $\mathit{RUNS}(\Theta)$ to Π. $\sharp$

It is clear that the correspondence $\Theta \mapsto \mathit{RUNS}(\Theta)$ defines a functor $\mathit{RUNS} : \mathbf{TI} \to \mathbf{P}$ and the correspondence $\Pi \mapsto T(\Pi)$ defines a functor $T : \mathbf{P} \to \mathbf{TI}$. Consequently, 9.6 can be formulated as follows.

9.7. Theorem. The functor $\mathit{RUNS} : \mathbf{TI} \to \mathbf{P}$ is the left adjoint of the functor $T : \mathbf{P} \to \mathbf{TI}$. $\sharp$

10

Recapitulation

The present paper has its origins in [Wink 82], where algebras of finite processes of Condition/Event Petri nets with invariant sets of admitted markings have been characterized and called behaviour algebras. The ideas of [Wink 82] have been extended in a way described in [Wink 07a]. The novelty of this extension consists in a new system of axioms such that a subsystem of this system does not require finiteness of processes or the existence of indivisible processes and thus allows one to model also continuous processes. The new system has been formulated due to discovery of the relation between independence of processes and existence of bicartesian squares in categories of bounded processes that has been described in [Wink 03]. It has been obtained from the characterization of algebras of bounded processes of finite Condition/Event Petri nets that has been described in [Wink 06] by omitting the axioms on decomposability of processes into atoms and on two only instances of each condition.

In [Wink 07b] we have presented a class of algebras of processes in universa of objects that contains also algebras with unbounded, continuous, and partially continuous processes. In [Wink 07a] and [Wink 07b] we have shown that such algebras are models of the new system of axioms and thus that they are behaviour algebras in the new sense. We have shown that there exists a correspondence between elements of behaviour algebras and lposets, and that in the case of a subclass of this class this correspondence results in a representation theorem. Finally, we have shown a way of extending the obtained results on algebras of processes with rich internal structures.

An early attempt of formulating an adequate system of axioms has been described in [Wink 05]. Its main line was to introduce a model of processes with context-dependent actions and rich internal structures and by defining and studying algebras of such processes in order to find out their characteristic properties.

Now, due to the results obtained for the new system of axioms, it seems that an adequate framework for modelling complex processes can be obtained with the aid of behaviour-oriented algebras and their subalgebras. For instance, processes with context-dependent components as in [MR 95] and [BBM 02] can be represented as elements of the subalgebra of an algebra of processes in a universe of objects that is generated by processes consisting of two concurrent components: one representing the proper process and the other representing the necessary context. Similarly, processes with rich internal structures as in [Wink 05] can be represented as elements of suitable subalgebras of behaviour-oriented algebras that are consistently endowed with the respective structures as it is described in section 8. For example, graph processes in the sense of [CMR 96] can be represented as proceses consistently provided with graph structures.

A problem that still remains open is how to come from the representation of processes of behaviour algebras with finite sets definable objects to a representation of processes of behaviour algebras with infinite sets of definable objects.

Behaviour-oriented algebras are thought as a framework for defining behaviours of concurrent systems. Behaviours of concrete systems can be defined as prefix-closed directed complete subsets of algebras of processes in suitable universes of objects. Such subsets inherit from the algebras they come from structures which reflect how processes compose, the prefix order, and possibly specific features of the represented behaviours. They can be constructed with the aid of operations similar to those in known algebras of behaviours in other similar calculi.

Many of the possibilities of behaviour-oriented algebras offer also partial algebras with one only operation of sequential composition, called behaviour-oriented partial categories, or briefly BOPCs. We have shown that some of such simplified algebras ca be represented as partial categories of global processes in some universes of objects. This result is interesting because it means that the proposed in the paper notion of a process is in a sense universal.

What we have presented in the paper about random behaviours suggests that algebras of processes in universes of objects and their subalgebras and reducts offer also an adequate framework for constructing models of concurrent systems with random behaviours. This framework seems to be universal enough to construct probabilistic models not only for discrete, but also for continuous and hybrid concurrent systems with random behaviours.

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Appendix A: Posets and their cross-sections

Given a partial order $\leq$ on a set X, i.e. a binary relation which is reflexive, anti-symmetric and transitive, we call $P = (X, \leq)$ a *partially ordered set*, or briefly a *poset*, by the *strict partial order* corresponding to $\leq$ we mean $<$, where $x < y$ iff $x \leq y$ and $x \neq y$, by a *chain* we mean a subset $Y \subseteq X$ such that $x \leq y$ or $y \leq x$ for all $x, y \in Y$, and by an *antichain* we mean a subset $Z \subseteq X$ such that $x < y$ does not hold for any $x, y \in Z$.

A.1. Definition. Given a poset $P = (X, \leq)$, by a *strong cross-section* of P we mean a maximal antichain Z of P that has an element in every maximal chain of P. By a *weak cross-section*, or briefly a *cross-section*, of P we mean a maximal antichain Z of P such that, for every $x, y \in X$ for which $x \leq y$ and $x \leq z'$ and $z'' \leq y$ with some $z', z'' \in Z$, there exists $z \in Z$ such that $x \leq z \leq y$. $\sharp$

A.2. Definition. We say that a partial order $\leq$ on X (and the poset $P = (X, \leq)$) is *strongly K-dense* (resp.: *weakly K-dense*) iff every maximal antichain of P is a strong (resp.: a weak) cross-section of P (cf. [Petri 80] and [Plue 85], where K-density is defined as the strong K-density in our sense). $\sharp$

A.3. Definition. For every cross-section Z of a poset $P = (X, \leq)$, we define $X^-(Z) =\leq Z (= \{x \in X : x \leq z \text{ for some } z \in Z\})$ and $X^+(Z) = Z \leq = (\{x \in X : z \leq x \text{ for some } z \in Z\})$, and we say that a cross-section Z' *precedes* a cross-section Z'' and write $Z' \preceq Z''$ iff $X^-(Z') \subseteq X^-(Z'')$. $\sharp$

A.4. Proposition. The relation $\preceq$ is a partial order on the set of cross-sections of $P = (X, \leq)$. For every two cross-sections Z' and Z'' of P there exist the greatest lower bound $Z' \wedge Z''$ and the least upper bound $Z' \vee Z''$ of Z' and Z'' with respect to $\preceq$, where $Z' \wedge Z''$ is the set of those $z \in Z' \cup Z''$ for which $z \leq z'$ for some $z' \in Z'$ and $z \leq z''$ for some $z'' \in Z''$, and $Z' \vee Z''$ is the set of those $z \in Z' \cup Z''$ for which $z' \leq z$ for some $z' \in Z'$ and $z'' \leq z$ for some $z'' \in Z''$. Moreover, the set of cross-sections of P with the operations thus defined is a distributive lattice. $\sharp$

Proof. The set $Z' \wedge Z''$ is an antichain since otherwise there would be $x < y$ for some x and y in this set. If $x \in Z'$ then there would be $y \in Z''$ and there would exist $z' \in Z'$ such that $y \leq z'$. However, this is impossible since Z' is an antichain. Similarly for $x \in Z''$.

The set $Z' \wedge Z''$ is a maximal antichain since otherwise there would exist x that would be incomparable with all the elements of this set. Consequently, there would not exist $z' \in Z'$ and $z'' \in Z''$ such that $z' \leq x \leq z''$, or $z'' \leq x \leq z'$, or $z', z'' \leq x$, and thus there would be $x \leq z'$ and $x \leq z''$ for some $z' \in Z'$ and $z'' \in Z''$ that are not in $Z' \wedge Z''$. Consequently, there would exist z, say in Z'', such that $x \leq z \leq z'$. Moreover, $z \in Z' \wedge Z''$ since otherwise there would be $t \in Z'$ such that $t \leq z \leq z'$, what is impossible.

In order to see that $Z' \wedge Z''$ is a cross-section we consider $x \leq y$ such that $x \leq t$ and $u \leq y$ for some $t \in Z' \wedge Z''$ and $u \in Z' \wedge Z''$, where $t \in Z'$ and $u \in Z''$. Without a loss of generality we can assume that $y \leq y'$ for some $y' \in Z'$ since otherwise we could replace y by an element of Z'. Consequently, there exists $z \in Z''$ such that $x \leq z \leq y$. On the other hand, $z \in Z' \wedge Z''$ since otherwise there would be $z' \in Z'$ such that $z' \leq z \leq y$, what is impossible. In a similar manner we can find $z \in Z' \wedge Z''$ for the other cases of t and u.

In order to see that $Z' \wedge Z''$ is the greatest lower bound of Z' and Z'' consider a cross-section Y which precedes Z' and Z'' and observe that $y \leq z' \in Z'$ and $y \leq z'' \in Z''$ with z' and z'' not in $Z' \wedge Z''$ and $y \in Y$ implies the existence of $t \in Z'$ such that $y \leq t \leq z'$ or $u \in Z''$ such that $y \leq u \leq z''$.

Similarly, $Z' \vee Z''$ is a cross-section and the least upper bound of Z' and Z''.

The last part of the proposition is a consequence of the easily verifiable inequality $Z \wedge (Z' \vee Z'') \preceq (Z \wedge Z') \vee (Z \wedge Z'')$ ♯

A.5. Definition. For cross-sections Z' and Z'' of a poset $P = (X, \leq)$ such that $Z' \preceq Z''$ we define a *segment* of P from Z' to Z'' as the restriction of P to the set $[Z', Z''] = X^+(Z') \cap X^-(Z'')$, written as $P|[Z', Z'']$. A segment $P|[Y', Y'']$ such that $Z' \preceq Y' \preceq Y'' \preceq Z''$ is called a *subsegment* of $P|[Z', Z'']$. If $Z' \neq Y'$ or $Y'' \neq Z''$ (resp.: if $Z' = Y'$, or if $Y'' = Z''$) then we call it a *proper* (resp.: an *initial*, or a *final*) subsegment of $P|[Z', Z'']$. ♯

The following proposition follows easily from definitions.

A.6. Proposition. For every strong or weak cross-section Z of a poset $P = (X, \leq)$ the reflexive and transitive closure of the union of the restrictions of the partial order $\leq$ to $X^-(Z)$ and to $X^+(Z)$ is exactly the partial order $\leq$. $\sharp$

A.7. Proposition. A poset $P = (X, \leq)$ is said to be *locally complete* if every segment $P|[Z', Z'']$ of P is a complete lattice. $\sharp$

A.8. Definition. Given a partial order $\leq$ on a set X and a function $l : X \to W$ that assigns to every $x \in X$ a label $l(x)$ from a set W, we call $L = (X, \leq, l)$ a *labelled partially ordered set*, or briefly an *lposet*, by a *chain* (resp.: an *antichain*, a *cross-section*) of L we mean a chain (resp.: an antichain, a cross-section) of $P = (X, \leq)$, by a *segment* of L we mean each restriction of L to a segment of P, and we say that L is *K-dense* (resp.: *weakly K-dense*, *locally complete*) iff $\leq$ is K-dense (resp.: weakly K-dense, locally complete). $\sharp$

By **LPOSETS** we denote the category of lposets and their morphisms, where a *morphism* from an lposet $L = (X, \leq, l)$ to an lposet $L' = (X', \leq', l')$ is defined as a mapping $b : X \to X'$ such that, for all x and y, $x \leq y$ iff $b(x) \leq' b(y)$, and, for all x, $l(x) = l'(b(x))$. In the category **LPOSETS** a morphism from $L = (X, \leq, l)$ to $L' = (X', \leq', l')$ is an *isomorphism* iff it is bijective, and it is an *automorphism* iff it is bijective and $L = L'$. If there exists an isomorphism from an lposet L to an lposet L' then we say that L and $L'$are *isomorphic*. A *partially ordered multiset*, or briefly a *pomset*, is defined as an isomorphism class ξ of lposets. Each lposet that belongs to such a class ξ is called an *instance* of ξ. The pomset corresponding to an lposet L is written as $[L]$.

Appendix B: Directed complete posets

Let $(X, \sqsubseteq)$ be a partially ordered set (poset). A subset $Y \subseteq X$ is said to be *downward closed* (resp. : *upward closed*) if $Y = \sqsubseteq Y (= \{x \in X : x \sqsubseteq y \text{ for some } y \in Y\})$ (resp. : $Y = Y \sqsubseteq (= \{x \in X : y \sqsubseteq x \text{ for some } y \in Y\})$). A nonempty subset $Y \subseteq X$ is said to be em bounded complete if every bounded subset of Y has a least upper bound. A nonempty subset $Y \subseteq X$ is said to be *directed* if for all $x, y \in Y$ there exists $z \in Y$ such that $x, y \sqsubseteq z$. The *Scott topology* of $(X, \sqsubseteq)$ is the topology on X in which a subset $U \subseteq X$ is open iff it is upward closed and disjoint with every directed $Y \subseteq X$ which has the least upper bound $\sqcup Y$. A poset is said to be *coherent* if every of its consistent subsets has a least upper bound. A poset is said to be a *directed complete partial order (DCPO)* if every of its directed subsets has a least upper bound.

Let $(X, \sqsubseteq)$ be a DCPO. An element $x \in X$ is said to *approximate* an element $y \in X$, or that x is *way below* y, if in every directed set Z such that $y \sqsubseteq \sqcup Z$ there exists z such that $x \sqsubseteq z$. An element $x \in X$ is said to be a *compact* if it approximates itself. A subset $B \subseteq X$ is called a *basis* of $(X, \sqsubseteq)$ if for every $x \in X$ the set of those elements of B which approximate x is directed and has the least upper bound equal to x. The DCPO $(X, \sqsubseteq)$ is said to be *continuous* if it has a basis, and ω-*continuous* if it has a countable basis. The DCPO $(X, \sqsubseteq)$ is said to be an *algebraic domain* if every $y \in X$ is the directed least upper bound of all compact elements x such that $x \sqsubseteq y$.

Appendix C: Probability spaces

Given a set X, by a σ-*algebra* of subsets of X we mean a set $\mathcal{F}$ of subsets of X such that $X \in \mathcal{F}$ and $\mathcal{F}$ is closed under complements and countable unions, and we call the pair $(X, \mathcal{F})$ a *measurable space*. If X is given with a topology τ then the least σ-algebra that contains τ is called the *Borel* σ-*algebra* of the topological space (X, τ).

Given measurable spaces $(X, \mathcal{F})$ and $(X', \mathcal{F}')$, a mapping $f : X \to X'$ is said to be $\mathcal{F}$-*measurable*, or a morphism from $(X, \mathcal{F})$ t $(X', \mathcal{F}')$, iff $f^{-1}(F') \in \mathcal{F}$ for every $F' \in \mathcal{F}'$.

By **MES** we denote the category of measurable spaces and their morphisms.

By a *probability space* we mean a triple $(\Omega, \mathcal{F}, \mu)$, where Ω is a set (the set of possible realizations of a random phenomenon), $\mathcal{F}$ is a σ-algebra of subsets of Ω, and μ is a real valued function on $\mathcal{F}$, called a *probability measure*, such that $0 \leq \mu(F) \leq 1$ for all $F \in \mathcal{F}$, $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, and $\mu(F_0 \cup F_1 \cup ...) = \mu(F_0) + \mu(F_1) + ...$ for mutually disjoint F_0, F_1,... from $\mathcal{F}$.

Given two probability spaces $\mathbf{\Omega} = (\Omega, \mathcal{F}, \mu)$ and $\mathbf{\Omega}' = (\Omega', \mathcal{F}', \mu')$ by a morphism from $\mathbf{\Omega}$ to $\mathbf{\Omega}'$ we mean a triple $f : \mathbf{\Omega} \to \mathbf{\Omega}'$, where f is a mapping from Ω to Ω' such that $f^{-1}(F') \in \mathcal{F}$ and $\mu(f^{-1}(F')) = \mu'(F')$ for every $F' \in \mathcal{F}'$.

By **PSPACES** we denote the category of probability spaces and their morphisms.

Given a probability space $\mathbf{\Omega} = (\Omega, \mathcal{F}, \mu)$ and a σ-algebra $\mathcal{E} \subseteq \mathcal{F}$, there exists a function $f : \mathcal{F} \times \Omega \to [0, 1]$ such that, for every $F \in \mathcal{F}$, the function $\omega \mapsto f(F|\omega)$ ($= f(F, \omega)$), is $\mathcal{E}$-measurable and for all $E \in \mathcal{E}$ it satisfies the equation

$$\int_E f(F|\omega) d\mu(\omega) = \mu(F \cap E).$$

Function f is called a *conditional probability distribution* in $(\Omega, \mathcal{F})$ with respect to $\mathcal{E}$. If f is such that $F \mapsto f(F|\omega)$ is a probability measure on $\mathcal{F}$ for every $\omega \in \Omega$ then it is called a *strict conditional probability distribution* in $(\Omega, \mathcal{F})$ with respect to $\mathcal{E}$. Every function $\omega \mapsto f(F|\omega)$ is called a *variant* of *conditional probability* of F with respect to $\mathcal{E}$.

Appendix D: Partial categories

A partial category can be defined in exactly the same way as an arrows-only category in [McL 71], except that sources and targets may be not defined for some arrows that are not identities and then the respective compositions are not defined. Limits and colimits in partial categories can be defined as in usual categories.

Let $\mathbf{A} = (A, ;)$ be a partial algebra with a binary partial operation $(\alpha, \beta) \mapsto \alpha; \beta$, where $\alpha; \beta$ is written also as $\alpha\beta$. An element $\iota \in A$ is called an *identity* if $\iota\phi = \phi$ whenever $\iota\phi$ is defined and $\psi\iota = \psi$ whenever $\psi\iota$ is defined. We call elements of A *arrows* or *morphisms* and say that $\mathbf{A}$ is a *partial category* if the following conditions are satisfied:

(1) For every α, β, and γ in A, if $\alpha\beta$ and $\beta\gamma$ are defined then $\alpha(\beta\gamma)$ and $(\alpha\beta)\gamma$ are defined and $\alpha(\beta\gamma) = (\alpha\beta)\gamma$; if $\alpha(\beta\gamma)$ is defined then $\alpha\beta$ is defined; if $(\alpha\beta)\gamma$ is defined then $\beta\gamma$ is defined.
(2) For every identity $\iota \in A$, $\iota\iota$ is defined.

The conditions (1) and (2) imply the following properties.

(3) For every $\alpha \in A$, there exists at most one identity $\iota \in A$, called the *source* or the *domain* of α and written as $dom(\alpha)$, such that $\iota\alpha$ is defined, and at most one identity $\kappa \in A$, called the *target* or the *codomain* of α and written as $cod(\alpha)$, such that $\alpha\kappa$ is defined.
(4) For every α and β in A, $\alpha\beta$ is defined if and only if $cod(\alpha) = dom(\beta)$. If $\alpha\beta$ is defined then $dom(\alpha\beta) = dom(\alpha)$ and $cod(\alpha\beta) = cod(\beta)$.

For (3) suppose that ι_1 and ι_2 are identities such that $\iota_1\alpha$ and $\iota_2\alpha$ are defined. Then $\iota_2\alpha = \alpha$ and $\iota_1(\iota_2\alpha) = \iota_1\alpha$. Hence, by (1), $\iota_1\iota_2$ is defined and $\iota_1 = \iota_2$. Similarly for identities ι_1 and ι_2 such that $\alpha\iota_1$ and $\alpha\iota_2$ are defined.

For (4) suppose that $cod(\alpha) = dom(\beta) = \iota$. Then $\alpha\iota$ and $\iota\beta$ are defined and, by (1), $(\alpha\iota)\beta = \alpha\beta$ is defined. Conversely, if $\alpha\beta$ is defined then taking $\iota = cod(\alpha)$ we obtain that $\alpha\iota$ is defined and, consequently, $\alpha\beta = (\alpha\iota)\beta = \alpha(\iota\beta)$; the existence of $\iota\beta$ implies $dom(\beta) = \iota$. In a similar way we obtain $dom(\alpha\beta) = dom(\alpha)$ and $cod(\alpha\beta) = cod(\beta)$.

As usual, a morphism α with the source $dom(\alpha) = s$ and the target $cod(\alpha)$ is represented in the form $s \xrightarrow{\alpha} t$.

Note that $\alpha \mapsto dom(\alpha)$ and $\alpha \mapsto cod(\alpha)$ are definable partial operations assigning to a morphism α respectively the source and the target of this morphism, if such a source or a target exists.

Dealing with arrows-only categories rather than with categories in the usual sense is sometimes more convenient since it allows us to avoid two sorted structures and more complicated denotations.

Given a morphism α, a morphism β such that $\alpha = \gamma\beta\varepsilon$ is called a *segment* of α.

Given a partial category $\mathbf{A} = (A, ;)$, let A' be the set of quadruples $(\alpha, \sigma, \tau, \beta)$ where $\sigma\alpha\tau$ is defined and $\sigma\alpha\tau = \beta$, or $dom(\alpha)$ and σ are not defined and $\alpha\tau$ is defined and $\alpha\tau = \beta$, or $cod(\alpha)$ and τ are not defined and $\sigma\alpha$ is defined and $\sigma\alpha = \beta$, or $dom(\alpha)$ and $cod(\alpha)$ are not defined and $\alpha = \beta$. The set A' thus defined and the partial operation

$((\alpha, \sigma, \tau, \beta), (\beta, \sigma', \tau', \gamma)) \mapsto (\alpha, \sigma'\sigma, \tau\tau', \gamma)$

form a category $occ(\mathbf{A})$, called the *category of occurrences of morphisms in morphisms* in $\mathbf{A}$.

Given a partial category $\mathbf{A} = (A, ;)$ and its morphism α, let A'_α be the set of triples (ξ_1, δ, ξ_2) such that $\xi_1\delta\xi_2 = \alpha$.
The set A'_α thus defined and the partial operation

$((\eta_1, \delta, \varepsilon\eta_2), (\eta_1\delta, \varepsilon, \eta_2)) \mapsto (\eta_1, \delta\varepsilon, \eta_2)$

form a category dec_α, called the *category of decompositions* of α. In this category each triple (ξ_1, δ, ξ_2) in which δ is an identity, and thus $\delta = cod(\xi_1) = dom(\xi_2)$, is essentially a decomposition of α into a pair (ξ_1, ξ_2) such that $\xi_1\xi_2 = \alpha$ and it can be identified with this decomposition.

Given partial categories $\mathbf{A} = (A, ;)$ and $\mathbf{A}' = (A', ;')$, a mapping $f : A \to A'$ such that $f(\alpha);' f(\beta)$ is defined and $f(\alpha);' f(\beta) = f(\alpha\beta)$ for every α and β such that $\alpha\beta$ is defined, and $f(\iota)$ is an identity for every identity ι, is called a *morphism* or a *functor* from $\mathbf{A}$ to $\mathbf{A}'$. Note that such a morphism becomes a functor in the usual sense if $\mathbf{A}$ and $\mathbf{A}'$ are categories.

Diagrams, limits and colimits in partial categories can be defined as in usual categories.

A *direct system* is a diagram $(a_i \overset{\alpha_{ij}}{\to} a_j : i \leq j, i, j \in I)$, where $(I, \leq)$ is a directed poset, α_{ii} is identity for every $i \in I$, and $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ for all $i \leq j \leq k$. The *inductive limit* of such a system is its colimit, i.e. a family $(a_i \overset{\alpha_i}{\to} a : i, j \in I)$ such that $\alpha_i = \alpha_{ij}\alpha_j$ for all $i \in I$ and for every family $(a_i \overset{\beta_i}{\to} b : i, j \in I)$ such that $\beta_i = \alpha_{ij}\beta_j$ for all $i \in I$ there exists a unique $a \overset{\beta}{\to} b$ such that $\beta_i = \alpha_i\beta$ for all $i \in I$.

A *projective system* is a diagram $(a_i \overset{\alpha_{ij}}{\leftarrow} a_j : i \leq j, i, j \in I)$, where $(I, \leq)$ is a directed poset, α_{ii} is identity for every $i \in I$, and $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ for all $i \leq j \leq k$. The *projective limit* of such a system is its limit, i.e. a family $(a_i \overset{\alpha_i}{\leftarrow} a : i, j \in I)$ such that $\alpha_i = \alpha_j\alpha_{ij}$ for all $i \in I$ and for every family $(a_i \overset{\beta_i}{\leftarrow} b : i, j \in I)$ such that $\beta_i = \beta_j\alpha_{ij}$ for all $i \in I$ there exists a unique $a \overset{\beta}{\leftarrow} b$ such that $\beta_i = \beta\alpha_i$ for all $i \in I$.

A *bicartesian square* is a diagram $(v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\to} w, v \overset{\alpha'_2}{\to} u' \overset{\alpha'_1}{\leftarrow} w)$ such that $v \overset{\alpha'_2}{\to} u' \overset{\alpha'_1}{\leftarrow} w$ is a pushout of $v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\to} w$ and $v \overset{\alpha_1}{\leftarrow} u \overset{\alpha_2}{\to} w$ is a pullback of

$v \xrightarrow{\alpha_2'} u' \xleftarrow{\alpha_1'} w$, i.e. such that for every $v \xrightarrow{\beta_1} u'' \xleftarrow{\beta_2} w$ such that $\alpha_1\beta_1 = \alpha_2\beta_2$ there exists a unique $u' \xrightarrow{\beta} u''$ such that $\beta_1 = \alpha_2'\beta$ and $\beta_2 = \alpha_1'\beta$, and for every $v \xleftarrow{\gamma_1} t \xrightarrow{\gamma_2} w$ such that $\gamma_1\alpha_2' = \gamma_2\alpha_1'$ there exists a unique $u \xleftarrow{\gamma} t$ such that $\gamma_1 = \gamma\alpha_1$ and $\gamma_2 = \gamma\alpha_2$.

The concept of a bicartesian square can be generalized to the concept of a bicartesian n-cube. This can be done as follows.

Given a partial graph G, by a *n-cube* in G we mean a subgraph G' of G whose nodes correspond to sequences $(a_1, ..., a_n)$ of binary coordinates $a_i =$ 0 or 1, and whose arrows lead from one node to another whenever one of the coordinates of the latter is obtained from the corresponding coordinate of the former by replacing 0 by 1. The arrow with all coordinates 0 and the arrows leading from this node to other nodes are termed *initial*. The node with all coordinates 1 and the arrows leading to this node from other nodes are termed *final*. Subgraphs of G' whose all nodes have some of the coordinates identical are m-cubes for the respective $m \leq n$, called *m-faces* of G'.

As partial categories are also partial graphs, all these notions apply to partial categories as well. In particular, one can define a *bicartesian* n-cube in a partial category C as an n-cube C' in $\mathbf{A}$ that commutes and is such that, for each face C'' of C', the family of initial arrows of C'' extends to a unique limiting cone for the remaining part of C'', and the family of final arrows of C'' extends to a unique colimiting cone for the remaining part of C''. For example, each bicartesian square is a bicartesian 2-cube.

Appendix E: Structures

By structures we mean slightly modified versions of structures in the sense of Bourbaki's Elements (cf. [Bou 57] and [BuDe 68]). We define them as follows.

Let **Ens** and **BijEns** denote respectively the category of sets and mappings and the category of sets and bijective mappings. Let $\mathcal{P} : \mathbf{Ens} \to \mathbf{Ens}$ be the powerset functor, i.e. the fuctor such that $\mathcal{P}(X)$ is the set of subsets of X and $(\mathcal{P}(f))(Z) = f(Z)$ for every mapping $f : X \to X'$ and every $Z \subseteq X$. Let $\times : \mathbf{Ens} \times \mathbf{Ens} \to \mathbf{Ens}$ be the bifunctor of cartesian product, i.e. the functor such that $\times(X, Y)$ is the cartesian product $X \times Y$ of X and Y and $(\times(f, g))(x, y) = (f(x), g(y))$ for every mappings $f : X \to X'$, $g : Y \to Y'$ and every $(x, y) \in X \times Y$. For every set A, let A denotes the constant functor from **Ens** to **Ens**, i.e., the functor that assigns the set A to every set X and the identity of A to every mapping $f : X \to X'$.

E.1. Definition. By a *structure form* we mean a functor
$F : \mathbf{Ens} \to \mathbf{Ens}$. $\sharp$

E.2. Definition. Given a structure form $F : \mathbf{Ens} \to \mathbf{Ens}$, by a *structure* of this form on a set X we mean an element s of the set $F(X)$. $\sharp$

For example, a binary relation ρ on a set X is a structure on X of the form $brel : X \mapsto \mathcal{P}(X \times X)$, a graph with a set V of vertices (nodes), a set E of edges (arrows) such that $E \cap V = \emptyset$, a source function $s : E \to V$, and a target function $t : E \to V$, is a structure $G = (V, E, s, t)$ on the set $X = V \cup E$ of the form $graphs : X \mapsto \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X \times X) \times \mathcal{P}(X \times X)$, a topology τ on a set X is a structure of the form $top : X \mapsto \mathcal{P}(\mathcal{P}(x))$ on X, etc.

In [Bou 57] only structures of such forms have been considered that can be built from the identity functor and constant functors using the powerset functor $\mathcal{P} : \mathbf{Ens} \to \mathbf{Ens}$ and the bifunctor $\times : \mathbf{Ens} \times \mathbf{Ens} \to \mathbf{Ens}$ of cartesian product. However, there is no real need of such a restriction.

E.3. Definition. By a *structure type* we mean a pair $T = (B, mor)$ that consists of a functor $B : \mathbf{BijEns} \to \mathbf{BijEns}$ (a specification of *structure species*), and of a family *mor* of sets $mor(X, s, X', s')$ of mappings $f : X \to X'$ called *morphisms* (a specification of morphisms), where

(1) $s \in B(X)$ and $s' \in B(X')$,

(2) the superposition $fg : X \to X''$ of $f \in mor(X, s, X', s')$
and $g \in mor(X', s', X'', s'')$ belongs to $mor(X, s, X'', s'')$,
(3) if $f : X \to X'$ is a bijection such that $s' = B(f)(s)$ then
$f \in mor(X, s, X', s')$ and $f^{-1} \in mor(X', s', X, s)$.

We say that such a structure type is a structure type of structures of a form F : **Ens** $\to$ **Ens** if $B(f) = F(f)$ for every bijection $f : X \to X'$ and $B(X) \subseteq F(X)$ for every set X. $\sharp$

For example, the type of binary relations can be defined as the pair $BREL = (B_{BREL}, mor_{BREL})$, where B_{BREL} : **BijEns** $\to$ **BijEns** with $B_{BREL}(X)$ being the set of binary relations on X, and where mor_{BREL} specifies morphisms in $mor_{BREL}(X, s, X', s')$ as mappings $f : X \to X'$ such that $(x, y) \in s$ implies $(f(x), f(y)) \in s'$.

The type of acyclic binary relations can be defined as the pair $ABREL = (B_{ABREL}, mor_{ABREL})$, where B_{ABREL} : **BijEns** $\to$ **BijEns** with $B_{ABREL}(X)$ being the set of acyclic binary relations on X, and where mor_{ABREL} specifies morphisms in $mor_{ABREL}(X, s, X', s')$ as mappings $f : X \to X'$ such that $(x, y) \in s$ implies $(f(x), f(y)) \in s'$.

The type of partial orders can be defined as the pair $PO = (B_{PO}, mor_{PO})$, where B_{PO} : **BijEns** $\to$ **BijEns** with $B_{PO}(X)$ being the set of partial orders on X, and where mor_{PO} specifies morphisms as order preserving mappings.

The type of graphs can be defined as the pair $GRAPHS = (B_{GRAPHS}, mor_{GRAPHS})$, where B_{GRAPHS} : **BijEns** $\to$ **BijEns** with $B_{GRAPHS}(X)$ being the set of quadruples $G = (V, E, s, t)$ of the form *graphs* : $X \mapsto \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X \times X) \times \mathcal{P}(X \times X)$ such that V and E are disjoint subsets of X, $X = V \cup E$, $s : E \to V$, $t : E \to V$, and where mor_{GRAPHS} specifies morphisms $f : G = (V, E, s, t) \to G' = (V', E', s', t')$ as mappings $f : X = V \cup E \to X' = V' \cup E'$ such that $f(V) \subseteq V'$, $f(E) \subseteq E'$, $f(s(x)) = s'(f(x)$, $f(t(x)) = t'(f(x))$.

The type of topologies can be defined as the pair $TOP = (B_{TOP}, mor_{TOP})$, where B_{TOP} : **BijEns** $\to$ **BijEns** with $B_{TOP}(X)$ being the set of topologies on X, and where mor_{TOP} specifies morphisms as continuous mappings.

The type of *algebras* of a signature Σ can be defined as the pair $ALG(\Sigma) = (B_{ALG(\Sigma)}, mor_{ALG(\Sigma)})$, where $B_{ALG(\Sigma)}$: **BijEns** $\to$ **BijEns** with $B_{ALG(\Sigma)}(X)$ being the set of systems of operations (possibly partial) on X, each operation corresponding to an element of the signature Σ, and where $mor_{ALG(\Sigma)}$ specifies morphisms in $mor_{ALG(\Sigma)}(X, s, X', s')$ as *homomorphisms* from (X, s) to (X', s'), that is mappings $f : X \to X'$ such that, for every operation ω from s and for the corresponding operation ω' from s', the result $\omega'(f(x), f(y), ...)$ is defined and equal $f(\omega(x, y, ...))$ whenever $\omega(x, y, ...)$ is defined. A homomorphisms f from (X, s) to (X', s') is said to be *strong* if also $\omega(x, y, ...)$ is defined whenever $\omega'(f(x), f(y), ...)$ is defined. Each (X, s) such that $s \in B_{ALG(\Sigma)}(X)$ is called a *partial algebra* of type $ALG(\Sigma)$, and each partial algebra (X', s') of this type such that $X' \subseteq X$ and this inclusion is a homomorphism from (X', s') to (X, s) is called a *subalgebra* of (X, s). By a *congruence* (resp.: a *strong congruence*) in a partial algebra (X, s) we mean an equivalence in X such that the natural mapping that assigns to every element the equivalence class containing this element is a homomorphism (resp. a strong homomorphism).

For $\Sigma = \{s, t\}$ and $B_{ALG(\Sigma)}(X)$ defined as the set of pairs of operations $s : X \to X$ and $t : X \to X$ such that $s(s(x)) = t(s(x)) = s(x)$ and $s(t(x)) = t(t(x))$ for all $x \in X$, $ALG(\Sigma)$ is the type of structures which can be called *algebraic graphs*. Consequently, each (X, s, t) such that $(s, t) \in B_{ALG(\Sigma)}$ is an algebraic graph (partial if s and t are partial functions) with all elements $x \in X$ playing the role of edges and those elements $x \in X$ for which $s(x) = t(x) = x$ playing also the role of vertices.

For $\Sigma = \{+\}$ and $B_{ALG(\Sigma)}$ defined as the set of operations $+ : X \times X \to X$ such that $x+(y+z) = (x+y)+z$ whenever either side is defined, $x+y = y+x$ whenever either side is defined, and such that there exists a neutral element 0 such that $x+0$ is defined and $x+0 = x$ for all $x \in X$, is the type of structures which can be called *partial commutative monoids*. Consequently, each $(X, +)$ such that $+ \in B_{ALG(\Sigma)}$ is a partial commutative monoid.

In a similar way one can define the type $RELS(\Sigma)$ of relational structures of a signature Σ.

In general, structure types specify structures on sets and their morphisms.

E.4. Definition. Given a structure type $T = (B, mor)$, by a *structure* of this type on a set X we mean an element s of the set $B(X)$, and by a *morphism* from a set X with a structure $s \in B(X)$ to a set X' with a structure $s' \in B(X')$ we mean a mapping $f : X \to X'$ such that $f \in mor(X, s, X', s')$. $\sharp$

By **STRUCT**(T) we denote the category of sets provided with structures of type T and the respective morphisms.

Appendix F: Transition systems and Petri nets

Transition systems are models of systems which operate in discrete steps.

A *transition system* is a structure $T = (S, L, \mathit{Tran})$ where S is a set of *states*, L is a set of *labels*, and $\mathit{Tran} \subseteq S \times L \times S$ is the *transition relation*. Equivalently, it is a graph with nodes representing states of the system represented by T, and labelled arcs represnting transitions from a state to a state due to executing actions represented by labels.

Usually, transition systems are considered together with an *initial state* $i \in S$.

Petri nets are models of concurrent systems, that is systems whose parts may operate independently.

A *Petri net* (or briefly a *net*) is a triple $N = (S, T, F)$ that consists of two disjoint sets S and T (a set S of S-*elements* and a set T of T-*elements*) and of a binary relation $F \subseteq S \times T \cup T \times S$ (a *flow relation*). Equivalently, it is a directed bipartite digraph with two types of nodes (S-elements represented as circles and T-elements represented as boxes) and with arcs running from S-elements to T-elements or from T-elements to S-elements (represented by elements of the flow relation F). Depending on interpretation, it is called a Place/Transition net or a Condition/Event net.

In a Place/Transition net $N = (S, T, F)$ each S-element $s \in S$ represents a *place* which may contain a number of marks, called *tokens*. Any distribution $M : S \to \{0, 1, 2, ...\}$ of tokens over places represents a state of the system represented by N, called a *marking*. Each T-element $t \in T$ represents a transition which may *fire* at a marking M if $M(s) > 0$ for every $s \in S$ such that sFt. When t fires at M then a new marking M' is obtained where $M'(s) = M(s) - 1$ if $s \in \mathit{pre}(t) - \mathit{post}(t)$, $M'(s) = M(s) + 1$ if $s \in \mathit{post}(t) - \mathit{pre}(t)$, and $M'(s) = M(s)$ otherwise, for the sets $\mathit{pre}(t) = \{s \in S : sFt\}$ and $\mathit{post}(t) = \{s \in S : tFs\}$.

Usually, Place/Transition nets are considered together with an *initial marking* and then they are called *net systems*.

In a Condition/Event net, written as $N = (B, E, F)$ instead of $N = (S, T, F)$ and called an *elementary net* if B and E are finite, each $b \in B$ represents a *condition* which may hold in the system represented by N, each subset $c \subseteq B$, called a *case*, represents the set of those conditions which hold in a state of this system, and each element $e \in E$ represents an *event* which may occur in c if $\mathit{pre}(e) \subseteq c$ and $\mathit{post}(e) \cap c = \emptyset$ for the set $\mathit{pre}(e) = \{b \in B : bFe\}$ and the set $\mathit{post}(e) = \{b \in B : eFb\}$. Each element of B can also be regarded as a place which carry a token when the corresponding condition holds and is empty otherwise, and a case can be regarded as the marking containig one token in every place of this case and no token in every other place.

Also Condition/Event nets and elementary nets are considered together with an initial marking and then they are called respectively *Condition/Event systems* or *elementary net systems*.

The behaviour of a net system can be represented by an acyclic net $N = (B, E, F)$ in which every $e \in E$ represents a unique occurrence of a T-element of the net system, and every $b \in B$ represnts the presence of a token in a place represented by an S-element of net system as the result of a unique occurrence of a T-element. Such a net, whose elements can be labelled with the corresponding elements of the net system, is called an *occurrence net* (see [RT 86] for formal definitions). When reduced to the occurrences of T-elements and provided with the relation that relates every two different occurrences of T-elements with a common predecessor representing the presence of a token in a place becomes what is called an *event structure*.

References

[AES 00] Alvarez-Manilla, M., Edalat, A., Saheb-Djahromi, N., *An Extension Result for Continuous Valuations*, J. London math. Soc. (2) 61 (2000) 629-640

[BBM 02] Baldan, P., Bruni, R., Montanari, U., *Pre-nets, read arcs and unfolding: a functorial presentation*, Proceedings of WADT'02, Wirsing, M., Pattison, D., Hennicker, R., (Eds.), Springer LNCS 2755 (2002) 145-164

[Bedn 88] Bednarczyk, M. A., *Categories of Asynchronous Systems*, PhD thesis in Computer Science, University of Sussex, Report no. 1/88 (1988)

[BK 84] Bergstra, J., Klop, J., *The algebra of recursively defined processes and the algebra of regular processes*, in Paradaens, J., (Ed.), Proc. of 11th ICALP, Springer LNCS 172 (1984) 82-95

[BD 87] Best, E., Devillers, R., *Sequential and Concurrent Behaviour in Petri Net Theory*, Theoret. Comput. Sci. 55 (1987) 87-136

[Bou 57] Bourbaki, N., *Éléments de mathématique, Livre I (Théorie des ensembles), Chapitre 4 (Structures)*, Act. Sci. Ind. 1258, Hermann, Paris, 1957

[BuDe 68] Bucur, I., Deleanu, A., *Introduction to the Theory of Categories and Functors*, John Wiley and Sons Ltd., Lozanna, New York, Sydney, 1968

[Carn 58] Carnap, R., *Introduction to Symbolic Logic and Its Applications, Chapter G: ASs of physics*, Dover Publications, Inc., New York, 1958

[CMR 96] Corradini, A., Montanari, U., Rossi, F., Graph Processes, Fundamenta Informaticae 26 (1996), 241-265

[DMM 89] Degano, P., Meseguer, J., Montanari, U., *Axiomatizing Net Computations and Processes*, in Proc. of 4th LICS Symposium, IEEE (1989) 175-185

[DS 01] Droste, M., Shortt, R. M., *Continuous Petri Nets and Transition Systems*, in Ehrig, H., et al. (Eds.), Unifying Petri Nets, Springer LNCS 2128 (2001) 457-484

[ER 90] Ehrenfeucht, A., Rozenberg, G., *Partial 2-structures*, Acta Informatica 27 (1990) 315-368

[EK 76] Ehrig, H., Kreowski, H. -J., *Parallelism of Manipulations in Multidimensional Information Structures*, in A. Mazurkiewicz (Ed.): Proc. of MFCS'76, Springer LNCS 45 (1976) 284-293

[Eng 91] Engelfriet, J., *Branching Processes of Petri Nets*, Acta Informatica 28 (1991) 575-591

[F 66] Feller, W., *An Introduction to Probability Theory and Its Applications, Vol. II*, John Wiley and Sons, Inc., 1966

[GHK 80] Gierz, G., Hofmann, k., H., Keimel, K., Lawson, J.,D., Mislove, M., and Scott, D.,S., *A compendium of continuous lattices*, Springer, Berlin, 1980

[GP 95] Glabbeek, R., J., van, Plotkin, G., D., *Configuration Structures*, Proceedings of LICS'95, Kozen, D., (Ed.), IEEE Computer Society Press (1995) 199-209

[HR 91] Hoogeboom, H. J., Rozenberg, G., *Diamond Properties of Elementary Net Systems*, Fundamenta Informaticae 14 (1991) 287-300

[HSP 83] Hart, S., Sharir, M., Pnueli, A., *Termination of Probabilistic Concurrent Programs*, ACM Trans. on Programming Languages and Systems, Vol. 5, No. 3, July 1083, 356-380

[JP 89] Jones, C., Plotkin, G. D., *A probabilistic powerdomain of evaluations*, Proceedings of 4th LICS, 1989, 186-195

[Kw 03] Kwiatkowska, M., *Model checking for probability and time: from theory to practice*, Proc. of 18th IEEE Symposium on Logic in Computer Science (LICS'03), IEEE Computer Society Press (2003), 351-360

[LSV 07] Lynch, N., Segala, R., Vaandrager, F., *Observing Branching Structure Through Probabilistic Contexts*, Siam Journal on Computing 37 (4), 977-1013, September 2007

[McL 71] Mac Lane, S., *Categories for the Working Mathematician*, Springer-Verlag New York Heidelberg Berlin 1971

[Maz 88] Mazurkiewicz, A., *Basic Notions of Trace Theory*, in J. W. de Bakker, W. P. de Roever and G. Rozenberg (Eds.): Linear Time, Branching Time and Partial Order in Logics and Models for Concurrency, Springer LNCS 354 (1988) 285-363

[MMS 96] Meseguer, J., Montanari, U., Sassone, V., *Process versus Unfolding semantics for Place/Transition Nets*, Theoret. Comput. Sci. 153, n. 1-2 (1996) 171-210

[Mey 66] Meyer, P. A., *Probability and Potentials*, Blaisdell Publishing Company, Waltham, Massachusetts, Toronto, London (1966)

[Miln 78] Milner, R., *Synthesis of Communicating Behaviour*, Proc. of MFCS'78, Winkowski, J. (Ed.), Springer LNCS 64 (1978) 71-83 (1980)

[Miln 80] Milner, R., *A Calculus of Communicating Systems*, Springer LNCS 92 (1980)

[Miln 96] Milner, R., *Calculi of interaction*, Acta Informatica 33 (1996) 707-737

[ML 07] Mitra, S., Lynch, N., *Trace-based Semantics for Probabilistic Timed I/O Automata*, Hybrid Systems: Computation and Control (HSCC 2007), Pisa, Italy, April 3-5, 2007, Springer LNCS 4416,

[MR 95] Montanari, U., Rossi, F., *Contextual Nets*, Acta Informatica 32 (1995) 545-596

[NK 93] Nerode, A., Kohn, W., *Models for Hybrid Systems: Automata, Topologies, Controllability, Observability*, Springer LNCS 736 (1993) 317-356

[NRT 90] Nielsen, M., Rozenberg, G., Thiagarajan, P. S., *Elementary Transition Systems*, Theoretical Computer Science (1992) 3-33

[Par 80] Parthasarathy, K. R., *Introduction to Probability and Measure*, New Delhi (1980)

[Petri 62] Petri, C. A., *Kommunikation mit Automaten*, PhD thesis, Institut fuer Instrumentelle Mathematik, Bonn, Germany (1962)

[Petri 77] Petri, C., A., *Non-Sequential Processes*, Interner Bericht ISF-77-5, Gesellschaft fuer Mathematik und Datenverarbeitung, 5205 St. Augustin, Germany (1977)

[Petri 80] Petri, C. A., *Introduction to General Net Theory*, in W. Brauer (Ed.): Net Theory and Applications, Springer LNCS 84 (1980) 1-19

[Plue 85] Pluenecke, H., *K-density, N-density and finiteness properties*, APN 84, Springer LNCS 188 (1985) 392-412

[Pn 86] Pnueli, A., *Applications of temporal logic to the specification and verification of reactive systems: a survey of current trends*, in: J. W. de Bakker, W.-P. de Roever and G. Rozenberg, eds., Lecture Notes in Comp. Sc. 224, Springer, Berlin, 1986, 510-584

[Re 85] Reisig, W., *Petri Nets: An Introduction*, Springer-Verlag (1985)

[RT 86] Rozenberg, G., Thiagarajan, P. S., *Petri Nets: Basic Notions, Structure, Behaviour*,
in J. W. de Bakker, W. P. de Roever and G. Rozenberg (Eds.): Current Trends in Concurrency, Springer LNCS 224 (1986) 585-668

[Sh 85] Shields, E. W., *Concurrent Machines*, Computer Journal, vol. 28 (1985) 449-465

[VVW 04] Varacca, D., Völzer, H., Winskel, G., *Probabilistic Event Structures and Domains*, in P. Gardner and N. Yoshida (eds.), CONCUR 2004, Springer LNCS 3170 (2004), 497-511

[Wink 80] Winkowski, J., *Behaviours of Concurrent Systems*, Theoret. Comput. Sci. 12 (1980) 39-60

[Wink 82] Winkowski, J., *An Algebraic Description of System Behaviours*, Theoret. Comput. Sci. 21 (1982) 315-340

[Wink 03] Winkowski, J., *An Algebraic Characterization of Independence of Petri Net Processes*, Information Processing Letters 88 (2003), 73-81

[Wink 05] Winkowski, J., *Towards a Framework for Modelling Systems with Rich Structures of States and Processes*, Fundamenta Informaticae 68 (2005), 175-206, http://www.ipipan.waw.pl/∼wink/winkowski.htm

[Wink 06] Winkowski, J., *An Axiomatic Characterization of Algebras of Processes of Petri Nets*, Fundamenta Informaticae 72 (2006), 407-420, http://www.ipipan.waw.pl/∼wink/winkowski.htm

[Wink 07a] Winkowski, J., *Behaviour Algebras*, Fundamenta Informaticae 75 (2007), 537-560 http://www.ipipan.waw.pl/∼wink/winkowski.htm

[Wink 07b] Winkowski, J., *Towards a Framework for Modelling Behaviours of Hybrid Systems*, Fundamenta Informaticae 80 (2007), 311-332, http://www.ipipan.waw.pl/∼wink/winkowski.htm

[Wink 08] Winkowski, J., *An Algebraic Framework for Defining Random Concurrent Behaviours*, Fundamenta Informaticae 85 (2008), 481-496

[Wink 09a] Winkowski, J., *An Algebraic Framework for Defining Behaviours of Concurrent Systems. Part 1: The Constructive Presentation*, Fundamenta Informaticae 97 (2009), 235-273

[Wink 09b] Winkowski, J., *An Algebraic Framework for Defining Behaviours of Concurrent Systems. Part 2: The Axiomatic Presentation*, Fundamenta Informaticae 97 (2009), 439-470

[Wink 11] Winkowski, J., *Multiplicative Transition Systems*, Fundamenta Informaticae 109, No 2 (2011), 201-222, http://www.ipipan.waw.pl/~wink/winkowski.htm

[WiMa 87] Winkowski, J., Maggiolo-Schettini, A., *An Algebra of Processes*, Journal of Comp. and System Sciences, Vol. 35, No. 2, October 1987, 206-228

[WN 95] Winskel, G., Nielsen, M., *Models for Concurrency*, in S. Abramsky, Dov M. Gabbay and T. S. E. Maibaum (Eds.): Handbook of Logic in Computer Science 4 (1995), 1-148

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