Concatenable weighted pomsets and their applications to modelling processes of Petri nets *

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Abstract. Structures called concatenable weighted pomsets are introduced which can serve as models of processes of Petri nets, including nets with time features. Operations on such structures are defined which allow to combine them sequentially and in parallel. These operations correspond to natural operations on processes. They make the universe of concatenable weighted pomsets a partial algebra which appears to be a symmetric strict monoidal category. Sets of processes of timed and time Petri nets are characterized as subsets of this algebra.

Keywords: concatenable weighted pomset, symmetry, table, sequential composition, parallel composition, interchange, algebra, timed Petri net, time Petri net, process, potential timed process, actual timed process.

1. Motivation

The idea of representing behaviours of concurrent systems with the aid of partial orders has appeared to be fruitful. From one side, it has allowed to develop an adequate theory of Petri nets (cf. [P 77], [Maz 77], [Wi 80], [Wi 82], [GR 86], [DMM 89], for example). From the other side, it has allowed to reduce dramatically the computational complexity of practical analysis of concurrent systems (cf. [GW 91], [GK 91], [Pe 93], [PP 95], for example).

The core of this idea is that all possible processes of a concurrent system are represented as partially ordered sets of executions of actions, or occurrences of state components, or both, where the lack of order between elements represents their causal independence. In [DMM 89] such sets are equipped with some extra arrangements of their minimal and maximal elements, which allows to define a sort of concatenation. However, with processes thus represented only those features of concurrent systems can be reflected which can be expressed in terms of causality and choice.

In this paper we enrich the existing partial order based models of processes by equipping them with features called weights. The latter are quantities assigned to pairs of elements in order to describe how the respective elements are related and to quantify in a sense the

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degree of relationship. In particular, they may represent delays with which elements follow their causal predecessors. Consequently, with processes represented with weights not only causality and branching but also the flow of time can be reflected.

The paper exploits some ideas of [Wi 80] and [Wi 92], and [Wi 94].

2. The concept

We are concerned in partially ordered multisets (pomsets in the terminology of [Pra 86]) with some extra features (arrangements of minimal and maximal elements similar to those in concatenable processes of [DMM 89], and weights). The respective structures, called concatenable weighted pomsets, can be defined as follows.

Given a partially ordered set (poset) $\mathcal{X} = (X, \leq)$, define a *cut* of $\mathcal{X}$ as a maximal antichain which has an element in each maximal chain, denote by X_{min} the set of minimal elements of $\mathcal{X}$ and by X_{max} the set of maximal elements of $\mathcal{X}$, for $Y \subseteq X$ denote by $\leq Y$ the set of $x \in X$ such that $x \leq y$ for some $y \in Y$ and by $Y \leq$ the set of $x \in X$ such that $y \leq x$ for some $y \in Y$, and for $x \leq y$ denote by $[x, y]$ the subposet of $\mathcal{X}$ that consists of all $z \in X$ such that $x \leq z \leq y$. Define a K-*dense* poset as a poset in which each maximal antichain is a cut. Denote by R the semiring of real numbers and infinities $-\infty$, $+\infty$ with the operation $(x, y) \mapsto max(x, y)$ playing the role of addition and the operation $(x, y) \mapsto x + y$, where $(-\infty) + (+\infty)$ is defined as $-\infty$, playing the role of multiplication. In this semiring $-\infty$ plays the role of zero and 0 plays the role of unit.

Let V be a set of labels. Let W be a semiring with addition $(x, y) \mapsto x + y$, multiplication $(x, y) \mapsto xy$, zero $\perp$, and unit ε.

2.1. Definition. A *concatenable weighted pomset* (or a *cw-pomset*) over V and W is an isomorphism class α of structures $\mathcal{A} = (X, \leq, d, e, s, t)$, where:

(1) $(X, \leq)$ is a finite *underlying poset*,
(2) $d : X \times X \rightarrow W$ is a *weight function* such that $d(x, y) = \perp$ whenever $x \leq y$ does not hold, $d(x, x) = \varepsilon$, and $d(x, y) = \Sigma(d(x, z)d(z, y) : z \in Z)$ for each cut Z of $[x, y]$ whenever $x \leq y$,
(3) $e : X \rightarrow V$ is a *labelling function*,
(4) $s = (s(v) : v \in V)$ is an *arrangement of minimal elements*, where each $s(v)$ is an enumeration of the set of minimal elements with the label v,
(5) $t = (t(v) : v \in V)$ is a *arrangement of maximal elements*, where each $t(v)$ is an enumeration of the set of maximal elements with the label v.

Each such a structure is called an *instance* of α, we write α as $[\mathcal{A}]$, and we use subscripts, $X_{\mathcal{A}}$, $\leq_{\mathcal{A}}$, $d_{\mathcal{A}}$, $e_{\mathcal{A}}$, $s_{\mathcal{A}}$, $t_{\mathcal{A}}$, when necessary. □

In this definition by an enumeration of a set we mean a sequence of elements of this set in which each element occurs exactly once, and by an isomorphism from $\mathcal{A}$ to $\mathcal{A}' = (X', \leq', d', e', s', t')$ we mean a bijection $b : X \rightarrow X'$ such that $x \leq y$ iff $b(x) \leq' b(y)$, $d'(b(x), b(y)) = d(x, y)$, $e'(b(x)) = e(x)$, $s'(v) = b(s(v))$, and $t'(v) = b(t(v))$, for all $x, y \in X$ and $v \in V$, where $b(x_1 ... x_n)$ denotes $b(x_1)...b(x_n)$. The condition of finiteness of the underlying poset is imposed in order to avoid technical problems with infinite partial orders which are of no use in the applications considered in this paper. The last condition imposed on the weight function is a generalization of a condition which says that, in the case of a cw-pomset with a K-dense underlying poset and with weights from the semiring R of real numbers and infinities, the weight $d(x, y)$ for each pair (x, y) such that $x \leq y$ is the maximum of sums of weights along maximal chains from x to y. The arrangements of minimal and maximal elements are needed for equipping minimal and maximal elements with identifiers which do not depend on conrete instances, where the identifier of an element x consists of the label

$e(x)$ and of the number indicating the position of this element in the respective sequence $s(e(x))$ or $t(e(x))$. Such identifiers allow to concatenate cw-pomsets by identifying maximal elements of one cw-pomset with minimal elements of another.

If the underlying poset $(X, \leq)$ of $\mathcal{A}$ is K-dense then also $\mathcal{A}$ and α are said to be K-dense. If $X = X_{min} \cup X_{max}$ then we call α a *slice*. If $X = X_{min} = X_{max}$, and thus the order $\leq$ reduces to the identity, then we call α a *symmetry*. If also $t = s$ then α becomes what we call an *identity*, and it can be identified with a multiset $ms(\alpha)$ of labels, namely with the multiset in which the multiplicity of each $v \in V$ is given by the cardinality of $e^{-1}(v) \cap X$. By $cwp(V,W)$, $dcwp(V,W)$, $sym(V,W)$, and $id(V,W)$, we denote respectively the set of cw-pomsets, the set of K-dense cw-pomsets, the set of symmetries and the set of identities over V and W.

Examples of cw-pomsets are shown in figures 2.1 and 2.2. In these examples A, B, C, D are labels and φ, ψ, σ, τ denote elements of the respective semiring. The arrangements of minimal elements and the arrangements of maximal elements are represented by endowing the labels of minimal elements with subscripts and the labels of maximal elements with superscripts, where each subscript (resp.: superscript) denotes the position of the corresponding element in the respective enumeration.

All the cw-pomsets in figure 2.1 are K-dense. The cw-pomset α' is a slice. It is obtained from α by ignoring elements which are neither minimal nor maximal. The cw-pomset β is both a slice and a symmetry. The cw-pomset γ in figure 2.2 is not K-dense since the occurrences of A_1 and D^2 in its instance constitute a maximal antichain which is not a cut.

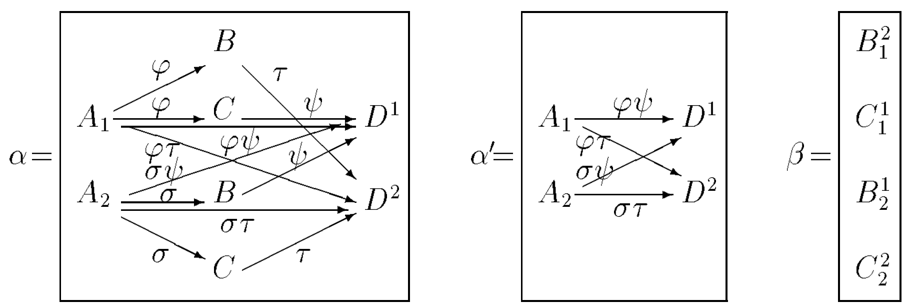

Figure 2.1

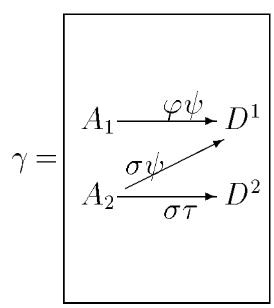

Figure 2.2

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of a cw-pomset. By a *cut* of $\mathcal{A}$ we mean a cut of the underlying poset $(X, \leq)$ and by $cuts(\mathcal{A})$ we denote the set of cuts of $\mathcal{A}$. Given $Y, Y' \in cuts(\mathcal{A})$, we write $Y \sqsubseteq Y'$ if $(\leq Y) \subseteq (\leq Y')$.

2.2. Proposition.; If $\mathcal{A}$ is K-dense then the relation $\sqsubseteq$ is a partial order on the set $cuts(\mathcal{A})$ such that $cuts(\mathcal{A})$ with this order is a lattice. $\square$

Proof: For $Y, Y' \in cuts(\mathcal{A})$ we define $Y \sqcup Y'$ as the set of all $x \in X$ of the form $max(TY, TY')$, where T is a maximal chain and TY, TY' denote the unique elements of this chain in Y and in Y', respectively. Then $Y \sqcup Y'$ cannot contain two different x, y such that $x \leq y$ (since each maximal chain containing x and y may have at most one member in $Y \sqcup Y'$) and each $x \in X$ must be comparable with $max(TY, TY') \in Y \sqcup Y'$ for each maximal chain T which contains x. Thus $Y \sqcup Y'$ is a maximal antichain. From the definition it follows that $Y \sqcup Y'$ has an element in each maximal chain and thus it is a cut. It is also obvious that $Y \sqcup Y'$ is the least upper bound of Y and Y', as required. Similarly for the greatest lower bounds. $\square$

2.3. Proposition. If $\mathcal{A}$ is K-dense then for each $Y \in cuts(\mathcal{A})$ the order $\leq$ is the transitive closure of the union of its restrictions to the subsets $\leq Y$ and $Y \leq$. $\square$

Proof: Let $\leq_1$ and $\leq_2$ be the restrictions of $\leq$ to $\leq Y$ and $Y \leq$, respectively. The fact that $\leq$ contains $(\leq_1 \cup \leq_2)^*$, the reflexive and transitive closure of $\leq_1 \cup \leq_2$, is immediate. Conversely, if $x \leq y$ then $x \leq_1 y$ whenever $x, y \in \leq Y$, $x \leq_2 y$ whenever $x, y \in Y \leq$, and $x \leq_1 z \leq_2 y$ with some $z \in Y$ whenever $x \in \leq Y$ and $y \in Y \leq$ since then there exists a maximal chain which contains x and y and this chain has an element z in Y. $\square$

2.4. Proposition. If $\mathcal{A}$ is K-dense then for each $Z \in cuts(\mathcal{A})$, each $x \in (\leq Z)$, and each $y \in (Z \leq)$ such that $x \leq y$, we have

$$d(x, y) = \Sigma(d(x, z)d(z, y) : z \in Z). \ \square$$

A proof follows immediately from the conditions in (2) of 2.1.

2.5. Proposition. If $\mathcal{A}$ is K-dense then the weight function d is determined uniquely by its values on the pairs of elements of which the first is an immediate predecessor of the second. $\square$

Proof: The proposition is trivially true if the underlying poset has not more than one element. Suppose that X has at least 2 elements.

Suppose that d and d' are two weight functions which have the same values on the pairs consisting of an element and its immediate successor, and that $d(x, y) \neq d'(x, y)$. Let Z be the set of maximal elements of the set $[x, y] - \{y\}$. Due to the properties of weight functions there must be $z \in Z$ such that $d(x, z) \neq d'(x, z)$. Similarly, in the set $[x, z] - \{z\}$ there must be a maximal element z' such that $d(x, z') \neq d'(x, z')$, etc. Thus, finally, there exists an immediate successor u of x such that $d(x, u) \neq d'(x, u)$, which contradicts to our assumption. $\square$

3. Operations

The set $cwp(V, W)$ of cw-pomsets can be made an algebra by equipping it with suitable operations. In this paper we consider operations of taking sources and targets of cw-pomsets, operations of composing cw-pomsets sequentially and in parallel, and so called interchanges (the latter similar to those in [DMM 89]).

We start with the simple observation that for each cw-pomset α and each instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of this cw-pomset, the restriction of $\mathcal{A}$ to X_{min} with t replaced by s and that to X_{max} with s replaced by t are instances of cw-pomsets. These cw-pomsets are identities. We write them respectively as $\partial_0(\alpha)$ and $\partial_1(\alpha)$ and call them respectively the *source* and the *target* of α.

The sequential composition of cw-pomsets can be defined by specifying how the result of composing two cw-pomsets is related to these cw-pomsets if it exists, and by showing that the respective relation defines a partial binary operation on cw-pomsets.

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of a cw-pomset.

The following proposition is a simple consequence of definitions.

3.1. Proposition. For each $Y \in cuts(\mathcal{A})$, and each arrangement of Y into a family $r = (r(v) : v \in V)$ of enumerations of the sets $e^{-1}(v) \cap Y$, the restriction of $\mathcal{A}$ to $\leq Y$ with r playing the role of arrangement of maximal elements, and that to $Y \leq$ with r playing the role of arrangement of minimal elements, are instances of cw-pomsets. We write them $head_{Y,r}(\mathcal{A})$ and $tail_{Y,r}(\mathcal{A})$, respectively. □

The cw-pomset $[\mathcal{A}]$ is said to *consist* of the cw-pomset $[head_{Y,r}(\mathcal{A})]$ *followed* by the cw-pomset $[tail_{Y,r}(\mathcal{A})]$.

Note that each cw-pomset α can be represented in the form $[head_{X_{max},t}(\mathcal{A})]$ and in the form $[tail_{X_{min},s}(\mathcal{A})]$, where $\mathcal{A} = (X, \leq, d, e, s, t)$ is any instance of α.

3.2. Proposition. For every two cw-pomsets α and β with $\partial_0(\beta) = \partial_1(\alpha)$ there exists a unique cw-pomset $\alpha;\beta$ which consists of α followed by β. This cw-pomset is K-dense whenever α and β are K-dense. It is a symmetry whenever α and β are symmetries. □

Proof: As $\partial_0(\beta) = \partial_1(\alpha)$ and instances of α and β may be chosen arbitrarily up to isomorphism, we may choose an instance $\mathcal{A} = (X_\mathcal{A}, \leq_\mathcal{A}, d_\mathcal{A}, e_\mathcal{A}, s_\mathcal{A}, t_\mathcal{A})$ of α and an instance $\mathcal{B} = (X_\mathcal{B}, \leq_\mathcal{B}, d_\mathcal{B}, e_\mathcal{B}, s_\mathcal{B}, t_\mathcal{B})$ of β such that $(t_\mathcal{A}(v))(i) = (s_\mathcal{B}(v))(i)$ for all v and i for which either side is defined and such that these are the only common elements of $X_\mathcal{A}$ and $X_\mathcal{B}$. Then we denote $X_\mathcal{A} \cap X_\mathcal{B}$ by U and define

$$X_\mathcal{C} = X_\mathcal{A} \cup X_\mathcal{B}$$

$$x \leq_\mathcal{C} y \text{ whenever } x \leq_\mathcal{A} y \text{ or } x \leq_\mathcal{B} y \text{ or } x \leq_\mathcal{A} z \leq_\mathcal{B} y \text{ for some } z \in U$$

$$d_\mathcal{C}(x,y) = \begin{cases} d_\mathcal{A}(x,y) & \text{for } x, y \in X_\mathcal{A} \\ d_\mathcal{B}(x,y) & \text{for } x, y \in X_\mathcal{B} \\ \Sigma(d_\mathcal{A}(x,u)d_\mathcal{B}(u,y) : u \in U) & \text{for } x \in X_\mathcal{A}, y \in X_\mathcal{B} \\ \perp & \text{for the remaining } x, y \in X_\mathcal{C} \end{cases}$$

$$e_\mathcal{C}(x) = \begin{cases} e_\mathcal{A}(x) \text{ for } x \in X_\mathcal{A} \\ e_\mathcal{B}(x) \text{ for } x \in X_\mathcal{B} \end{cases}$$

$$s_\mathcal{C} = s_\mathcal{A} \text{ and } t_\mathcal{C} = t_\mathcal{B}.$$

In order to prove that $\mathcal{C} = (X_\mathcal{C}, \leq_\mathcal{C}, d_\mathcal{C}, e_\mathcal{C}, s_\mathcal{C}, t_\mathcal{C})$ is an instance of a cw-pomset it suffices to consider $x \in X_\mathcal{A}$ and $y \in X_\mathcal{B}$ such that $x \leq_\mathcal{C} y$, to take a cut Z of $[x, y]$, and to show that $d_\mathcal{C}(x,y) = \Sigma(d_\mathcal{A}(x,z)d_\mathcal{B}(z,y) : z \in Z)$. To this end we exploit the fact that $U \cap [x,y]$ is a cut of $[x,y]$, define U_1 as the set of $u \in U \cap [x,y]$ such that $z \leq_\mathcal{C} u$ for some $z \in Z$ such that $z \neq u$, U_2 as the set of $u \in U \cap [x,y]$ such that $u \leq_\mathcal{C} z$ for some $z \in Z$, Z_1 as the set of $z \in Z$ such that $z \leq u$ for some $u \in U$ such that $u \neq z$, Z_2 as the set of $z \in Z$ such that $u \leq z$ for some $u \in U$, and make use of the following equalities:

$$d_\mathcal{C}(x,y) = \Sigma(d_\mathcal{A}(x,u)d_\mathcal{B}(u,y) : u \in U \cap [x,y])$$

$$= \Sigma(d_\mathcal{A}(x,u_1)d_\mathcal{B}(u_1,y) : u_1 \in U_1)\Sigma(d_\mathcal{A}(x,u_2)d_\mathcal{B}(u_2,y) : u_2 \in U_2)$$

$$= \Sigma(\Sigma(d_\mathcal{A}(x,z_1)d_\mathcal{A}(z_1,u_1) : z_1 \in Z_1)d_\mathcal{B}(u_1,y) : u_1 \in U_1)$$

$$\Sigma(d_\mathcal{A}(x,u_2)\Sigma(d_\mathcal{B}(u_2,z_2)d_\mathcal{B}(z_2,y) : z_2 \in Z_2) : u_2 \in U_2)$$

$$= \Sigma(d_{\mathcal{A}}(x, z_1)d_{\mathcal{A}}(z_1, u_1)d_{\mathcal{B}}(u_1, y) : z_1 \in Z_1, u_1 \in U_1)$$
$$\Sigma(d_{\mathcal{A}}(x, u_2)d_{\mathcal{B}}(u_2, z_2)d_{\mathcal{B}}(z_2, y) : z_2 \in Z_2), u_2 \in U_2)$$
$$= \Sigma(d_{\mathcal{A}}(x, z_1)\Sigma(d_{\mathcal{A}}(z_1, u_1)d_{\mathcal{B}}(u_1, y) : u_1 \in U_1) : z_1 \in Z_1)$$
$$\Sigma(\Sigma(d_{\mathcal{A}}(x, u_2)d_{\mathcal{B}}(u_2, z_2) : u_2 \in U_2)d_{\mathcal{B}}(z_2, y) : z_2 \in Z_2)$$
$$= \Sigma(d_{\mathcal{C}}(x, z_1)d_{\mathcal{C}}(z_1, y) : z_1 \in Z_1)\Sigma(d_{\mathcal{C}}(x, z_2)d_{\mathcal{C}}(z_2, y) : z_2 \in Z_2)$$
$$= \Sigma(d_{\mathcal{C}}(x, z)d_{\mathcal{C}}(z, y) : z \in Z).$$

In order to prove that $\mathcal{C}$ is an instance of $\alpha;\beta$ it suffices to note that U is a cut of $\mathcal{C}$ and apply 2.3 and 2.4.

In order to prove that $\mathcal{C}$ is K-dense if $\mathcal{A}$ and $\mathcal{B}$ are K-dense we have to prove that in this case $Z \cap T$ is nonempty for each maximal antichain Z and each maximal chain T. To this end we prove first that $P = (Z - X_{\mathcal{B}}) \cup ((\leq Z) \cap U)$ and $Q = (Z - X_{\mathcal{A}}) \cup ((Z \leq) \cap U)$ are maximal antichains.

It is clear that P is an antichain. Suppose that P is not a maximal antichain. Then there exists x, say in $X_{\mathcal{A}}$, which is incomparable with the elements of P. For such x there exists $z \in Z$ which is comparable with x and such z must belong to $Z - X_{\mathcal{A}}$. By the definition of $\leq_{\mathcal{C}}$ there exists $u \in U$ such that $x \leq_{\mathcal{C}} u \leq_{\mathcal{C}} z$ and it must belong to $(\leq Z) \cap U$ since otherwise it would be an element of $(Z \leq) \cap U$ and z would be comparable with an element of $(\leq Z) \cap U$. Consequently, x is comparable with an element of $(\leq Z) \cap U$, which contradicts to our assumption. For similar reasons we cannot have any $x \in X_{\mathcal{B}}$ which would be incomparable with the elements of P. Thus P is a maximal antichain. Similarly, Q is a maximal antichain.

Now, $T \cap X_{\mathcal{A}}$ is a maximal chain of $\mathcal{A}$ and $T \cap X_{\mathcal{B}}$ is a maximal chain of $\mathcal{B}$. Thus $T \cap X_{\mathcal{A}} \cap P \neq \emptyset$ and $T \cap X_{\mathcal{B}} \cap Q \neq \emptyset$. Let $T \cap X_{\mathcal{A}} \cap P \neq \emptyset$. If $(T \cap X_{\mathcal{A}}) \cap (Z - X_{\mathcal{B}})$ is empty then $(T \cap X_{\mathcal{A}}) \cap ((\leq Z) \cap U)$ is nonempty and hence $(T \cap X_{\mathcal{A}}) \cap Z \neq \emptyset$ or $(T \cap X_{\mathcal{A}}) \cap (Z \leq) = \emptyset$. In the first case we have $T \cap Z \neq \emptyset$. In the second case we have $(T \cap X_{\mathcal{B}}) \cap Q = (T \cap X_{\mathcal{B}}) \cap ((Z - X_{\mathcal{A}}) \cup ((Z \leq) \cap U))$ with $(T \cap X_{\mathcal{B}}) \cap ((Z \leq) \cap U) = \emptyset$, so that $(T \cap X_{\mathcal{B}}) \cap (Z - X_{\mathcal{A}}) \neq \emptyset$, i.e. $T \cap Z \neq \emptyset$, as required. Similarly for $T \cap X_{\mathcal{B}} \cap Q \neq \emptyset$. Thus $\mathcal{C}$ is K-dense.

Finally, it is obvious that $\alpha;\beta$ is a symmetry if α and β are symmetries. This ends the proof. $\square$

The operation $(\alpha, \beta) \mapsto \alpha;\beta$ is called the *sequential composition* of cw-pomsets. Examples of application of this operation are shown in figures 3.1 and 3.2.

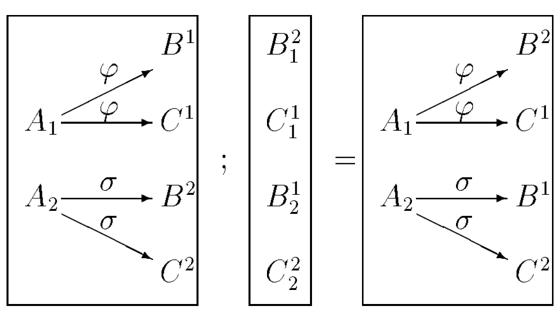

Figure 3.1

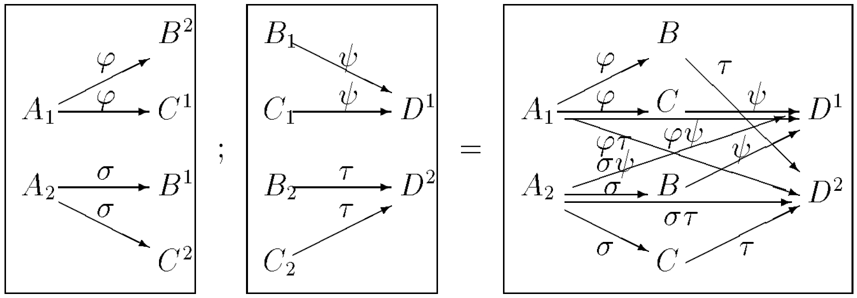

Figure 3.2

From the fact that $\alpha;\beta$ consists of α followed by β we obtain immediately the following proposition.

3.3. Proposition. The sequential composition is defined for all pairs (α,β) of cw-pomsets with $\partial_0(\beta) = \partial_1(\alpha)$, it is associative and such that $\partial_0(\alpha;\beta) = \partial_0(\alpha)$ and $\partial_1(\alpha;\beta) = \partial_1(\beta)$ and $\partial_0(\alpha);\alpha = \alpha;\partial_1(\alpha) = \alpha$ for all cw-pomsets α,β. $\square$

The parallel composition of cw-pomsets can be defined by specifying how the result of composing two cw-pomsets is related to these cw-pomsets, and by showing that the respective relation defines a binary operation on cw-pomsets.

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of a cw-pomset.

By a *splitting* of $\mathcal{A}$ we mean a partition $p = (X', X'')$ of X into two disjoint subsets X', X'' which are *independent* in the sense that x', x'' are incomparable whenever $x' \in X'$ and $x'' \in X''$, each $s(v)$ is $(s(v)|X')(s(v)|X'')$, the concatenation of the restrictions of $s(v)$ to X' and X'', and each $t(v)$ is $(t(v)|X')(t(v)|X'')$, the concatenation of the restrictions of $t(v)$ to X' and X''. By *splittings*$(\mathcal{A})$ we denote the set of splittings of $\mathcal{A}$.

The following proposition is a simple consequence of definitions.

3.4. Proposition. For each $p = (X', X'') \in splittings(\mathcal{A})$ the restrictions of $\mathcal{A}$ to X' and X'' with arrangements of minimal elements given respectively by $s|X' = (s(v)|X' : v \in V)$ and $s|X'' = (s(v)|X'' : v \in V)$, and arrangements of maximal elements given respectively by $t|X' = (t(v)|X' : v \in V)$ and $t|X'' = (t(v)|X'' : v \in V)$, are instances of cw-pomsets. We write them as $left_p(\mathcal{A})$ and $right_p(\mathcal{A})$, respectively. $\square$

The cw-pomset $[\mathcal{A}]$ is said to *consist* of the cw-pomset $[left_p(\mathcal{A})]$ *accompanied* by the cw-pomset $[right_p(\mathcal{A})]$.

Note that each cw-pomset α can be represented in the form $[left_{(X,\emptyset)}(\mathcal{A})]$ and in the form $[right_{(\emptyset,X)}(\mathcal{A})]$, where $\mathcal{A} = (X, \leq, d, e, s, t)$ is any instance of α.

3.5. Proposition. For every two cw-pomsets α and β there exists a unique cw-pomset $\alpha \otimes \beta$ which consists of α accompanied by β. This cw-pomset is K-dense whenever α and β are K-dense, and it is a symmetry whenever α and β are symmetries. $\square$

Proof: As instances of α and β may be chosen arbitrarily up to isomorphism, we may choose an instance $\mathcal{A} = (X_\mathcal{A}, \leq_\mathcal{A}, d_\mathcal{A}, e_\mathcal{A}, s_\mathcal{A}, t_\mathcal{A})$ of α and an instance $\mathcal{B} = (X_\mathcal{B}, \leq_\mathcal{B}, d_\mathcal{B}, e_\mathcal{B}, s_\mathcal{B}, t_\mathcal{B})$ of β such that $X_\mathcal{A}$ and $X_\mathcal{B}$ are disjoint. Then we define

$$X_\mathcal{C} = X_\mathcal{A} \cup X_\mathcal{B}$$

$$p = (X_\mathcal{A}, X_\mathcal{B})$$

$$x \leq_{\mathcal{C}} y \text{ whenever } x \leq_{\mathcal{A}} y \text{ or } x \leq_{\mathcal{B}} y$$

$$d_{\mathcal{C}}(x,y) = \begin{cases} d_{\mathcal{A}}(x,y) & \text{for } x,y \in X_{\mathcal{A}} \\ d_{\mathcal{B}}(x,y) & \text{for } x,y \in X_{\mathcal{B}} \\ \perp & \text{for the remaining } x,y \in X_{\mathcal{C}} \end{cases}$$

$$e_{\mathcal{C}}(x) = \begin{cases} e_{\mathcal{A}}(x) & \text{for } x \in X_{\mathcal{A}} \\ e_{\mathcal{B}}(x) & \text{for } x \in X_{\mathcal{B}} \end{cases}$$

$$(s_{\mathcal{C}})(v) = ((s_{\mathcal{A}})(v))((s_{\mathcal{B}})(v)) \text{ for all } v \in V$$

$$(t_{\mathcal{C}})(v) = ((t_{\mathcal{A}})(v))((t_{\mathcal{B}})(v)) \text{ for all } v \in V.$$

It is straightforward to verify that the structure $\mathcal{C} = (X_{\mathcal{C}}, \leq_{\mathcal{C}}, d_{\mathcal{C}}, e_{\mathcal{C}}, s_{\mathcal{C}}, t_{\mathcal{C}})$ is an instance of $\alpha \otimes \beta$, as required. $\square$

The operation $(\alpha,\beta) \mapsto \alpha\otimes\beta$ is called the *parallel composition* of cw-pomsets. An example of application of this operation is shown in figure 3.3.

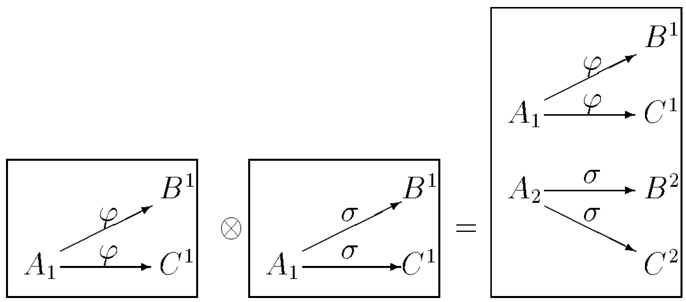

Figure 3.3

3.6. Proposition. The parallel composition is defined for all pairs (α,β) of cw-pomsets, it is associative, and has a neutral element *nil*, where *nil* is the unique cw-pomset with the empty instance. $\square$

A proof follows from the fact that $\alpha \otimes \beta$ consists of α accompanied by β.

In the sequel we assume that the sequential composition ; binds stronger than the parallel composition $\otimes$.

3.7. Proposition. The parallel composition is *functorial* in the sense that

$$\alpha;\beta \otimes \gamma;\delta = (\alpha \otimes \gamma);(\beta \otimes \delta)$$

whenever $\alpha;\beta$ and $\gamma;\delta$ are defined. $\square$

Proof: Let $\mathcal{C} = (X, \leq, d, e, s, t)$ be an instance of $\alpha;\beta \otimes \gamma;\delta$. Then $\alpha;\beta = [left_p(\mathcal{C})]$ and $\gamma;\delta = [right_p(\mathcal{C})]$ for some $p = (X', X'') \in splittings(\mathcal{C})$, $\alpha = [head_{Y',r'}(left_p(\mathcal{C}))]$ and $\beta = [tail_{Y',r'}(left_p(\mathcal{C}))]$ for some Y' and r', and $\gamma = [head_{Y'',r''}(right_p(\mathcal{C}))]$ and $\delta = [tail_{Y'',r''}(right_p(\mathcal{C}))]$ for some Y'' and r''. Consequently, the restriction of $\mathcal{C}$ to $(\leq Y') \cup (\leq Y'')$ with s playing the role of arrangement of minimal elements and $r = (r'(v)r''(v) : v \in V)$ playing the role of arrangement of maximal elements is an instance $\mathcal{A}_0$ of $\alpha \otimes \gamma$ and that to $(Y' \leq) \cup (Y'' \leq)$ with r playing the role of arrangement of minimal elements and t playing the role of arrangement of maximal elements is an instance $\mathcal{C}_1$ of $\beta \otimes \delta$. As $Y = Y' \cup Y''$ is a cut and $(\leq Y) = (\leq Y') \cup (\leq Y'')$, $Y \leq = (Y' \leq) \cup (Y'' \leq)$, $\mathcal{C}$ is an instance of $(\alpha \otimes \gamma); (\beta \otimes \delta)$, as required. $\square$

The interchanges are operations which produce symmetries from identities. They can be defined as follows.

Let $a_1, ..., a_n$ be identities and let p be a permutation of the sequence $1, ..., n$. By combining disjoint instances of $a_1, ..., a_n$ we obtain an instance $\mathcal{A}$ of a symmetry $I_p(a_1, ..., a_n)$.

3.8. Proposition. There exists a unique symmetry $I_p(a_1, ..., a_n)$ such that each instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of this symmetry can be partitioned into instances $\mathcal{A}_i = (X_i, \leq_i, d_i, e_i, s_i, t_i)$ of the respective a_i, where X is a disjoint union of all X_i, $\leq$ is a disjoint union of all $\leq_i$, d is a disjoint union of all d_i, e is a disjoint union of all e_i, each $s(v)$ is $s_1(v)...s_n(v)$, the concatenation of $s_1(v), ..., s_n(v)$, and each $t(v)$ is $s_{p(1)}(v)...s_{p(n)}(v)$, the concatenation of $s_{p(1)}(v), ..., s_{p(n)}(v)$. $\square$

The operation $(a_1, ..., a_n) \mapsto I_p(a_1, ..., a_n)$ is called the *interchange* of identities according to p. By $*$ and I_* we denote respectively the permutation $1 \mapsto 2, 2 \mapsto 1$ and the corresponding interchange.

An example of application of this operation is shown in figure 3.4.

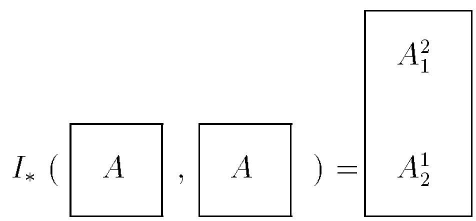

Figure 3.4

The following proposition is a simple consequence of definitions.

3.9. Proposition. The interchanges enjoy the following properties:

$$I_p(a_1, ..., a_n); I_{p^{-1}}(a_{p(1)}, ..., a_{p(n)}) = a_1 \otimes ... \otimes a_n$$

$$(I_*(a_1, a_2) \otimes a_3); (a_2 \otimes I_*(a_1, a_3)) = I_*(a_1, a_2 \otimes a_3). \; \square$$

As each permutation is a superposition of transpositions of elements which are neighbours, by applying I_* and the parallel composition to one-element cw-pomsets we obtain all the possible symmetries over V and W.

The interchanges are related as follows with the parallel composition.

3.10. Proposition. The parallel composition is *coherent* in the sense that

$$I_p(\partial_0(\alpha_1), ..., \partial_0(\alpha_n)); \alpha_{p(1)} \otimes ... \otimes \alpha_{p(n)} = \alpha_1 \otimes ... \otimes \alpha_n; I_p(\partial_1(\alpha_1), ..., \partial_1(\alpha_n))$$

for all $\alpha_1, ..., \alpha_n \in cwp(V, W)$ and for each permutation p of the sequence $1, ..., n$. $\square$

Proof: Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of $\alpha_1 \otimes ... \otimes \alpha_n$. For $i = 1, ..., n$ there exist instances $\mathcal{A}_i = (X_i, \leq_i, d_i, e_i, s_i, t_i)$ of the respective α_i such that all X_i are mutually disjoint, and $X_1 \cup ... \cup X_n = X$. Then $\mathcal{A}$ with t replaced by $t' = (t_{\mathcal{A}_{p(1)}}(v)...t_{\mathcal{A}_{p(n)}}(v) : v \in V)$ is an instance of $I_p(\partial_0(\alpha_1), ..., \partial_0(\alpha_n)); \alpha_{p(1)} \otimes ... \otimes \alpha_{p(n)}$ and an instance of $\alpha_1 \otimes ... \otimes \alpha_n; I_p(\partial_1(\alpha_1), ..., \partial_1(\alpha_n))$ as well, which implies the required equality. $\square$

The following proposition is a simple consequence of the respective definitions.

3.11. Proposition. The subset of K-dense cw-pomsets and the subset of symmetries are closed w.r. to the compositions and interchanges. $\square$

The stated properties of operations on cw-pomsets can be summarized in a brief way using notions of category theory.

3.12. Theorem. The (partial) algebra

$$CWP(V,W) = (cwp(V,W), \partial_0, \partial_1, ;, \otimes, nil, I_*)$$

is a symmetric strict monoidal category with cw-pomsets playing the role of morphisms, identities playing the role of object identities, and I_* playing the role of a natural transformation from the functor $(\alpha,\beta) \mapsto \alpha \otimes \beta$ to the functor $(\alpha,\beta) \mapsto \beta \otimes \alpha$. This algebra, called *the algebra of cw-pomsets*, contains $DCWP(V,W)$, the subalgebra of K-dense cw-pomsets, and $SYM(V,W)$, the subalgebra of symmetries. $\square$

The subalgebra $DCWP(V,W)$ of K-dense cw-pomsets is situated as follows in the algebra $CWP(V,W)$ of cw-pomsets.

3.13. Proposition. Each K-dense cw-pomset α can be obtained with the aid of interchanges and compositions from *atomic* cw-pomsets of the following two types:

(1) *one-element* cw-pomsets, one for each $v \in V$, namely the one-element cw-pomset with v being the label of the only element of its instance,
(2) *prime* cw-pomsets of the form $\pi = [\mathcal{P}]$, where $X_{\mathcal{P}} = (X_{\mathcal{P}})_{min} \cup (X_{\mathcal{P}})_{max}$, $(X_{\mathcal{P}})_{min}$ and $(X_{\mathcal{P}})_{max}$ are nonempty and disjoint, and each $x \in (X_{\mathcal{P}})_{min}$ is comparable with each $y \in (X_{\mathcal{P}})_{max}$.

In order to obtain α one needs always the same number of copies of each prime cw-pomset. $\square$

Proof: We start with recalling that each permutation is a superposition of transpositions of elements which are neighbours. Consequently, by applying interchanges and the parallel composition to one-element cw-pomsets we obtain all the possible symmetries over V and W.

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of α and let $Y_0 \sqsubseteq Y_1 \sqsubseteq ... \sqsubseteq Y_{n-1} \sqsubseteq Y_n$ be a maximal chain of maximal antichains of $\mathcal{A}$. Due to the maximality of this chain each $x \in Y_i - Y_{i+1}$ is comparable with each $y \in Y_{i+1} - Y_i$ (since otherwise between Y_i and Y_{i+1} there would be a maximal antichain containing x and y and it would be different from Y_i and Y_{i+1}).

There exists a symmetry $\sigma_1 = [\mathcal{S}_1]$ rearranging the minimal elements of X such that for each $v \in V$ the elements of $e^{-1}(v) \cap Y_0 \cap Y_1$ precede those of $e^{-1}(v) \cap (Y_0 - Y_1)$ and the orders of elements in $e^{-1}(v) \cap Y_0 \cap Y_1$ are consistent with an enumeration $y_{11}y_{12}...y_{1i_1}$ of entire $Y_0 \cap Y_1$. Besides, there exists an arrangement t_1 of elements of Y_1 which is identical with the arrangement of maximal elements of $\mathcal{S}_1$ in $Y_0 \cap Y_1$ and such that for each $v \in V$ the elements of $e^{-1}(v) \cap Y_0 \cap Y_1$ precede those of $e^{-1}(v) \cap (Y_1 - Y_0)$.

The restriction of $\mathcal{A}$ to $(Y_0 \leq) \cap (\leq Y_1)$ with the arrangement of minimal elements given by the arrangement of maximal elements of $\mathcal{S}_1$ and the arrangement of maximal elements given by t_1 is an instance of a cw-pomset $\alpha_1 = [\mathcal{A}_1]$.

Now, α_1 can be represented in the form

$$\alpha_1 = u_{11} \otimes ... \otimes u_{1i_1} \otimes \pi_1$$

where $u_{11}, ..., u_{1i_1}, \pi_1$ correspond to the respective restrictions of $\mathcal{A}$ to the subsets $\{y_{11}\}$, ..., $\{y_{1i_1}\}$, $(Y_0 - Y_1) \cup (Y_1 - Y_0)$. Thus we obtain a decomposition of α_1 into the one-element cw-pomsets u_{11},...,u_{1i_1} and the prime cw-pomset π_1.

Similarly, for Y_1,Y_2 we can define a symmetry rearranging the maximal elements of $\mathcal{A}_1$, the corresponding restriction $\mathcal{A}_2$ of $\mathcal{A}$, and a representation of $\alpha_2 = \mathcal{A}_2$ in the form

$$\alpha_2 = u_{21} \otimes ... \otimes u_{2i_2} \otimes \pi_2$$

and so on, until reaching

$$\alpha_n = u_{n1} \otimes ... \otimes u_{ni_n} \otimes \pi_n.$$

Finally, we define σ_{n+1} as the symmetry rearranging the maximal elements of $\mathcal{A}_n$ to t. Thus we obtain a sequence

$$\sigma_1, \alpha_1, \sigma_2, \alpha_2, ..., \sigma_n, \alpha_n, \sigma_{n+1}$$

such that $\sigma_1; \alpha_1; \sigma_2; \alpha_2; ...; \sigma_n; \alpha_n; \sigma_{n+1}$ is defined and equal to α, as required. Moreover, the subsets of X to which the prime cw-pomsets π_1,...,π_n correspond are determined uniquely by $\mathcal{A}$ and thus they do not depend on the particular choice of the maximal chain $Y_0 \sqsubseteq Y_1 \sqsubseteq ... \sqsubseteq Y_{n-1} \sqsubseteq Y_n$. Consequently, the number of copies of each prime process π which is needed in order to construct α depends only on α. $\square$

By *atomic*(V, W), *one_element*(V, W), *prime*(V, W) we denote respectively the set of atomic, one-element, and prime cw-pomsets over V. For each subset P of cw-pomsets over V and W by *closure*(P) we denote the least subset of *cwp*(V, W) that contains P and is closed w.r. to interchanges and compositions. With these notions we obtain the following result.

3.14. Theorem. The subalgebras $DCWP(V, W)$ and $SYM(V, W)$ of $CWP(V, W)$ are generated respectively by the subset of atomic cw-pomsets and the subset of one-element cw-pomsets in the sense that

$$dcwp(V, W) = closure(atomic(V, W))$$

$$sym(V, W) = closure(one\_element(V, W)) \quad \square$$

4. Tables

By ignoring in cw-pomsets elements which are neither minimal nor maximal we obtain slices. Such slices can be visualized in the form of rectangular tables with labelled rows and columns such that elements with the same label of row and the same label of column form a block. For example, the slices α' and β in figure 2.1 can be visualized as shown in figure 4.1.

	D^1	D^2
A_1	$\varphi\psi$	$\varphi\tau$
A_2	$\sigma\psi$	$\sigma\tau$

	B^1	B^2	C^1	C^2
B_1	$\perp$	ε	$\perp$	$\perp$
B_2	ε	$\perp$	$\perp$	$\perp$
C_1	$\perp$	$\perp$	ε	$\perp$
C_2	$\perp$	$\perp$	$\perp$	ε

Figure 4.1

The notion of table can be formalized as follows.

Denote by $mat(W)$ the set of rectangular matrices with elements from the semiring W. For each matrix M from this set denote by $height(M)$ the number of rows and by $width(M)$ the number of columns of M.

4.1. Definition. A *table* over V and W is a mapping A from $V \times V$ to $mat(W)$ such that $height(A(v, v')) = height(A(v, v''))$ and $width(A(v', v)) = width(A(v'', v))$ for all v, v', v''. $\square$

By $height(A(v, .))$ we denote the common value of all $height(A(v, v'))$, and by $width(A(., v))$ we denote the common value of all $width(A(v', v))$. If $height(A(v, .)) = width(A(., v))$ for all v and there exists a permutation φ_v of the sequence $1, ..., height(A(v, .))$ such that $(A(v, v))(i, j)$ is different from $\perp$ only for $j = \varphi_v(i)$ and then it coincides with ε, and if each $A(v, v')$ with $v \neq v'$ is a zero matrix, then we call A a *table symmetry.* If also all φ_v are identity permutations then we call A a *table identity.* By $tab(V, W)$, $tsym(V, W)$, and $tid(V, W)$ we denote respectively the set of tables, the set of table symmetries, and the set of table identities over V and W.

The set $tab(V, W)$ can be equipped with operations similar to those on cw-pomsets.

For each table A we have two identity tables: a *source* $\partial_0'(A)$ and a *target* $\partial_1'(A)$, where $(\partial_0'(A))(v, v)$ is the identity matrix of the size $height(A(v, .)) \times height(A(v, .))$, $(\partial_0'(A))(v, v')$ with $v \neq v'$ is the zero matrix of the size $height(A(v, .)) \times height(A(v', .))$, $(\partial_1'(A))(v, v)$ is the identity matrix of the size $width(A(., v)) \times width(A(., v))$, and $(\partial_1'(A))(v, v')$ with $v \neq v'$ is the zero matrix of the size $width(A(., v)) \times width(A(., v'))$.

For tables A and B such that $\partial_1'(A) = \partial_0'(B)$ we have a *sequential composition* $A;' B$, where

$$(A;' B)(v, v') = \Sigma(A(v, v'')B(v'', v') : v'' \in V)$$

for all $v, v' \in V$, that is

$$((A;' B)(v, v'))(i, j) =$$
$$\Sigma((A(v, v''))(i, k)(B(v'', v'))(k, j) : v'' \in V, k \in \{1, ..., width(A(., v''))\})$$

for all the respective v, v', i, j.

Thus the sequential composition of tables is an operation similar to matrix multiplication. An example of application of this operation is shown in figure 4.2.

$$\begin{array}{c|cccc} & B^1 & B^2 & C^1 & C^2 \\ \hline A_1 & \varphi & \perp & \varphi & \perp \\ A_2 & \perp & \sigma & \perp & \sigma \end{array} \;;'\; \begin{array}{c|cc} & D^1 & D^2 \\ \hline B_1 & \psi & \perp \\ B_2 & \perp & \tau \\ C_1 & \psi & \perp \\ C_2 & \perp & \tau \end{array} = \begin{array}{c|cc} & D^1 & D^2 \\ \hline A_1 & 2\varphi\psi & \perp \\ A_2 & \perp & 2\sigma\tau \end{array}$$

Figure 4.2

For arbitrary matrices A and B we have a *parallel composition* $A \otimes' B$, where each $(A \otimes' B)(v, v')$ denotes the matrix of the form

$$\begin{bmatrix} A(v, v') & zero\ matrix \\ zero\ matrix & B(v, v') \end{bmatrix}$$

Thus the parallel composition of tables is an operation related to building matrices from blocks. An example of application of this operation is shown in figure 4.3.

$$
\begin{array}{c|cc}
 & B^1 & C^1 \\ \hline
A_1 & \varphi & \varphi
\end{array}
\otimes'
\begin{array}{c|cc}
 & B^1 & C^1 \\ \hline
A_1 & \sigma & \sigma
\end{array}
=
\begin{array}{c|cccc}
 & B^1 & B^2 & C^1 & C^2 \\ \hline
A_1 & \varphi & \perp & \varphi & \perp \\
A_2 & \perp & \sigma & \perp & \sigma
\end{array}
$$

Figure 4.3

For arbitrary table identities A and B we have an *interchange* $I'_*(A, B)$, where each $(I'_*(A, B))(v, v')$ denotes the matrix of the form

$$\begin{bmatrix} zero\ matrix & A(v, v') \\ B(v, v') & zero\ matrix \end{bmatrix}$$

Finally, we define nil' as the table consisting of empty matrices.

When endowed with the operations just introduced the set $tab(V, W)$ of tables over V and W forms a partial algebra. From the properties of operations on matrices we obtain the following result.

4.2. Theorem. The (partial) algebra

$$TAB(V, W) = (tab(V, W), \partial'_0, \partial'_1, ;', \otimes', nil', I'_*)$$

is a symmetric strict monoidal category with tables playing the role of morphisms, table identities playing the role of object identities, and I'_* playing the role of a natural transformation from the functor $(\alpha, \beta) \mapsto \alpha \otimes' \beta$ to the functor $(\alpha, \beta) \mapsto \beta \otimes' \alpha$. This algebra, called the *algebra of tables* over V and W, contains the subalgebra $TSYM(V, W)$ of table symmetries. $\square$

The relation between cw-pomsets and tables can be described as follows.

4.3. Theorem. For each cw-pomset α there exists a table $table(\alpha)$ such that

$$((table(\alpha))(v, v'))(i, j) = d((s(v))(i), (t(v'))(j))$$

for each instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α, all $v, v' \in V$, each $i \in \{1, ..., length(s(v))\}$, and each $j \in \{1, ..., length(t(v'))\}$. The correspondence

$$table : CWP(V, W) \rightarrow TAB(V, W)$$

is a surjective homomorphism. $\square$

Proof: Given a cw-pomset α, we choose an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ and for each pair $v, v' \in V$ we define a matrix $[\mathcal{A}](v, v')$ of the size $length(s(v)) \times length(t(v'))$ by

$$([\mathcal{A}](v, v'))(i, j) = d((s(v))(i), (t(v'))(j)).$$

For each instance $\mathcal{A}' = (X', \leq', d', e', s', t')$ of α we have an isomorphism b from $\mathcal{A}$ to $\mathcal{A}'$. This implies

$$([\mathcal{A}'](v, .))(i, j) = d((s'(v))(i), (t'(v'))(j)) =$$
$$d((b(s(v)))(i), (b(t(v')))(j)) = d((s(v))(i), (t(v'))(j)) = ([\mathcal{A}](v, .))(i, j).$$

Hence $[\mathcal{A}] = [\mathcal{A}']$ and we can define $table(\alpha) = [\mathcal{A}]$. It is straightforward that the correspondece thus defined enjoys the required properties and that it is surjective. Moreover, from the definitions of operations on cw-pomset and operations on tables it follows easily that the correspondence $\alpha \mapsto table(\alpha)$ is a homomorphism. $\square$

5. The case $W = R$

The semiring R of real numbers and infinities is ordered and the operation $(x, y) \mapsto max(x, y)$ which plays the role of addition is idempotent. This implies some particular properties of cw-pomsets with weights from R, called also numerical weights.

First of such properties is a sort of criticity of weight functions of K-dense cw-pomsets.

5.1. Proposition. If $\mathcal{A} = (X, \leq, d, e, s, t)$ is any instance of a K-dense cw-pomset α with weights from the semiring R of real numbers and infinities then for all $x, y \in X$ such that $x \leq y$ the weight $d(x, y)$ is the maximum of sums $d(x, x_1) + ... + d(x_n, y)$ over all maximal chains $x \leq x_1 \leq ... \leq x_n \leq y$ from x to y. $\square$

Proof: The proposition can be proved by induction on the number of elements in $[x, y]$. Suppose that the required property holds for the number of elements not exceeding n and consider x, y such that the cardinality of $[x, y]$ is $n + 1$. Choose any $z \in [x, y]$ which is an immediate predecessor of y. Choose in $[x, y]$ a maximal antichain Z that contains z. As $[x, y]$ is K-dense, Z is a cut of $[x, y]$ and thus $d(x, y) = max(d(x, t) + d(t, y) : t \in Z)$ with $d(x, y) = d(x, u) + d(u, y)$ for some $u \in Z$. As $[x, u]$ has at most n elements, $d(x, u)$ is the maximum of sums $d(x, x_1) + ... + d(x_k, u)$ over all maximal chains $x \leq x_1 \leq ... \leq x_k \leq u$. Consequently, $d(x, y) = d(x, x_1) + ... + d(x_k, u) + d(u, y)$ for a maximal chain $x \leq x_1 \leq ... \leq x_k \leq u \leq y$. On the other hand, each maximal chain from x to y is of the form $x \leq x_1 \leq ... \leq x_k \leq t \leq y$ for some $t \in Z$ and we have $d(x, x_1) + ... + d(x_k, t) + d(t, y) \leq d(x, y)$. Hence $d(x, y)$ is the maximum of sums $d(x, x_1) + ... + d(x_k, t) + d(t, y)$ over maximal chains from x to y. $\square$

Another property characterizes candidates for weight functions of K-dense cw-pomsets with numerical weights.

5.2. Proposition. For each K-dense finite poset $\mathcal{X} = (X, \leq)$ and each function $d : X^2 \to R$ such that $d(x, y) = -\infty$ if $x \leq y$ does not hold, $d(x, x) = 0$, and $d(x, y)$ is the maximum of sums $d(x, x_1) + ... + d(x_n, y)$ over all maximal chains $x \leq x_1 \leq ... \leq x_n \leq y$ from x to y if $x \leq y$, there exists an instance $\mathcal{A}$ of a cw-pomset with the underlying poset $\mathcal{X}$ and the weight function d. $\square$

Proof: It suffices to show that for every x, y such that $x \leq y$ and for each cut Z of the poset $[x, y]$ we have $d(x, y) = max(d(x, z) + d(z, y) : z \in Z)$. The proof can be carried out by an easy induction on the cardinality of $[x, y]$ noting that each maximal chain from x to y must consist of a maximal chain from x to some $z \in Z$ and of a maximal chain from z to y. $\square$

Finally, the properties of weight functions of K-dense cw-pomsets with numerical weights imply uniqueness of some functions defined with the aid of weights.

5.3. Proposition. If $\mathcal{A} = (X, \leq, d, e, s, t)$ is an instance of a K-dense cw-pomset and $f : X \to R$ and $g : X \to R$ are functions such that $f(x) = g(x)$ for all $x \in X_{min}$, $f(y) = max(f(x) + d(x, y) : x \leq y, x \neq y)$ for all $y \in X - X_{min}$, and $g(y) = max(g(x) + d(x, y) : x \text{ immediately precedes } y)$ for all $y \in X - X_{min}$, then $f = g$. $\square$

Proof: By induction on the number of predecessors of an element it can be shown that $g(x) \leq f(x)$ for all $x \in X$. In order to prove that also $f(x) \leq g(x)$ for all $x \in X$ suppose that $g(y) < f(y)$ for some $y \in X$. Without a loss of generality we may assume that y is a minimal element such that $g(y) < f(y)$. This implies that $g(x) = f(x)$ for all $x \leq y$ such that $x \neq y$. From the properties of f it follows that $f(y) = f(t) + d(t, y)$ for some $t \leq y$ such that $t \neq y$. As $f(t) = g(t)$, we obtain $f(y) = g(t) + d(t, y)$. By 5.1 there exists a maximal chain $t \le x_1 \le ... \le x_n \le y$ from t to y such that $d(t,y) = d(t,x_1) + ... + d(x_n,y)$. From the properties of g we obtain $g(t) + d(t,x_1) \le g(x_1)$,...,$g(x_n) + d(x_n,y) \le g(y)$, which implies $g(t) + d(t,y) \le g(y)$. Consequently, $f(y) \le g(y)$, which contradicts to our assumption. $\Box$

6. Applications

In this section we describe how to use concatenable weighted pomsets as models of processes of Petri nets.

Place/transition Petri nets.
Let N be a place/transition Petri net with a set Pl of places of infinite capacities, a set Tr of transitions, and input and output functions $pre, post : Tr \to Pl^+$, where Pl^+ denotes the set of multisets of places. The multiset $pre(\tau)$ represents a collection of tokens, $pre(\tau,p)$ tokens in each place p, which must be consumed in order to execute a transition τ. The multiset $post(\tau)$ represents a collection of tokens, $post(\tau,p)$ tokens in each place p, which is produced by executing τ. We admit many concurrent nonconflicting executions of the same transition.

A process of N is either a presence of a token in a place, or an execution of a transition, or a combination of such processes. It may be represented as a cw-pomset α, where each instance $\mathcal{A} = (X, \le, d, e, s, t)$ of α represents a concrete process execution, elements of X represent the tokens which take part in this execution, the partial order $\le$ specifies the causal succession of tokens, the weight function d specifies the sets of sequences of transitions which must be executed in order to reach tokens from their causal predecessors, the labelling function e specifies places where tokens appear, and s and t are respectively arrangements of the tokens which the process receives from its environment and an arrangement of the tokens which the process delivers to its environment.

For each place $p \in Pl$ we have a process of presence of a token in p. This process, $pr(p)$, may be defined as the one-element cw-pomset with the label p.

For each transition $\tau \in Tr$, we have a process of executing τ with a collection $X_{in} = \{x(p,i) : p \in Pl, 1 \le i \le pre(\tau,p)\}$ of consumed tokens and a collection $X_{out} = \{y(q,j) : q \in Pl, 1 \le j \le post(\tau,p)\}$ of produced tokens, where each $x(p,i)$ comes from the place p and each $y(q,j)$ is produced in the place q. This process, $pr(\tau)$, may be defined as the prime cw-pomset with the instance $\mathcal{A} = (X, \le, d, e, s, t)$, where

$$X = X_{min} \cup X_{max} \text{ with } X_{min} = X_{in} \text{ and } X_{max} = X_{out}$$

$$d(x,y) = \begin{cases} \tau \ \text{ for } x \in X_{in} \text{ and } y \in X_{out} \\ \varepsilon \ \text{ for } x = y \\ \perp \text{ for the remaining } x, y \end{cases}$$

$$e(z) = \begin{cases} p \text{ for } z = x(p,i) \in X_{in} \\ q \text{ for } z = y(q,j) \in X_{out}, \end{cases}$$

and where $s(p)$ denotes the sequence $x(p,1)...x(p,pre(\tau,p))$ and $t(q)$ denotes the sequence $y(q,1)...y(q,post(\tau,q))$.

Processes which are combinations of presences of tokens in places and executions of transitions may be defined as cw-pomsets which can be obtained from the respective atomic cw-pomsets of the above two forms $pr(p)$ and $pr(\tau)$ with the aid of compositions and interchanges. Thus we obtain a set $proc(N)$ of cw-pomsets representing all processes of N.

For example, the cw-pomset α in figure 2.1 can be considered as a process of the net in figure 6.1.

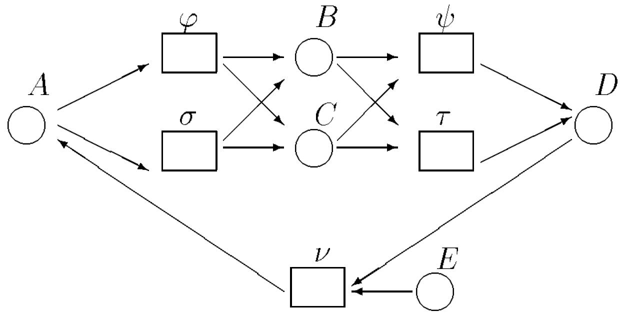

Figure 6.1

By definition, the set *proc*(N) of processes of N is closed with respect to the considered operations on cw-pomsets. When equipped with the respective restrictions of these operations, it becomes a subalgebra $PROC(N)$ of the algebra $DCWP(Pl,(Tr^*)^+)$ of K-dense cw-pomsets over Pl and $(Tr^*)^+$, where $(Tr^*)^+$ is the free semiring generated by the set Tr. We call this subalgebra the *algebra of processes* of N.

To each process $\alpha \in proc(N)$ there corresponds a table similar to causal streams of [FMM 91], namely $table(\alpha)$. Due to 4.2 we obtain the following result.

6.1. Theorem. The correspondence $\alpha \mapsto table(\alpha)$ is a homomorphism from the algebra $PROC(N)$ to $TAB(Pl,(Tr^*)^+)$. □

By replacing in processes of N weights by their images under the homomorphism which assigns to each different from zero element of $(Tr^*)^+$ the unit of the semiring $\{\perp, \varepsilon\}$ we obtain less informative models of processes.

Petri nets with time features.
In the case of Petri nets with time features we consider *timed* Petri nets similar to those in [Ram 74], and *time* Petri nets similar to those in [MerFa 76].

Timed Petri nets, called in the sequel *timed nets of the first type*, are place/transition nets in which executions of transitions start as soon as they are enabled and have some indeterministic durations, where the duration of execution of a transition may vary within some bounds depending on the transition.

Time Petri nets, called in the sequel *timed nets of the second type*, are place/transition nets in which executions of transitions are instantaneous and they start after some indeterministic periods of continuous enabling, where the necessary period of continuous enabling of a transition may vary within some bounds depending on the transition.

We exploit the fact that timed nets of the two types have the same form of place/transition nets with intervals assigned to transitions, and the fact that the difference between them reduces to interpretation, that is to way of defining their processes.

Processes of timed nets of both the types, called in the sequel *timed processes*, are described within a common framework. To this end, timed processes of a timed net are defined in two steps. In the first step we define timed processes which may be considered as possible in this net under some of its interpretations, and we call them *potential timed processes*. In the second step we characterize those of the processes thus defined which can be regarded as possible according to the interpretation of interest, and we call them *actual timed processes*.

The respective formalization can be as follows.

Denote by I the set of closed intervals of the semiring R of real numbers and infinities. Given an interval $i \in I$, denote by i_{min} the left bound of i and by i_{max} the right bound of i.

Let N be a timed net of the first or of the second type with a set Pl of places of infinite capacities, a set Tr of transitions, input and output functions $pre, post : Tr \to Pl^+$, where Pl^+ denotes the set of multisets of places, and with a function $D : Tr \to I$, called a specification of delays. For each transition τ, $D(\tau)$ represents the interval of variation of delays between enablings and completions of τ. We assume that the left bound $(D(\tau))_{min}$ of such an interval is positive.

A potential timed process of N is either a presence of a token in a place, or an execution of a transition, or a combination of such processes. It may be represented as a cw-pomset α, where each instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α represents a concrete process execution, elements of X represent the tokens which take part in this execution, the partial order $\leq$ specifies the causal succession of tokens, the weight function d specifies the delays with which tokens appear after their causal predecessors, the labelling function e characterizes each of the tokens, and s and t are respectively arrangements of the tokens which the process receives from its environment and an arrangement of the tokens which the process delivers to its environment. We assume that the labelling function e consists of a *place part*, e_{place}, and of a *time part* or *timing*, e_{time}, that is $e(x) = (e_{proper}(x), e_{time}(x))$ for each token x, where $e_{place}(x)$ specifies the place in which x appears and $e_{time}(x)$ specifies the appearance time of x. We assume also that the arrangement of the tokens which the process receives from its environment is given by a family $\vartheta = (\vartheta(p) : p \in Pl)$, where each $\vartheta(p)$ is an arbitrarily ordered (not necessarily monotonic) sequence of the appearance times of the tokens received in the respective place p. We call such a family a *delivery timing*. If the process receives only one token, and this token appears in a place p at instant u, then we identify the respective delivery timing ϑ with the single element of its only nonempty sequence, that is with u.

For each place $p \in Pl$ and each delivery timing u representing a delivery of a token to p at instant u we have a potential timed process of presence of the delivered token in p starting from u. This process, $tpr(p, u)$, may be defined as the one-element cw-pomset with the label (p, u).

For each transition $\tau \in Tr$, each delivery timing $\vartheta = (\vartheta(p) : p \in Pl)$ representing a delivery of tokens to places of N such that the length of each $\vartheta(p)$ is $pre(\tau, p)$, and each delay $\delta \in D(\tau)$, we have a potential process of executing τ with a collection $X_{in} = \{x(p, i) : p \in Pl, 1 \leq i \leq pre(\tau, p)\}$ of delivered tokens and a collection $X_{out} = \{y(q, j) : q \in Pl, 1 \leq j \leq post(\tau, p)\}$ of produced tokens, where each $x(p, i)$ appears at instant $\xi(p, i) = (\vartheta(p))(i)$ and each $y(q, j)$ appears with the delay δ after enabling of τ, that is at instant $\eta(q, j) = max((\vartheta(p))(k) : p \in Pl, 1 \leq k \leq pre(\tau, p)) + \delta$. This process, $tpr(\tau, \vartheta, \delta)$, may be defined as the prime cw-pomset with the instance $\mathcal{A} = (X, \leq, d, e, s, t)$, where

$$X = X_{min} \cup X_{max} \text{ with } X_{min} = X_{in} \text{ and } X_{max} = X_{out}$$

$$d(x, y) = \begin{cases} \delta & \text{for } x \in X_{in} \text{ and } y \in X_{out} \\ 0 & \text{for } x = y \\ -\infty & \text{for the remaining } x, y \end{cases}$$

$$e(x) = (e_{place}(x), e_{time}(x))$$

with

$$e_{place}(z) = \begin{cases} p \text{ for } z = x(p, i) \in X_{in} \\ q \text{ for } z = y(q, j) \in X_{out} \end{cases}$$

$$e_{time}(z) = \begin{cases} \xi(p, i) \text{ for } z = x(p, i) \in X_{in} \\ \eta(q, j) \text{ for } z = y(q, j) \in X_{out} \end{cases}$$

and where $s(p, u)$ denotes the subsequence of the sequence $x(p, 1)...x(p, pre(\tau, p))$ consisting of those $x(p, i)$ for which $\xi(p, i) = u$, and $t(q, w)$ denotes the subsequence of the sequence $y(q, 1)...y(q, post(\tau, q))$ consisting of those $y(q, j)$ for which $\eta(q, j) = w$. (Note that all $\eta(q, j)$ are equal, which implies that either $t(q, w)$ is entire sequence $y(q, 1)...y(q, post(\tau, q))$ or $t(q, w)$ is empty.)

Potential timed processes which are combinations of presences of tokens in places and executions of transitions may be defined as cw-pomsets which can be obtained from the respective atomic cw-pomsets of the above two forms $tpr(p, u)$ and $tpr(\tau, \vartheta, \delta)$ with the aid of compositions and interchanges. Thus we obtain a set $ptproc(N)$ of cw-pomsets representing all potential timed processes of N.

An example of a potential timed process of a timed net wich is given by the net in figure 6.1 and by the specification of delays such that $D(\varphi) = D(\psi) = D(\tau) = [1, 2]$ and $D(\sigma) = D(\nu) = [2, 3]$, is shown in figure 6.2.

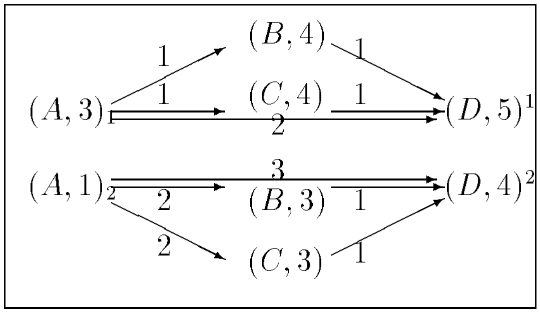

Figure 6.2

By definition, the set $ptproc(N)$ of potential timed processes of N is closed with respect to the considered operations on cw-pomsets. When equipped with the respective restrictions of these operations, it becomes a subalgebra $PTPROC(N)$ of the algebra $DCWP(Pl \times R, R)$ of K-dense cw-pomsets over $Pl \times R$ and R. We call this subalgebra the *algebra of potential timed processes* of N.

By reducing the labelling function e of an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of a potential timed process $\alpha \in ptproc(N)$ to its place part e_{place} we obtain an instance of a K-dense cw-pomset $free(\alpha)$, called the *free process* of N corresponding to α, and a table $deltab(\alpha)$, where $deltab(\alpha) = table(free(\alpha))$, called the *delay table* of α. Such a table plays the role of execution time of α, and it is more adequate in this role than a number as in [BG 92]. Due to the obvious properties of the correspondence $\alpha \mapsto free(\alpha)$ and 4.2 we obtain the following result.

6.2. Theorem. The correspondence $\alpha \mapsto deltab(\alpha)$ is a homomorphism from the algebra $PTPROC(N)$ to $TAB(Pl, R)$. $\square$

Actual timed processes of N are defined as such potential timed processes of N which are possible in N according to the assumed interpretation of N. In order to make it precise we need some auxiliary notions.

Each potential timed process $\alpha \in ptproc(N)$ has a unique *beginning of enabling*, $be(\alpha)$, and a unique *end of enabling*, $ee(\alpha)$, where

$$be(\alpha) = inf(sup(e_{time}(x) : x \leq y, x \neq y) : y \in X - X_{min})$$

and

$$ee(\alpha) = sup(e_{time}(x) : x \leq y \text{ and } x \neq y \text{ for some } y \in X_{max} - X_{min})$$

for each instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α. In particular, $be(\alpha) = +\infty$ if α is a symmetry (since then $X - X_{min}$ is empty), and $ee(\alpha) = -\infty$ if α is a symmetry (since then $X_{max} - X_{min}$ is empty). Intuitively, $be(\alpha)$ and $ee(\alpha)$ are respectively the earliest and the latest instants at which some of the transitions represented in α are enabled.

Given an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of a potential timed process $\alpha \in ptproc(N)$, an antichain $Y \subseteq X$, and a transition $\tau \in Tr$, we say that Y *enables* τ in α if the restriction of $\mathcal{A}$ to Y (with any identical arrangements of minimal and maximal elements) is an instance of the source of a prime potential timed process $\pi = tpr(\tau, \vartheta, \delta)$ with $be(\pi) \leq ee(\alpha)$, and we say that Y *originates* τ in α if Y is the set of minimal elements of a subset $X' \subseteq X$ such that the restriction of $\mathcal{A}$ to X' (with suitable arrangements of minimal and maximal elements) is an instance of a prime potential timed process $\varrho = tpr(\tau, \vartheta, \delta)$.

Actual timed processes of the timed net N are defined depending on the type of N. If the timed net N is interpreted as a timed net of the first type (that is as a standard timed net) then the respective actual processes are called *actual timed processes of the first type*. If it is interpreted as a timed net of the second type (that is as a time net) then the respective actual processes are called its *actual processes of the second type*.

6.3. Definition. A potential timed process $\alpha \in ptproc(N)$ is an actual process of N of the first type if the following inequality is satisfied for each antichain of an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α such that Y enables a transition $\pi \in Tr$ in α and for some antichain $Z(Y)$ of $\mathcal{A}$ such that $Y \cap Z(Y) \neq \emptyset$ and $Z(Y)$ originates a transition $\tau' \in Tr$ in α:

$$max(e_{time}(z) : z \in Z(Y)) \leq max(e_{time}(y) : y \in Y). \ \Box$$

Intuitively, α is an actual timed process of N of the first type if transitions represented in it cannot be prevented from execution by other transitions of N due to earlier enabling. In general, the set of actual timed processes of N of the first type, $atproc_1(N)$, is not closed under the considered operations on cw-pomsets and thus it does not form a subalgebra of the algebra of potential timeed processes of N.

6.4. Definition. A potential timed process $\alpha \in ptproc(N)$ is an actual process of N of the second type if the following inequality is satisfied for each antichain of an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α such that Y enables a transition $\pi \in Tr$ in α and for some antichain $Z(Y)$ of $\mathcal{A}$ such that $Y \cap Z(Y) \neq \emptyset$ and $Z(Y)$ originates a transition $\tau' \in Tr$ in α and for each z having an immediate predecessor in $Z(Y)$:

$$e_{time}(z) \leq max(e_{time}(y) : y \in Y) + (D(\pi))_{max}. \ \Box$$

Intuitively, α is an actual timed process of N of the second type if transitions represented in it cannot be prevented from execution by other transitions of N due to obligatory completion. In general, the set of actual timed processes of N of the second type, $atproc_2(N)$, is not closed under the considered operations on cw-pomsets and thus it does not form a subalgebra of the algebra of potential timeed processes of N.

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