Fundamenta Informaticae 80 (2007) 311–332
IOS Press

Towards a Framework for Modelling Behaviours of Hybrid Systems

Józef Winkowski*
Institute of Computer Science
Polish Academy of Sciences
Ordona 21, 01-237 Warszawa, Poland
wink@ipipan.waw.pl

Abstract. The paper is devoted to characterizing hybrid systems by specifying their possible runs, called processes, where each process is represented by a pomset in an intrinsic, global time independent way and can possibly be obtained by composing sequentially and in parallel other processes.

Keywords: Hybrid systems, processes, states, sequential composition, parallel composition, category, partial monoid, structure, random behaviour.

1. Introduction

In everyday life we have often to do with hybrid systems in which installations acting in a continuous way are controlled by automata acting in a discrete way. In order to design and analyse such systems we need a model of system behaviour universal enough to reflect both continuous and discrete way of acting.

The existing models of hybrid systems are based mainly on the ideas from automata theory (see [13]) and from the theory of Petri nets (see [6] and [8]). The main feature of these models is that the behaviours of systems are described as reactions to global time dependent control paths.

In the present paper we propose to characterize hybrid systems by specifying their possible runs, called processes, where each process is represented by a pomset in an intrinsic, global time independent way and can possibly be obtained by composing sequentially and in parallel other processes. More precisely, we propose to use the model of processes and the respective behaviour algebras introduced in [19], and to specify processes of a system as elements of a subset of a behaviour algebra.

Address for correspondence: Józef Winkowski, Instytut Podstaw Informatyki PAN, Ordona 21, 01-237 Warszawa, Poland
*This work of the author has been supported by the Institute of Computer Science of the Polish Academy of Sciences.

Note that our understanding of term process as a run of a system, as in the theory of Petri nets (cf. for example [3], [16], and [7]), is different from that in CCS and other similar calculi (cf. for example [10], [2], and [11]), where processes are understood as evolving objects.

2. Preliminaries

Given a partial order $\leq$ on a set X, we call $\mathcal{X} = (X, \leq)$ a *partially ordered set*, or briefly a *poset*, by the *strict partial order* corresponding to $\leq$ we mean $<$, where $x < y \text{ iff } x \leq y \text{ and } x \neq y$.

2.1. Definition. By a *strong cross-section* of a poset $\mathcal{X} = (X, \leq)$ we mean a maximal antichain Z of $\mathcal{X}$ that has an element in every maximal chain of $\mathcal{X}$. By a *weak cross-section*, or briefly a *cross-section*, of $\mathcal{X}$ we mean a maximal antichain Z of $\mathcal{X}$ such that, for every $x, y \in X$ for which $x \leq y$ and $x \leq z'$ and $z'' \leq y$ with some $z', z'' \in Z$, there exists $z \in Z$ such that $x \leq z \leq y$. We say that $\leq$ (and $\mathcal{X}$) is K*-dense* (resp.: *weakly* K*-dense*) iff every maximal antichain of $\mathcal{X}$ is a strong cross-section (resp.: (weak) cross-section) of $\mathcal{X}$ (cf. [14] and [15]). For every cross-section Z of $\mathcal{X}$, we define $X^-(Z) = \{x \in X : x \leq z \text{ for some } z \in Z\}$ and $X^+(Z) = \{x \in X : z \leq x \text{ for some } z \in Z\}$, and we say that a cross-section Z' *precedes* a cross-section Z'' and write $Z' \preceq Z''$ iff $X^-(Z') \subseteq X^-(Z'')$. $\sharp$

2.2. Proposition. The relation $\preceq$ is a partial order on the set of cross-sections of $\mathcal{X}$. Moreover, for every two cross-sections Z' and Z'' of $\mathcal{X}$ there exist the greatest lower bound $Z' \bigtriangleup Z''$ and the least upper bound $Z' \bigtriangledown Z''$ of Z' and Z'' with respect to $\preceq$, where $Z' \bigtriangleup Z''$ is the set of those $z \in Z' \cup Z''$ for which $z \leq z'$ for some $z' \in Z'$ and $z \leq z''$ for some $z'' \in Z''$, and $Z' \bigtriangledown Z''$ is the set of those $z \in Z' \cup Z''$ for which $z' \leq z$ for some $z' \in Z'$ and $z'' \leq z$ for some $z'' \in Z''$. $\sharp$

Proof. The set $Z' \bigtriangleup Z''$ is an antichain since otherwise there would be $x < y$ for some x and y in this set. If $x \in Z'$ then there would be $y \in Z''$ and there would exist $z' \in Z'$ such that $y \leq z'$. However, this is impossible since Z' is an antichain. Similarly for $x \in Z''$.

The set $Z' \bigtriangleup Z''$ is a maximal antichain since otherwise there would exist x that would be incomparable with all the elements of this set. Consequently, there would not exist $z' \in Z'$ and $z'' \in Z''$ such that $z' \leq x \leq z''$, or $z'' \leq x \leq z'$, or $z', z'' \leq x$, and thus there would be $x \leq z'$ and $x \leq z''$ for some $z' \in Z'$ and $z'' \in Z''$ that are not in $Z' \bigtriangleup Z''$. Consequently, there would exist z, say in Z'', such that $x \leq z \leq z'$. Moreover, $z \in Z' \bigtriangleup Z''$ since otherwise there would be $t \in Z'$ such that $t \leq z \leq z'$, what is impossible.

In order to see that $Z' \bigtriangleup Z''$ is a cross-section we consider $x \leq y$ such that $x \leq t$ and $u \leq y$ for some $t \in Z' \bigtriangleup Z''$ and $u \in Z' \bigtriangleup Z''$, where $t \in Z'$ and $u \in Z''$. Without a loss of generality we can assume that $y \leq y'$ for some $y' \in Z'$ since otherwise we could replace y by an element of Z'. Consequently, there exists $z \in Z''$ such that $x \leq z \leq y$. On the other hand, $z \in Z' \bigtriangleup Z''$ since otherwise there would be $z' \in Z'$ such that $z' \leq z \leq y$, what is impossible. In a similar manner we can find $z \in Z' \bigtriangleup Z''$ for the other cases of t and u.

In order to see that $Z' \bigtriangleup Z''$ is the greatest lower bound of Z' and Z'' consider a cross-section Y which precedes Z' and Z'' and observe that $y \leq z' \in Z'$ and $y \leq z'' \in Z''$ with z' and z'' not in $Z' \bigtriangleup Z''$ and $y \in Y$ implies the existence of $t \in Z'$ such that $y \leq t \leq z'$ or $u \in Z''$ such that $y \leq u \leq z''$.

Similarly, $Z' \bigtriangledown Z''$ is a cross-section and the least upper bound of Z' and Z''. $\sharp$

For cross-sections Z' and Z'' of a poset $\mathcal{X} = (X, \leq)$ such that $Z' \preceq Z''$ we define a *segment* of $\mathcal{X}$ from Z' to Z'' as the restriction of $\mathcal{X}$ to the set $[Z', Z''] = X^{+}(Z') \cap X^{-}(Z'')$, written as $\mathcal{X}|[Z', Z'']$. A segment $\mathcal{X}|[Y', Y'']$ such that $Z' \preceq Y' \preceq Y'' \preceq Z''$ is called a *subsegment* of $\mathcal{X}|[Z', Z'']$. If $Z' \neq Y'$ or $Y'' \neq Z''$ (resp.: if $Z' = Y'$, or if $Y'' = Z''$) then we call it a *proper* (resp.: an *initial*, or a *final*) subsegment of $\mathcal{X}|[Z', Z'']$.

Note that for every strong or weak cross-section Z of a poset $\mathcal{X} = (X, \leq)$ the reflexive and transitive closure of the union of the restrictions of the partial order $\leq$ to $X^{-}(Z)$ and to $X^{+}(Z)$ is exactly the partial order $\leq$.

Given a partial order $\leq$ on a set X and a function $l : X \to W$ that assigns to every $x \in X$ a label $l(x)$ from a set W, we call $\mathbf{X} = (X, \leq, l)$ a *labelled partially ordered set*, or briefly an *lposet*, by a *chain* (resp.: an *antichain*, a *cross-section*) of $\mathbf{X}$ we mean a chain (resp.: an antichain, a cross-section) of $\mathcal{X} = (X, \leq)$, by a *segment* of $\mathbf{X}$ we mean each restriction of $\mathbf{X}$ to a segment of $\mathcal{X}$, and we say that $\mathbf{X}$ is K*-dense* (resp.: *weakly* K*-dense*) iff $\leq$ is K-dense (resp.: weakly K-dense).

By $\mathbf{LPOSETS}$ we denote the category of lposets and their morphisms, where a *morphism* from an lposet $\mathbf{X} = (X, \leq, l)$ to an lposet $\mathbf{X}' = (X', \leq', l')$ is defined as an injection $b : X \to X'$ such that, for all x and y, $x \leq y$ iff $b(x) \leq' b(y)$, and, for all x, $l(x) = l'(b(x))$. In the category $\mathbf{LPOSETS}$ a morphism from $\mathbf{X}$ to $\mathbf{X}'$ is an *isomorphism* iff it is bijective, and it is an *automorphism* iff it is bijective and $\mathbf{X} = \mathbf{X}'$. If there exists an isomorphism from an lposet $\mathbf{X}$ to an lposet $\mathbf{X}'$ then we say that $\mathbf{X}$ and $\mathbf{X}'$are *isomorphic*. A *partially ordered multiset*, or briefly a *pomset*, is defined as an isomorphism class ξ of lposets. Each lposet that belongs to such a class ξ is called an *instance* of ξ. The pomset corresponding to an lposet $\mathbf{X}$ is written as $[\mathbf{X}]$.

2.3. Definition. By a *partial category* we mean a partial algebra $\mathcal{C} = (C, dom, cod, ;)$ that is defined in exactly the same way as the morphisms-only category with the set C of morphisms, the source and the target functions "dom" and "cod", and the composition ";", except that sources and targets may be not defined for some morphisms that are not identities and then the respective compositions are not defined. $\sharp$

We call ";" the sequential composition and write composits $c; d$ as cd. By *functors* between partial categories we mean strong homomorphisms. By *subalgebras* and *congruences* of partial categories we mean subalgebras and congruences in the strong sense. Limits and colimits in partial categories are defined as in usual categories.

2.4. Definition. By a *partial commutative monoid* we mean a partial algebra $\mathcal{M} = (M, +, 0)$ that is defined in exactly the same way as the commutative monoid with the set M of elements, the summation operation "$+$", and the neutral element "0", except that the summation is not always defined. We assume that the sum with the neutral element is always defined, that the summation is associative in the sense that $(m + n) + p = m + (n + p)$ whenever either side of such an equation is defined, and that it is commutative in the sense that $m + n = n + m$ whenever either side of such an equation is defined. $\sharp$

We call "$+$" the *parallel composition*. By *homomorphisms* between partial commutative monoids we mean strong homomorphisms. By *subalgebras* and *congruences* of partial commutative monoids we mean subalgebras and congruences in the strong sense.

3. Processes

We think of processes as of activities in a universe of objects, each object with a set of possible internal states and instances corresponding to these states, each activity changing states of some objects (see [17]).

A universe of objects and processes in such a universe can be defined as follows.

3.1. Definition. By a *universe of objects* we mean a structure $U = (W, V, ob)$, where V is a set of *objects*, W is a set of *instances* of objects from V (a set of *object instances*), and ob is a mapping that assigns the respective object to each of its instances. $\sharp$

3.2. Example (after [6]. Suppose that a machine M produces a coated copper wire from uncovered copper wire and plastic, 1 metre of product from 1 metre of uncovered wire and 0,05 kilogram of plastic. Suppose that the machine M is equipped with a switch S to resume production (the position ON) and to break it (the position OFF). Define an instance of M to be a quadruple (M, a, e, b), where $a \geq 0$ and $e \geq 0$ are respectively the available amount of uncovered wire and of plastic, and $b \geq 0$ is the amount of coated wire. Define an instance of S to be a pair (S, s), where s is ON or OFF. Define $V_1 = \{M, S\}$, $W_1 = W_M \cup W_S$, where $W_M = \{(M, a, e, b) : a, e, b \geq 0\}$, and $W_S = \{(S, ON), (S, OFF)\}$. Define $ob_1(w) = M$ for $w = (M, a, e, b) \in W_M$ and $ob_1(w) = S$ for $w = (S, s) \in W_S$. Then $U_1 = (W_1, V_1, ob_1)$ is a universe of objects. $\sharp$

3.3. Example. Suppose that a producer p produces some material for a distributor d. Define an instance of p to be a pair (p, q), where $q \geq 0$ is the amount of material at disposal of p. Define an instance of d to be a pair (d, r), where $r \geq 0$ is the amount of material at disposal of d. Define $V_2 = \{p, d\}$, $W_2 = W_p \cup W_d$, where $W_p = \{(p, q) : q \geq 0\}$, $W_d = \{(d, r) : r \geq 0\}$. Define $ob_2(w) = p$ for $w = (p, q) \in W_p$ and $ob_2(w) = d$ for $w = (d, r) \in W_d$. Then $U_2 = (W_2, V_2, ob_2)$ is a universe of objects. $\sharp$

3.4. Definition. Given a universe $U = (W, V, ob)$ of objects, by a *concrete process* in U we mean a labelled partially ordered set $P = (X, \leq, ins)$, where

(1) X is a set (of *occurrences* of objects from V, called *object occurrences*),

(2) $ins : X \to W$ is a mapping (a *labelling* that assigns an object instance to each occurrence of the respective object),

(3) $\leq$ is a partial order (the *flow order* of P) such that

- (3.1) for every object $v \in V$, the set $X|v = \{x \in X : ob(ins(x)) = v\}$ is either empty or it is a maximal chain and has an element in every cross-secton,
- (3.2) every element of X belongs to a cross-section,
- (3.3) no segment of P is isomorphic to its proper subsegment. $\sharp$

Condition (3.1) means that P contains all information on the behaviour within P of every object which has in P an occurrence, and that every potential global state of P contains an element of this information. Condition (3.2) guarantees that every occurrence of an object in P belongs to a potential global state of P. Condition (3.3) allows one to distinguish every segment of P even if P is considered up to isomorphism. Note that (3.3) holds if for an object v with nonempty $X|v$ there is no flow order and labelling preserving bijection from an interval of $X|v$ to its proper subinterval.

Remark (Correction to [19] **).** The author would like to take this opportunity to correct an error in the paper "Behaviour Algebras" (item [19] of the references). In the definition of a process in a universe of objects in 2.3 of [19] the condition (3.1) should be replaced by the stronger condition (3.1) of the present definition of a process.

3.5. Example. Let $U_1 = (W_1, V_1, ob_1)$ be the universe described in 3.2.

The work of the machine M in an interval $[t', t'']$ of global time is a concrete process that when considered without taking into account the switch can be defined as $B = (X_B, \leq_B, ins_B)$, where

X_B is the set $\{b(t) : t \in [t', t'']\}$ of values of the real-valued function $t \mapsto b(t)$ that specifies the amount of coated wire that has been produced until $t \in [t', t'']$,

$\leq_B$ is the restriction of the usual order of numbers to X_B,

$ins_B(x) = (M, a(t), e(t), b(t)))$ for $x = b(t)$, where $a(t)$ and $e(t)$ are respectively the amount of uncovered wire and the amount of plastic available at $t \in [t', t'']$.

Defining X_B as above instead of taking simply X_B equal to $[t', t'']$ is necessary in order to ensure the property (3.3) of 3.4 (this property could not be ensured with $X_B = [t', t'']$ if the function $t \mapsto b(t)$ were constant on subsegments of $[t', t'']$). Note that $a(t') - a(t) = b(t) - b(t')$ and $e(t') - e(t) = 0,05(b(t) - b(t'))$ for every $t \in [t', t'']$.

Switching on the machine M in a state $s_0 = (M, a_0, e_0, b_0)$ is a concrete process that can be defined as $I = (X_I, \leq_I, ins_I)$, where

$X_I = \{x_1, x_2, x_3, x_4\}$,

$x_1 <_I x_3, \quad x_1 <_I x_4, \quad x_2 <_I x_3, \quad x_2 <_I x_4$,

$ins_I(x_1) = ins_I(x_3) = s_0$, $ins_I(x_2) = (S, OFF)$,
$ins_I(x_4) = (S, ON)$.

Switching off the machine M in a state $s_1 = (M, a_1, e_1, b_1)$ is a concrete process that can be defined as $J = (X_J, \leq_J, ins_J)$, where

$X_J = \{x_1, x_2, x_3, x_4\}$,

$x_1 <_J x_3, \quad x_1 <_J x_4, \quad x_2 <_J x_3, \quad x_2 <_J x_4$,

$ins_J(x_1) = ins_J(x_3) = s_1$, $ins_J(x_2) = (S, ON)$,
$ins_J(x_4) = (S, OFF)$.

Switching on the machine M in a state s_0 followed by a work of M and by switching off M in a state s_1 is a concrete process that can be defined as $K = (X_K, \leq_K, ins_K)$, where

$X_K = X_{B'} \cup X_{I'} \cup X_{J'}$,

$\leq_K$ is the transitive closure of $\leq_{B'} \cup \leq_{I'} \cup \leq_{J'}$,

$ins_K = ins_{B'} \cup ins_{I'} \cup ins_{J'}$,

for a variant B' of B, a variant I' of I, and a variant J' of J, such that the maximal element of $X_{I'}$ with the label (S, ON) coincides the minimal element of $X_{J'}$ with the label (S, ON), the maximal element of $X_{I'}$ with the label s_0 coincides the minimal element of $X_{B'}$ with the label s_0, the maximal element of $X_{B'}$ with the label s_1 coincides the minimal element of $X_{J'}$ with the label s_1, and these are the only common elements of pairs of sets from among $X_{B'}$, $X_{I'}$, $X_{J'}$.

Isomorphism classes of lposets corresponding to processes B, I, J, and K, are represented graphically in Figure 1. $\sharp$

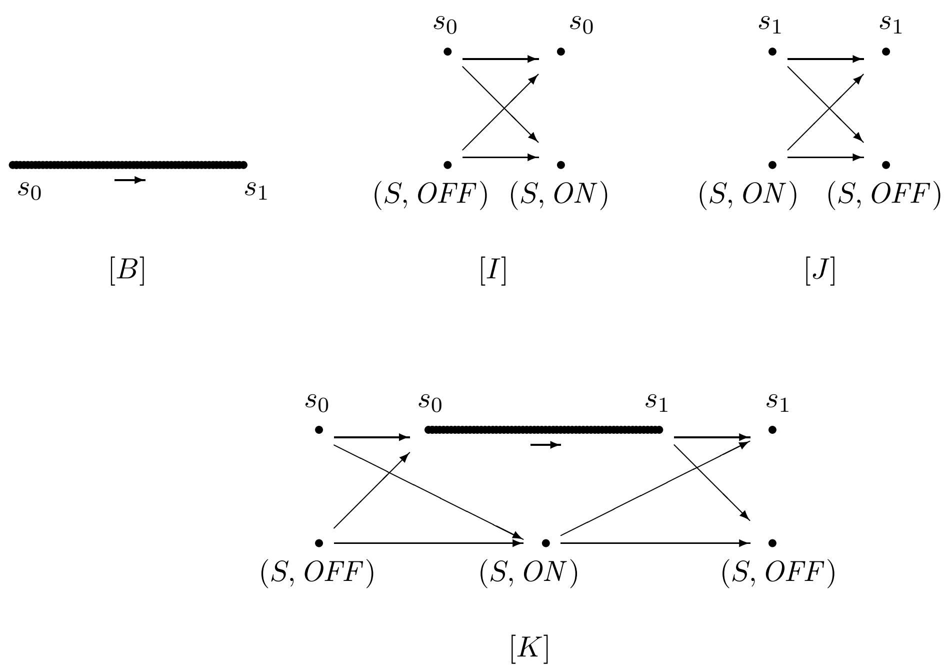

Figure 1: $[B]$, $[I]$, $[J]$, $[K]$

3.6. Example. Let $U_2 = (W_2, V_2, ob_2)$ be the universe described in 3.3.

Undisturbed production of material by the producer p in an interval $[t', t'']$ of global time is a concrete process that can be defined as $Q = (X_Q, \leq_Q, ins_Q)$, where

X_Q is the set of numbers equal to variations $var(t \mapsto q(t); t', t)$ in $[t', t] \subseteq [t', t'']$ of the real valued function $t \mapsto q(t)$ that specifies the amount of material at disposal of p at every moment of $[t', t'']$,

$\leq_Q$ is the restriction of the usual order of numbers to X_Q,

$ins_Q(x) = (p, q(t))$ for $x = var(t \mapsto q(t); t', t)$.

(We recall that the variation of a real-valued function f on an interval $[a, b]$, written as $var(f; a, b)$, is the least upper bound of the set of numbers $|f(a_1) - f(a_0)| + ... + |f(a_n) - f(a_{n-1})|$ corresponding to subdivisions $a = a_0 < a_1 < ... < a_n = b$ of $[a, b]$. In the case of more than one real-valued function the concept of variation turns into the concept of the length of the curve defined by these functions.).

Undisturbed distribution of material by the distributor d in an interval $[t', t'']$ of global time is a concrete process that can be defined as $R = (X_R, \leq_R, ins_R)$, where

X_R is the set of numbers equal to variations $var(t \mapsto r(t); t', t)$ in $[t', t] \subseteq [t', t'']$ of the real valued function $t \mapsto r(t)$ that specifies the amount of material at disposal of d at every moment of $[t', t'']$,

$\leq_R$ is the restriction of the usual order of numbers to X_R,

$ins_R(x) = (d, r(t))$ for $x = var(t \mapsto r(t); t', t)$.

Transfer of an amount m of material from the producer p to the distributor d is a concrete process that can be defined as $D = (X_D, \leq_D, ins_D)$, where

$X_D = \{x_1, x_2, x_3, x_4\}$,

$x_1 <_D x_3, \quad x_1 <_D x_4, \quad x_2 <_D x_3, \quad x_2 <_D x_4$,

$ins_D(x_1) = (d, r)$, $ins_D(x_2) = (p, q)$, $ins_D(x_3) = (d, r + m)$,
$ins_D(x_4) = (p, q - m)$.

Transfer of an amount of material from the producer p to the distributor d followed by independent behaviour of p and d and by another transfer of material from p to d is a concrete process
$L = (X_L, \leq_L, ins_L)$, where

$X_L = X_{Q'} \cup X_{R'} \cup X_{D'} \cup X_{D''}$,

$\leq_L$ is the transitive closure of $\leq_{Q'} \cup \leq_{R'} \cup \leq_{D'} \cup \leq_{D''}$,

$ins_L = ins_{Q'} \cup ins_{R'} \cup ins_{D'} \cup ins_{D''}$,

for a variant Q' of Q, a variant R' of R, and variants D' and D'' of D, such that one maximal element of $X_{D'}$ coincides the minimal element of $X_{Q'}$ with the same label and the other maximal element coincides with the minimal element of $X_{R'}$ with the same label, one minimal element of $X_{D''}$ coincides the maximal element of $X_{Q'}$ with the same label and the other minimal element coincides with the maximal element of $X_{R'}$ with the same label, and these are the only common elements of pairs of sets from among $X_{Q'}$, $X_{R'}$, $X_{D'}$, $X_{D''}$.

Isomorphism classes of lposets corresponding to processes Q, R, D, and L, are represented graphically in Figure 2. $\sharp$

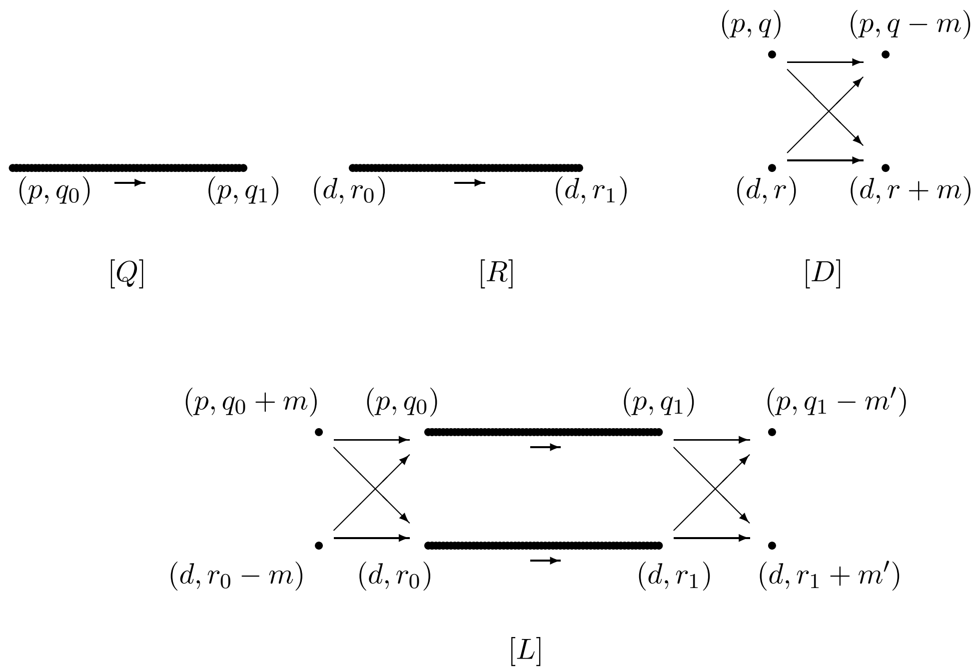

Figure 2: $[Q]$, $[R]$, $[D]$, $[L]$

Let $U = (W, V, ob)$ be a universe of objects.

Let $P = (X, \leq, ins)$ be a concrete process in U.

Every cross-section of P contains an occurrence of each object v with nonempty $X|v$, and it is called a ***cross-section*** of P. By $csections(P)$ we denote the set of cross-sections of P. This set is partially ordered by the relation $\preceq$, and for every two cross-sections Z' and Z'' from $csections(P)$ there exist in $csections(P)$ the greatest lower bound $Z' \bigtriangleup Z''$ and the least upper bound $Z' \bigtriangledown Z''$ of Z' and Z'' with respect to $\preceq$. From (3.1) and (3.2) of 3.4 it follows that the set of objects occurring in a cross-section is the same for all cross-sections of P. We call it the ***range*** of P and write it as $objects(P)$. We say that P is ***global*** if $objects(P) = V$. We say that P is ***bounded*** if the set of elements of P that are minimal with respect to $\leq$ and the set of elements of P that are maximal with respect to $\leq$ are cross-sections; the respective cross-sections are then called the ***origin*** and the ***end*** of P, and they are written as $origin(P)$ and $end(P)$.

As concrete processes are lposets, their morphisms are defined as morphisms of lposets, that is as injections that preserve the ordering and the labelling (see section 2).

3.7. Proposition. If P is a process then for every segment Q of P, every isomorphism between initial or final subsegments of Q is an identity. $\sharp$

Proof. Let R and S be two initial subsegments of Q.

Suppose that $f : R \to S$ is an isomorphism that it is not an identity. Then there exists an initial subsegment T of R such that the image of T under f, say T', is different from T. By (3.3) of 3.4 neither T' is a subsegment of T nor T is a subsegment of T'. Define T'' to be the least segment containing both T and T', and consider $f' : T \to T''$, where $f'(x) = f(x)$ for $x \leq f(x)$ and $f'(x) = x$ for $f(x) < x$. In order to derive a contradiction, and thus to prove that f is an identity, it suffices to verify, that f' is an isomorphism. It can be done as follows.

For injectivity suppose that $f'(x) = f'(y)$. If $x \leq f(x)$ and $y \leq f(y)$ then $f(x) = f'(x) = f'(y) = f(y)$ and thus $x = y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) = f'(y) = y$. The case $x \leq f(x)$ and $f(y) < y$ is excluded by $f'(x) = f'(y)$ since $x \leq f(x) = f'(x) = f'(y) = y$ and, on the other hand, $f(y) < y = f(x)$ implies $y < x$. Similarly, the case $f(x) < x$ and $y \leq f(y)$ is excluded. Consequently, f' is injective.

For surjectivity suppose that y is in T''. If $y \leq f(y)$ then $y = f(t)$ for some $t \leq y$ and thus $y = f'(t)$ since $t \leq y = f(t)$ and thus $f'(t) = f(t)$. If $f(y) < y$ then $y = f'(y)$. Consequently, f' is surjective.

For monotonicity suppose that $x \leq y$. If $x \leq f(x)$ and $y \leq f(y)$ then $f'(x) = f(x) \leq f(y) = f'(y)$. If $f(x) < x$ and $f(y) < y$ then $f'(x) = x \leq y = f'(y)$. If $x \leq f(x)$ and $f(y) < y$ then $f'(x) = f(x) \leq f(y) < y = f'(y)$. If $f(x) < x$ and $y \leq f(y)$ then $f'(x) = x \leq y \leq f(y) = f'(y)$. Consequently, f' is monotonic.

For monotonicity of the inverse suppose that $f'(x) < f'(y)$. If $x \leq f(x)$ and $y \leq f(y)$ then $f(x) = f'(x) < f'(y) = f(y)$ and thus $x < y$. If $f(x) < x$ and $f(y) < y$ then $x = f'(x) < f'(y) = y$. If $x \leq f(x)$ and $f(y) < y$ then $x \leq f(x) = f'(x) < f'(y) = y$. If $f(x) < x$ and $y \leq f(y)$ then $f(x) < x = f'(x) < f'(y) = f(y)$ and thus $x < y$. Consequently, the inverse of f' is monotonic.

Verification for final subsegments is similar. $\sharp$

3.8. Corollary. For every segment Q of a process P, every isomorphism between initial or final subsegments of Q has an extension to an automorphism of the whole segment Q. $\sharp$

3.9. Definition. An *abstract process* is an isomorphism class of concrete processes. $\sharp$

For every concrete process P' such that P and P' are isomorphic we have $objects(P') = objects(P)$. Consequently, for the abstract process $[P]$ that corresponds to a concrete process P we define $objects([P]) = objects(P)$. We say that an abstract process is *global* (resp.: *bounded*, *K-dense*, *weakly K-dense*) if the instances of this process are global (resp.: bounded, K-dense, weakly K-dense).

By $PROC(U)$ and $Proc(U)$ we denote respectively the set of all processes in U and the subset of all bounded processes in U. Similarly, By $KPROC(U)$ and $KProc(U)$ we denote respectively the set of all K-dense processes in U and the subset of all bounded K-dense processes in U.

4. Operations on processes

Let $U = (W, V, ob)$ be a universe of objects.

In the set $PROC(U)$ of processes in U there exists a bounded process with the empty set of object instances, called the *empty process* and denoted by 0.

Processes from $PROC(U)$ with flow orders reducing to identities are bounded, they are called *states*, or *identities*, and we can identify with the sets of instances of occurring objects.

For each process π from $PROC(U)$ with a cross-section $origin(P)$ (resp.: with a cross-section $end(P)$) for each $P \in \pi$ there exists a unique identity, called the *source* or the *domain* of π and written as $dom(\pi)$ (resp.: a unique identity, called the *target* or the *codomain* of π and written as $cod(\pi)$), whose instance can be obtained from an instance P of π by restricting P to the cross-section $origin(P)$ (resp.: to the cross-section $end(P)$).

Thus we have two partial unary operations on processes: the operation dom of taking the source (the domain), and the operation cod of taking the target (the codomain).

We have also a sequential composition and a parallel composition.

The sequential composition allows one to combine two processes whenever one of them is a continuation of the other. It can be defined due to the following proposition.

4.1. Proposition. For each cross-section c of a concrete process $P = (X, \leq, ins)$, the restrictions of P to the subsets $X^-(c) = \{x \in X : x \leq z \text{ for some } z \in c\}$ and $X^+(c) = \{x \in X : z \leq x \text{ for some } z \in c\}$ are concrete processes, called respectively the *head* and the *tail* of P with respect to c, and written respectively as $head(P, c)$ and $tail(P, c)$. $\sharp$

A proof is straightforward.

4.2. Definition. A process π is said to *consist* of a process π_1 *followed* by a process π_2, and we say that π_1 is a *prefix* of π, iff an instance P of π has a cross-section c such that $head(P, c)$ is an instance of π_1 and $tail(P, c)$ is an instance of π_2. $\sharp$

4.3. Proposition. For every two processes π_1 and π_2 such that $cod(\pi_1)$ and $dom(\pi_2)$ are defined and $cod(\pi_1) = dom(\pi_2)$ there exists a unique process, written as $\pi_1;\pi_2$, or as $\pi_1\pi_2$, that consists of π_1 followed by π_2. $\sharp$

Proof. Take $P_1 = (X_1, \leq_1, ins_1) \in \pi_1$ and $P_2 = (X_2, \leq_2, ins_2) \in \pi_2$ with $X_1 \cap X_2 = end(P_1) = origin(P_2)$ and with the restriction of P_1 to $end(P_1)$ identical with the restriction of P_2 to $origin(P_2)$, and equip $X_1 \cup X_2$ with the least common extension of the flow orders and labellings of P_1 and P_2.

Let P be the lposet thus obtained. It suffices to prove that P is a process and notice that $head(P, c) = P_1$ and $tail(P, c) = P_2$.

In order to prove that P is a process it suffices to show that P does not contain a segment with isomorphic proper subsegment. To this end suppose the contrary.

Suppose that $f : Q \to R$ is an isomorphism from a segment Q of P to a proper subsegment R of Q, where Q consists of a part Q_1 contained in P_1 and a part Q_2 contained in P_2. By applying twice the method described in the proof of 3.7 we can modify f to an isomorphism $f' : Q \to R$ such that the image of Q_1 under f', say R_1, is contained in Q_1, and the image of Q_2 under f', say R_2, is contained in

Q_2. As R is a proper subsegment of Q, one of these images, say R_1, is a proper part of the respective Q_i. By taking the greatest lower bounds and the least upper bounds of appropriate cross-sections we can extend Q_1 and R_1 to segments Q_1' and R_1' of P_1 such that R_1' is a proper subsegment of Q_1' and there exists an isomorphism from Q_1' to R_1'. This is in a contradiction with the fact that P_1 is a process and implies that P is a process. $\sharp$

4.4. Definition. The operation $(\pi_1, \pi_2) \mapsto \pi_1\pi_2$ is called the *sequential composition* of processes. $\sharp$

The parallel composition allows one to combine processes with disjoint sets of involved objects. It can be defined as follows.

4.5. Definition. Given a concrete process $P = (X, \leq, ins)$, by a *splitting* of P we mean an ordered pair $s = (X^F, X^S)$ of two disjoint subsets X^F and X^S of X such that $X^F \cup X^S = X$, $x' \leq x''$ only if x' and x'' are both in one of these subsets. $\sharp$

4.6. Proposition. For each splitting $s = (X^F, X^S)$ of a concrete process $P = (X, \leq, ins)$, the restrictions of P to the subsets X^F and X^S are concrete processes, called respectively the *first part* and the *second part* of P with respect to s, and written respectively as $\mathit{first}(P, s)$ and $\mathit{second}(P, s)$. $\sharp$

A proof is straightforward.

4.7. Definition. A process π is said to *consist* of two *parallel* processes π_1 and π_2 iff its instance P has a splitting s such that $\mathit{first}(P, s)$ is an instance of π_1 and $\mathit{second}(P, s)$ is an instance of π_2. $\sharp$

4.8. Proposition. For every two processes π_1 and π_2 such that $objects(\pi_1) \cap objects(\pi_2) = \emptyset$ there exists a process π with an instance P that has a splitting s such that $\mathit{first}(P, s)$ is an instance of π_1 and $\mathit{second}(P, s)$ is an instance of π_2. If such a process π exists then it is unique, we write it as $\pi_1 + \pi_2$, and we say that the processes π_1 and π_2 are *parallel*. $\sharp$

For a proof it suffices to take $P_1 = (X_1, \leq_1, ins_1) \in \pi_1$ and $P_2 = (X_2, \leq_2, ins_2) \in \pi_2$ with $X_1 \cap X_2 = \emptyset$, and to equip $X_1 \cup X_2$ with the least common extension of the flow orders and labellings of P_1 and P_2.

4.9. Definition. The operation $(\pi_1, \pi_2) \mapsto \pi_1 + \pi_2$ is called the *parallel composition* of processes. $\sharp$

The introduced operations on processes allow one to represent complex processes in terms of their components. For example, in the case of processes in 3.6 we can represent $[L]$ as $[D']([Q'] + [R'])[D'']$. They allow one to turn the sets $PROC(U)$ and $Proc(U)$ into algebras.

4.10. Definition. We call $\mathbf{PROC}(U) = (PROC(U), dom, cod, ;, +, 0)$ the *algebra of processes* in U. We call the restriction of this algebra to the subset $Proc(U)$ of $PROC(U)$ the *algebra of bounded processes* in U and write it as $\mathbf{Proc}(U) = (Proc(U), dom, cod, ;, +, 0)$. $\sharp$

The following theorems follow easily from the definitions of operations (cf. [18] for proofs of similar theorems for processes of Petri nets).

4.11. Proposition. The reduct $(PROC(U), dom, cod, ;)$ of the algebra $\mathbf{PROC}(U)$ is a partial category $pcat(\mathbf{PROC}(U))$ such that if $\sigma\tau$ is an identity then σ and τ are also identities. $\sharp$

4.12. Proposition. The reduct $(PROC(U), +, 0)$ of the algebra $\mathbf{PROC}(U)$ is a partial commutative monoid $pmon(\mathbf{PROC}(U))$, and it enjoys the following properties:

(1) if $\pi + \sigma$ and $\pi + \sigma'$ are defined and $\pi + \sigma = \pi + \sigma'$ then $\sigma = \sigma'$,

(2) $\pi + \pi$ is defined only for $\pi = 0$,

(3) given a family $(\pi_i : i \in \{1, ..., n\})$, where $n \geq 2$, if $\pi_i + \pi_j$ are defined for all $i, j \in \{1, ..., n\}$ such that $i \neq j$ then $\pi_1 + ... + \pi_n$ is defined,

(4) the following relation $\sqsubseteq$ is a partial order:

$\pi_1 \sqsubseteq \pi_2$ iff π_2 contains π_1 in the sense that $\pi_2 = \pi_1 + \rho$ for some ρ,

(5) for all π_1 and π_2 there exists the greatest lower bound of π_1 and π_2 with respect to $\sqsubseteq$, written as $\pi_1 \sqcap \pi_2$,

(6) if $\pi_1 + \pi_2$ is defined then $(\pi_1 \sqcap \sigma) + (\pi_2 \sqcap \sigma)$ is defined and $(\pi_1 \sqcap \sigma) + (\pi_2 \sqcap \sigma) = (\pi_1 + \pi_2) \sqcap \sigma$,

(7) if $\pi_1 \sqcap \pi_2 = 0$ and $\pi_1 \sqsubseteq \pi$ and $\pi_2 \sqsubseteq \pi$ for some π then $\pi_1 + \pi_2$ is defined,

(8) each $\pi \neq 0$ contains some α that is a $(+)$-*atom* in the sense that $\alpha \neq 0$ and $\alpha = \pi_1 + \pi_2$ only if either $\pi_1 = \alpha$ and $\pi_2 = 0$ or $\pi_1 = 0$ and $\pi_2 = \alpha$; in particular, each identity of the partial category $pcat(\mathbf{PROC}(U))$ contains a $(+)$-*atom* and this $(+)$-*atom* is an identity of $pcat(\mathbf{PROC}(U))$, called an *atomic identity*.

(9) each π is determined uniquely by the set $h(\pi)$ of $(+)$-atoms it contains in the sense that $h(\pi_1) = h(\pi_2)$ implies $\pi_1 = \pi_2$; in particular, each identity u is determined uniquely by the set $h(u)$ of atomic identities it contains. $\sharp$

.

4.13. Proposition. The reducts $pcat(\mathbf{PROC}(U))$ and $pmon(\mathbf{PROC}(U))$ are related such that:

(1) $dom(\pi_1 + \pi_2)$ and $dom(\pi_1) + dom(\pi_2)$ are defined and $dom(\pi_1 + \pi_2) = dom(\pi_1) + dom(\pi_2)$ whenever $\pi_1 + \pi_2$, $dom(\pi_1)$, $dom(\pi_2)$ are defined,

(2) $cod(\pi_1 + \pi_2)$ and $cod(\pi_1) + cod(\pi_2)$ are defined and $cod(\pi_1 + \pi_2) = cod(\pi_1) + cod(\pi_2)$ whenever $\pi_1 + \pi_2$, $cod(\pi_1)$, $cod(\pi_2)$ are defined,

(3) $dom(\pi) = 0$ implies $\pi = 0$ and $cod(\pi) = 0$ implies $\pi = 0$,

(4) if $(\pi_{11}\pi_{12}) + (\pi_{21}\pi_{22})$ is defined then $\pi_{11} + \pi_{21}$, $\pi_{11} + \pi_{22}$, $\pi_{12} + \pi_{21}$, $\pi_{12} + \pi_{22}$ are also defined and $(\pi_{11}\pi_{12}) + (\pi_{21}\pi_{22}) = (\pi_{11} + \pi_{21})(\pi_{12} + \pi_{22})$,

(5) if $\pi_{11}\pi_{12}$ and $\pi_{21}\pi_{22}$ are defined, and $\pi_{11} + \pi_{21}$ is defined, or $\pi_{11} + \pi_{22}$ is defined, or $\pi_{12} + \pi_{21}$ is defined, or $\pi_{12} + \pi_{22}$ is defined, then $(\pi_{11}\pi_{12}) + (\pi_{21}\pi_{22})$ is defined,

(6) $\pi_1 + \pi_2 = \sigma_1\sigma_2$ implies the existence of unique π_{11}, π_{12}, π_{21}, π_{22} such that $\pi_1 = \pi_{11}\pi_{12}$, $\pi_2 = \pi_{21}\pi_{22}$, $\sigma_1 = \pi_{11} + \pi_{21}$, $\sigma_2 = \pi_{12} + \pi_{22}$. $\sharp$

4.14. Proposition. In $pmon(\mathbf{PROC}(U))$ there exists the least congruence $\sim$ such that, $\pi \sim dom(\pi)$ for all π such that $dom(\pi)$ is defined, and $\pi \sim cod(\pi)$ for all π such that $cod(\pi)$ is defined. $\sharp$

4.15. Proposition. A diagram $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ is a bicartesian square in $pcat(\mathbf{PROC}(U))$ if and only if there exist c, φ_1, φ_2 such that c is an identity, $c+\varphi_1+\varphi_2$ is defined, $\pi_1 = c+\varphi_1+dom(\varphi_2)$, $\pi_2 = c + dom(\varphi_1) + \varphi_2$, $\pi_1' = c + \varphi_1 + cod(\varphi_2)$, $\pi_2' = c + cod(\varphi_1) + \varphi_2$. $\sharp$

4.16. Proposition. For all ξ_1, ξ_2, η_1, η_2 such that $\xi_1\xi_2 = \eta_1\eta_2$ there exist unique σ_1, σ_2, and a unique bicartesian square $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$, such that $\xi_1 = \sigma_1\alpha_1$, $\xi_2 = \pi_2'\sigma_2$, $\eta_1 = \sigma_1\pi_2$, $\eta_2 = \pi_1'\sigma_2$. $\sharp$

4.17. Proposition. The restriction $\mathbf{KPROC}(U)$ of $\mathbf{PROC}(U)$ to the subset $KPROC(U)$ of K-dense processes enjoys the following property:

> Given π such that $dom(\pi)$ contains an atomic identity p and $cod(\pi)$ contains an atomic identity q, if π cannot be represented as $(p + \pi_1)(q + \pi_2)$ then for every ξ and η such that $\pi = \xi\eta$ the state $cod(\xi) = dom(\eta)$ contains an atomic identity m such that ξ cannot be represented as $(p + \xi_1)(m + \xi_2)$ and η cannot be represented as $(m + \eta_1)(q + \eta_2)$. $\sharp$

4.18. Proposition. The algebra $\mathbf{Proc}(U) = (Proc(U), dom, cod, ;, +, 0)$ of bounded processes in U and its restriction $\mathbf{KProc}(U)$ to the subset $KProc(U)$ of K-dense bounded processes enjoy all the properties stated in 4.11 - 4.17. Moreover, its reduct $pcat(\mathbf{Proc}(U))$ is a category and it enjoys the following properties:

(1) if $\sigma\pi$ and $\sigma'\pi$ are defined and $\sigma\pi = \sigma'\pi$ then $\sigma = \sigma'$,

(2) if $\pi\tau$ and $\pi\tau'$ are defined and $\pi\tau = \pi\tau'$ then $\tau = \tau'$,

(3) if $\sigma\pi\tau$ is defined and $\sigma\pi\tau = \pi$ then σ and τ are identities. $\sharp$

Proof. It suffices to prove the last part. To this end we proceed as follows.

From 3.8 we obtain that in the case of processes $\sigma\pi = \sigma'\pi$ implies $\sigma = \sigma'$. Indeed, if i is an isomorphism from an instance Q of $\sigma\pi$ to an instance Q' of $\sigma'\pi$, where $S = head(Q, c)$ is an instance of σ, $P = tail(Q, c)$ is an instance of π, $S' = head(Q', c')$ is an instance of σ', $P' = tail(Q', c')$ is an instance of π, and j is an isomorphism from P' to P, then P' is isomorphic to the image of P under i and, consequently, the composite $j \circ (i|P)$ has an extension to an automorphism k of Q'. Hence S' is isomorphic to the image of S under i and thus to S, too, and this implies $\sigma = \sigma'$.

Similarly, $\pi\tau = \pi\tau'$ implies $\tau = \tau'$.

From (3.3) of 3.4 we obtain also that if $\sigma\pi\tau$ is defined and $\sigma\pi\tau = \pi$ then σ and τ are identities. $\sharp$

In particular, the structure $\mathbf{Proc}(U) = (Proc(U), dom, cod, ;, +, 0)$ is a behaviour algebra in the sense of [Wink 06b] and the algebra $\mathbf{Proc}(U) = (Proc(U), dom, cod, ;, +, 0)$ of K-dense bounded processes is a subalgebra of this algebra.

Algebras of processes in universa of objects are domains whose subsets may serve to represent processes possible in concrete systems. For example, the algebra $\mathbf{Proc}(U_1)$, where U_1 is the universe in 3.2, contains the subalgebra generated by all the possible variants of processes $[B]$, $[I]$, $[J]$ in 3.5, and the underlying set of this subalgebra can be regarded as the set of bounded processes in the system consisting of the machine M and the switch S. Some subalgebras of algebras of processes can be interpreted as restrictions of the monoidal categories of concatenable processes of P/T Petri nets as in [7].

5. Endowing processes with structures

Now we want to show how some processes can be endowed with additional structures.

By structures we mean slightly modified versions of structures in the sense of Bourbaki's Elements (cf [4]). We define them as follows.

Let $\mathbf{Ens}$ and $\mathbf{BijEns}$ denote respectively the category of sets and mappings and the category of sets and bijective mappings. Let $\mathcal{P} : \mathbf{Ens} \to \mathbf{Ens}$ be the powerset functor, i.e. the fuctor such that $\mathcal{P}(X)$ is the set of subsets of X and $(\mathcal{P}(f))(Z) = f(Z)$ for every mapping $f : X \to X'$ and every $Z \subseteq X$. Let $\times : \mathbf{Ens} \times \mathbf{Ens} \to \mathbf{Ens}$ be the bifunctor of cartesian product, i.e. the functor such that $\times(X, Y)$ is the cartesian product $X \times Y$ of X and Y and $(\times(f, g))(x, y) = (f(x), g(y))$ for every mappings $f : X \to X'$, $g : Y \to Y'$ and every $(x, y) \in X \times Y$. For every set A let A denotes the constant functor from $\mathbf{Ens}$ to $\mathbf{Ens}$, i.e., the functor that assigns the set A to every set X and the identity of A to every mapping $f : X \to X'$.

5.1. Definition. By a *structure form* we mean a functor $F : \mathbf{Ens} \to \mathbf{Ens}$ that can be built from the identity functor and constant functors using the powerset functor $\mathcal{P} : \mathbf{Ens} \to \mathbf{Ens}$ and the bifunctor $\times : \mathbf{Ens} \times \mathbf{Ens} \to \mathbf{Ens}$ of cartesian product. $\sharp$

5.2. Definition. Given a structure form F, by a *structure* of the form F on a set X we mean an element S of the set $F(X)$. $\sharp$

For example, a binary relation ρ on a set X is a structure of the form $BREL : X \mapsto \mathcal{P}(X \times X)$, a topology τ on a set X is a structure of the form $\mathcal{T} : X \mapsto \mathcal{P}(\mathcal{P}(x))$ on X, etc.

5.3. Definition. Given a structure form F, by a *morphism* from a structure $S \in F(X)$ of the form F on X to a structure $S' \in F(X')$ of the same form F on X' we mean an injection $f : X \to X'$ such that S' is the image of S under the mapping $F(f)$. $\sharp$

By $\mathbf{STR}(F)$ we denote the category of structures of a form F and their morphisms.

5.4. Definition. By a *structure type* we mean a pair $T = (F, G)$, where F is a structure form $F : \mathbf{Ens} \to \mathbf{Ens}$ and G is a functor $G : \mathbf{BijEns} \to \mathbf{BijEns}$ such that $G(b) = F(b)$ for every bijection $b : X \to X'$ and $G(X) \subseteq F(X)$ for every set X (cf. [5]). $\sharp$

For example, the type of partial orders can be defined as the pair $PO = (BREL, Po)$, where $Po : \mathbf{BijEns} \to \mathbf{BijEns}$ with $Po(X)$ being the set of partial orders on X.

By $\mathbf{STRUCT}(T)$ we denote the category of structures of type T.

Let $U = (W, V, ob)$ be a universe of objects.

Given a subalgebra $\mathcal{A} = (A, dom, cod, ;, +, 0)$ of the algebra $\mathbf{KProc}(U)$ of K-dense bounded processes in U, each instance of each process of $\mathcal{A}$ can be endowed with a structure of type T on its underlying set. However, the choice of such a structure cannot be arbitrary since processes of the subalgebra $\mathcal{A}$ and their instances can be related and then we expect also the corresponding structures to be related in a similar way. Consequently, we propose to formalize such a choice by assigning to each process $\pi \in A$ a canonical instance $P(\pi) = (X_\pi, \leq_\pi, l_\pi)$, by endowing the assigned instances with a suitable structures str_π in a way consistent with the operations on processes, and by transporting the structures thus introduced from the canonical instances of processes to arbitrary instances with the aid of the respective isomorphisms. This can be done as follows (cf. [19]).

Let $\pi \in A$ be a process.

5.5. Definition. By a *cut* of π we mean a pair (π_1, π_2) such that $\pi_1\pi_2 = \pi$. $\sharp$

Cuts of every $\pi \in A$ are partially ordered by the relation $\preceq_\pi$, where $x \preceq_\pi y$ with $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ means that $\eta_1 = \xi_1\delta$ with some δ. From 4.18 it follows that $\preceq_\pi$ is a partial order, and that for $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ such that $x \preceq_\pi y$ there exists a unique δ such that $\eta_1 = \xi_1\delta$, written as $x \to y$. From 4.16 it follows that this partial order makes the set of cuts of π a lattice L_π. Given two cuts x and y, by $x \bigtriangledown_\pi y$ and $x \bigtriangleup_\pi y$ we denote respectively the least upper bound and the greatest lower bound of x and y. From (A6) it follows that $(x \leftarrow x \bigtriangleup_\pi y \to y, x \to x \bigtriangledown_\pi y \leftarrow y)$ is a bicartesian square.

Let $P = (X, \leq, ins)$ be an instance of π.

5.6. Lemma. There exists a bijective correspondence $\lambda_{\pi,P}$ between cuts of π and cross-sections of P. $\sharp$

For a proof it suffices to apply 3.7.

Given a cut $x = (\xi_1, \xi_2)$ of π and an atomic identity p, we say that p *occurs* in x and call (x, p) an *occurrence* of p in x if p is contained in $cod(\xi_1) = dom(\xi_2)$.

Given an occurrence (x, p) of an atomic identity p in a cut $x = (\xi_1, \xi_2)$ of π and an occurrence (y, q) of an atomic identity q in a cut $y = (\eta_1, \eta_2)$ of π, we say that these occurrences are *adjoint* and write $(x, p) \sim_\pi (y, q)$ if $p = q$ and $p \sqsubseteq (x \bigtriangleup_\pi y \to x \bigtriangledown_\pi y)$, that is if $p = q$ and $(x \bigtriangleup_\pi y \to x \bigtriangledown_\pi y) = c + \varphi_1 + \varphi_2$ with an identity c that contains p and with $(x \bigtriangleup_\pi y \to x) = c + \varphi_1 + dom(\varphi_2)$, $(x \bigtriangleup_\pi y \to y) = c + dom(\varphi_1) + \varphi_2$, $(y \to x \bigtriangledown_\pi y) = c + \varphi_1 + cod(\varphi_2)$, $(x \to x \bigtriangledown_\pi y) = c + cod(\varphi_1) + \varphi_2$.

5.7. Lemma. To every occurrence (x, p) of an object instance p there corresponds a unique element $\mu_{\pi,P}(x, p)$ of the cross-section $\lambda_{\pi,P}(x)$ such that $ins(\mu_{\pi,P}(x, p)) = p$. $\sharp$

A proof is immediate.

5.8. Lemma. Occurrences (x, p) and (y, q) of object instances are adjoint iff $\mu_{\pi,P}(x, p) = \mu_{\pi,P}(y, q)$. $\sharp$

A proof follows easily due to 5.6, 5.7, 4.15, and 4.16.

5.9. Corollary. The relation $\sim_\pi$ is an equivalence relation. $\sharp$

The elements of the underlying set X_π of the canonical instance of a process π can be defined as equivalence classes of $\sim_\pi$.

5.10. Definition. Given an atomic identity p, by an *occurrence* of p in π we mean an equivalence class of occurrences of p in cuts of π. $\sharp$

5.11. Definition. The set of occurrences of atomic identities in π, written as X_π, is called the *canonical underlying set* of π. $\sharp$

5.12. Definition. The correspondence $[(x, p)] \mapsto p$ between occurrences of atomic identities in π and the atomic identities themselves, written as ins_π, is called the *canonical labelling* of (occurrences of atomic identities in) π. $\sharp$

The partial order $\leq_\pi$ on X_π can be defined as follows.

Given an occurrence (x, p) of an atomic identity p in a cut $x = (\xi_1, \xi_2)$ of π and an occurrence (y, q) of an atomic identity q in a cut $y = (\eta_1, \eta_2)$ of π, we say that (x, p) *precedes* (y, q) and write $(x, p) <_\pi (y, q)$ if $x \preceq_\pi y$, p occurs in x, q occurs in y, and there is no cut v of $x \to y$ such that $(x, p) \sim_\pi (v, p)$ and $(y, q) \sim_\pi (v, q)$.

5.13. Lemma. The relation $(x, p) <_\pi (y, q)$ holds iff $\mu_{\pi,P}(x, p) < \mu_{\pi,P}(y, q)$. $\sharp$

A proof follows from the definition of $(x, p) <_\pi (y, q)$ due to the K-density of P.

5.14. Corollary. For each $\pi \in A$ the relation $\leq_\pi$ on X_π, where $u \leq_\pi v$ iff $u \sim_\pi v$ or $(x, p) <_\pi (y, q)$ for some $(x, p) \in u$ and $(y, q) \in v$, is a partial order. $\sharp$

5.15. Definition. The partial order $\leq_\pi$ is called the *canonical partial order* of (occurrences of atomic identities in) π. The triple $P(\pi) = (X_\pi, \leq_\pi, l_\pi)$ is called the *canonical instance* of π. $\sharp$

It is straightforward that the correspondence $P : \pi \mapsto (X_\pi, \leq_\pi, l_\pi)$ just described between processes of $\mathbf{KProc}(U)$ and their canonical instances enjoys the following properties.

5.16. Lemma. If $\gamma = \alpha + \beta$ then $P(\gamma)$ is a coproduct object in $\mathbf{KProc}(U)$ of $P(\alpha)$ and $P(\beta)$ with the canonical morphisms given by the correspondences

$$i_{\alpha,\alpha+\beta} : [((\xi_1, \xi_2), p)] \mapsto [((\xi_1 + dom(\beta), \xi_2 + \beta), p)]$$

$$i_{\beta,\alpha+\beta} : [((\eta_1, \eta_2), p)] \mapsto [((dom(\alpha) + \eta_1, \alpha + \eta_2), p)] \quad \sharp$$

5.17. Lemma. If $\gamma = \alpha\beta$ with $cod(\alpha) = dom(\beta) = c$ then $P(\gamma)$ is the pushout object in $\mathbf{KProc}(U)$ of the injections of $P(c)$ in $P(\alpha)$ and in $P(\beta)$ given by

$$k_{c,\alpha} : [((c, c), p] \mapsto ((\alpha, c), p)]$$
$$k_{c,\beta} : [((c, c), p] \mapsto [((c, \beta), p]$$

with the canonical morphisms given by the correspondences

$$j_{\alpha,\alpha\beta} : [((\xi_1, \xi_2), p)] \mapsto [((\xi_1, \xi_2\beta), p)]$$

$$j_{\beta,\alpha\beta} : [((\eta_1, \eta_2), p)] \mapsto [((\alpha\eta_1, \eta_2), p)] \quad \sharp$$

This suggests that structures for the canonical instances of processes should be related as follows to the structures for the canonical instances of the components of these processes.

5.18. Definition. Processes of a subalgebra $\mathcal{A} = (A, dom, cod, ;, +, 0)$ of the algebra $\mathbf{KProc}(U)$ are said to be consistently endowed with structures of type T if there exists a correspondence $\pi \mapsto str_\pi$ such that, for every $\pi \in A$, str_π is a structure of type T on the canonical underlying set X_π of π and the following conditions are fulfilled:

(1) if $\alpha + \beta$ is defined then $str_{\alpha+\beta}$ is the coproduct object in $\mathbf{STRUCT}(T)$ of str_α and str_β with the canonical injections $i_{\alpha,\alpha+\beta}$ and $i_{\beta,\alpha+\beta}$ as in 5.16,

(2) if $\alpha\beta$ is defined and $cod(\alpha) = dom(\beta) = c$ then $str_{\alpha\beta}$ is the pushout object in $\mathbf{STRUCT}(T)$ of the injections $k_{c,\alpha}$ and $k_{c,\beta}$ of str_c in str_α and in str_β as in 5.17 with the canonical injections $j_{\alpha,\alpha\beta}$ and $j_{\beta,\alpha\beta}$ as in 5.17. $\sharp$

Examples that follow illustrate the idea.

Let LPO be the structure type of labelled partial orders. Let $\mathcal{A} = (A, dom, cod, ;, +, 0)$ be a subalgebra of the algebra $\mathbf{KProc}(U)$. To each process π of $\mathcal{A}$ we can assign the structure $lpo_\pi = (\leq_\pi, l_\pi)$ on the canonical underlying set X_π. Then 5.16 and 5.17 imply that the correspondence $\pi \mapsto lpo_\pi$ fulfils the conditions (1) and (2) of 5.18 for the structure type LPO.

Let WPO be the structure type of weighted partial orders defined as pairs $wpo = (\leq, d)$, where $\leq$ is a partial order on a set X and $d : X \times X \to Real \cup \{-\infty, +\infty\}$ is a function such that

(a) $d(x, x) = 0$,

(b) $d(x, y) = -\infty$ if x and y are incomparable with respect to $\leq$,

(c) $d(x, y) = sup\{d(x, z) + d(z, y) : z \neq x, z \neq y, x \leq z \leq y\}$ if there exists z such that $z \neq x$, $z \neq y$, $x \leq z \leq y$.

Let $\mathcal{A} = (A, dom, cod, ;, +, 0)$ a subalgebra of the algebra $\mathbf{KProc}(U)$ generated by a set A_0 of $(+, ;)$-atoms, where by $(+, ;)$-atoms we mean processes that are indecomposable in a nontrivial manner with respect to the operations "+" and ";". Then to each process π of the subalgebra $\mathcal{A}$ we can assign a structure $wpo_\pi = (\leq_\pi, d_\pi)$ of the type WPO. To this end it suffices to define d_π on $(+, ;)$-atoms generating $\mathcal{A}$ and then extend it on entire $\mathcal{A}$ such that the conditions (1) and (2) of 5.18 are fulfilled for the structure type WPO. Values of functions d_π can be interpreted as delays between elements of the canonical underlying set X_π of π. Together with data about occurrence times of minimal elements of X_π they determine occurrence times of all elements of X_π. For instance, in the case of a process π with a linear flow order the occurrence time of each $x \in X_\pi$ is $t' + d_\pi(x', x)$, where x' is the minimal element of X_π and t' is the occurrence time of x'.

Let $ABREL$ be the structure type of acyclic binary relations. Let $\mathcal{A} = (A, dom, cod, ;, +, 0)$ be a subalgebra of $\mathbf{Proc}(U)$ generated by a set A_0 of processes which need not to be $(+, ;)$-atoms. Suppose that we can assign to each process $\pi \in A_0$ an acyclic binary relation cxt_π on X_π (a *context relation* in the sense of [17]) such that, for all elements of X_π, $(x, y) \in cxt_\pi$ excludes both $x \leq_\pi y$ and $y \leq_\pi x$, and the reflexive and transitive closure of the following relation R, where cxt_π^+ denotes the transitive closure of cxt_π, is a partial order:

$$(x, y) \in R \text{ iff } x \leq_\pi y \text{ or } (x <_\pi z \text{ and } (z, y) \in cxt_\pi^+ \text{ for some } z)$$
$$\text{or } (x <_\pi t \text{ and } z <_\pi y \text{ and } (z, t) \in cxt_\pi \text{ for some } z \text{ and } t).$$

Then we can extend the correspondence $\pi \mapsto cxt_\pi$ on instances of processes from A such that the conditions (1) and (2) of 5.18 are fulfilled for the structure type $ABREL$.

For instance, in the case of processes in 3.5 and their combinations, we can consider the subalgebra generated by variants of $([B] + \{(S, ON)\})$, $[I]$, $[J]$, and endow $([B] + \{(S, ON)\})$ with a context relation as it is illustrated in Figure 3 with a dotted arrow.

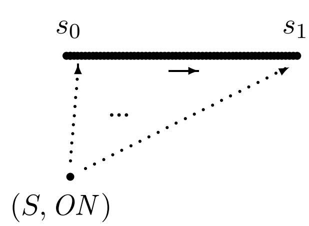

Figure 3: $[B] + \{(S, ON)\}$ endowed with a context relation

6. Random behaviours

Processes representing runs of a system with random behaviour can be regarded as effects of activities of a random mechanism. We should think of such processes as of elements of a suitable probability space. Now we want to define such a space.

Let $U = (W, V, ob)$ be a universe of objects. Let $\mathcal{A} = (A, dom, cod, ;)$ be a subalgebra of the partial category of global processes in U. Let $\Omega(\mathcal{A})$ be the set of all those global processes from $\mathcal{A}$ which have a source and are not proper prefixes of other processes, where by a proper prefix of a process we mean a prefix that is not identical with the process itself. Our aim is to show how to endow $\Omega(\mathcal{A})$ with a reasonable σ-field $\mathcal{F}$ of subsets and with a reasonable probability measure μ on this σ-field. Our idea is to define $\mathcal{F}$ and μ from σ-fields and probability measures characterizing bounded segments of the represented behaviour.

First of all, we have to define a partially ordered set that might play the role of a time scale. This can be done as follows.

6.1. Definition. Two processes from $\mathcal{A}$ are said to be *confluent* iff they are prefixes of a process from $\mathcal{A}$. $\sharp$.

6.2. Definition. A set I of bounded processes from $\mathcal{A}$ is called a *confluence-free set* of processes iff it does not contain confluent processes. $\sharp$.

Note that each set of all states belonging to $\mathcal{A}$ is confluence-free.

From Kuratowski - Zorn Lemma we obtain the following property.

6.3. Proposition. Each confluence-free set of processes is contained in a maximal confluence-free set of processes. $\sharp$.

6.4. Definition. We say that a maximal confluence-free set I of processes *precedes* another such a maximal confluence-free set J, and we write $I \ll J$, iff each process from I is a prefix of a process from J. $\sharp$.

Note that the set of all states from $\mathcal{A}$ is a maximal confluence-free set of processes.

6.5. Proposition. The set of all maximal confluence-free sets of processes with the partial order $\ll$ is a directed set $\mathcal{T}(\mathcal{A})$. $\sharp$.

For a proof it suffices to consider two maximal confluence-free sets of processes, to make their union confluence-free by replacing every possible pair of confluent processes by the least processes that has the component processes of the pair as prefixes, and to extend the set thus obtained to a maximal confluence-free set of processes. The existence of the respective least processes follows from 2.2 and 4.18.

Now, assuming the directed set $\mathcal{T}(\mathcal{A})$ as a time scale we think of the required probability space $\mathcal{X} = (\Omega(\mathcal{A}), \mathcal{F}, \mu)$ as of a limit in a sense of a directed family $\mathbf{X} = (\mathcal{X}_I : I \in \mathcal{T}(\mathcal{A}))$ of simpler probability spaces $\mathcal{X}_I = (\Omega_I, \mathcal{F}_I, \mu_I)$. As members of such a family are supposed to approximate $\mathbf{X}$ with a growing accuracy, we require the family to be consistent in the following sense.

6.6. Definition. A family $\mathbf{X} = (\mathcal{X}_I : I \in \mathcal{T}(\mathcal{A}))$ of probability spaces $\mathcal{X}_I = (\Omega_I, \mathcal{F}_I, \mu_I)$ is said to be *consistent* iff the following conditions are fulfilled:

(1) $\Omega_I = I$ for all $I \in \mathcal{T}(\mathcal{A})$,

(2) for every I and J such that $I \ll J$, for the relation $pref_{IJ} \subseteq \Omega_I \times \Omega_J$, where $i pref_{IJ} j$ iff i is a prefix of j, and for every $F \in \mathcal{F}_I$, the image $F pref_{IJ} = \{j \in \Omega_J : i pref_{IJ} j \text{ for some } i \in F\}$ of F under $pref_{IJ}$ is a member of $\mathcal{F}_J$ and $\mu_I(F) = \mu_J(F pref_{IJ})$. $\sharp$

The required σ-field $\mathcal{F}$ is defined for a consistent family of probability spaces by means of subsets called cylinders.

6.7. Definition. Given a consistent family $\mathbf{X} = (\mathcal{X}_I : I \in \mathcal{T}(\mathcal{A}))$ of probability spaces $\mathcal{X}_I = (\Omega_I, \mathcal{F}_I, \mu_I)$, by an $(\mathbf{X}, I)$-*cylinder* of $\Omega(\mathcal{A})$ we mean each $C(F) = \{\omega \in \Omega(\mathcal{A}) : \omega = \xi\eta \text{ for some } \xi \in F\}$ such that $F \in \mathcal{F}_I$, by $\mathcal{C}(\mathbf{X})$ we denote the set the set of all $(\mathbf{X}, I)$-cylinders, by $\mathcal{F}(\mathbf{X})$ we denote the σ-field generated by the set $\mathcal{C}(\mathbf{X})$, and we define $\mathcal{F}$ as $\mathcal{F}(\mathbf{X})$. $\sharp$.

Finally, the required probability measure μ is defined for a consistent family of probability spaces by transporting the probability measures of the probability spaces of the family from the σ-fields of the family to the corresponding cylinders and by extending the function thus obtained to a probability measure on entire $\mathcal{F}$, if such an extension exists.

6.8. Definition. Given a consistent family $\mathbf{X} = (\mathcal{X}_I : I \in \mathcal{T}(\mathcal{A}))$ of probability spaces $\mathcal{X}_I = (\Omega_I, \mathcal{F}_I, \mu_I)$,
by the *combination* of the probability measures μ_I we mean the real valued function $\mu(\mathbf{X})$ defined as follows on the set $\mathcal{C}(\mathbf{X})$ of $(\mathbf{X}, I)$-cylinders:

$$(\mu(\mathbf{X}))(C(F)) = \mu_I(F) \text{ for } F \in \mathcal{F}_I$$

If the function thus defined has an extension μ to a probability measure on entire $\mathcal{F}$ then this extension is unique and we call the probability space $\mathcal{X} = (\Omega(\mathcal{A}), \mathcal{F}, \mu)$ a *random behaviour* in $\mathcal{A}$ defined by $\mathbf{X}$. $\sharp$.

Conversely, to each probability space $\mathcal{X} = (\Omega(\mathcal{A}), \mathcal{F}, \mu)$ with the underlying set $\Omega(\mathcal{A})$ and a σ-field $\mathcal{F}$ that is generated by a suitable family $(\mathcal{G}_I : I \in \mathcal{T}(\mathcal{A}))$ of σ-fields there corresponds a consistent family of probability spaces that defines $\mathcal{X}$.

6.9. Proposition. Let $\mathcal{X} = (\Omega(\mathcal{A}), \mathcal{F}, \mu)$ be a probability space with the underlying set $\Omega(\mathcal{A})$ and a σ-field $\mathcal{F}$ that is generated by a family $(\mathcal{G}_I : I \in \mathcal{T}(\mathcal{A}))$ of σ-fields that enjoys the following properties:

(1) $\mathcal{G}_I \subseteq \mathcal{G}_J$ whenever $I \ll J$,

(2) for every $I \in \mathcal{T}(\mathcal{A})$, every set $G \in \mathcal{G}_I$ that contains an element of $\Omega(\mathcal{A})$ with a prefix belonging to I contains also every element of $\Omega(\mathcal{A})$ that has this prefix.

Then $\mathcal{X}$ is a random behaviour in $\mathcal{A}$ that is defined by a consistent family $\mathbf{X}$ of probability spaces. $\sharp$.

For a proof it suffices to notice that the correspondence between elements of $\Omega(\mathcal{A})$ and their prefixes belonging to $I \in \mathcal{T}(\mathcal{A})$ induces an isomorphism between $\mathcal{G}_I$ and a σ-field of subsets of I.

Though the idea of defining a random behaviour in $\mathcal{A}$ from a consistent family of probability spaces which characterize bounded segments of this behaviour is similar to the idea of defining classical stochastic processes, details are more sophisticated. Consequently, we cannot exploit directly the well known Kolmogoroff theorem on the existence of the required resulting probability measure (cf. [9]) and we have to prove the existence in each concrete case.

Moreover, also the probability spaces characterizing bounded segments of behaviours are reasonably simple only under some specific assumptions.

A natural reasonably simple class of random behaviours is the class whose members consists of independet increments. In our case this property of random behaviours means that the following equation is satisfied for all maximal confluence-free sets I and J such that the set $IJ = \{ij : i \in I \text{ and } j \in J\}$ is a maximal confluence-free set, and for all $F \in \mathcal{F}_{IJ}$:

$$\mu_{IJ}(F) = \int \mu_J(\{j : ij \in F\}|dom(j) = cod(i))d\mu_I$$

where $\mu_J(\{j : ij \in F\}|dom(j) = cod(i))$ denotes the respective conditional probability.

This equation allows us to define explicitly the probability measures μ_I for various systems that are discrete in the sense that their bounded global processes can be obtained by composing sequentially finitely many global processes which are $(;)$-atoms, i.e. elements indecomposable in a notrivial manner with respect to the sequential composition $(;)$.

7. Conclusions

It seems that algebras of processes in universa of objects and their subalgebras offer an adequate framework for modelling processes of hybrid systems. In particular, processes with rich internal structures can be represented as elements of suitable subalgebras of algebras of bounded processes in universa of objects, the elements consistently endowed with the respective structures as it is described in section 5. For example, elements of subalgebras generated by sets of atoms can be endowed with structures representing the flow of time. Processes with context-dependent actions as in [12] and [1] can be represented as elements of the subalgebra of the algebra of K-dense processes in a universe of objects that is generated by processes consisting of two concurrent components: one representing the proper action and the other representing the necessary context, each such a process endowed with an acyclic context relation. Finally, random behaviours of systems can be modelled by endowing sets of possible system runs with the structure of a probability space.

Acknowledgements. The author is grateful to the anonymous referees for their remarks which helped to improve the paper.

References

[1] Baldan, P., Bruni, R., Montanari, U., *Pre-nets, read arcs and unfolding: a functorial presentation*, Proceedings of WADT'02, Wirsing, M., Pattison, D., Hennicker, R., (Eds.), Springer LNCS 2755 (2002) 145-164

[2] Bergstra, J., Klop, J., *The algebra of recursively defined processes and the algebra of regular processes*, in Paradaens, J., (Ed.), Proc. of 11th ICALP, Springer LNCS 172 (1984) 82-95

[3] Best, E., Devillers, R., *Sequential and Concurrent Behaviour in Petri Net Theory*, Theoret. Comput. Sci. 55 (1987) 87-136

[4] Bourbaki, N., *Éléments de mathématique, Livre I (Théorie des ensembles), Chapitre 4 (Structures)*, Act. Sci. Ind. 1258, Hermann, Paris, 1957

[5] Bucur, I., Deleanu, A., *Introduction to the Theory of Categories and Functors*, John Wiley and Sons Ltd., Lozanna, New York, Sydney, 1968

[6] David, R., *Modeling of Dynamic Systems by Petri Nets*, in Proc. of European Control Conference, Grenoble, France, July 2-5 1991, 136-147

[7] Degano, P., Meseguer, J., Montanari, U., *Axiomatizing Net Computations and Processes*, in Proc. of 4th LICS Symposium, IEEE (1989) 175-185

[8] Droste, M., Shortt, R. M., *Continuous Petri Nets and Transition Systems*, in Ehrig, H., et al. (Eds.), Unifying Petri Nets, Springer LNCS 2128 (2001) 457-484

[9] Feller, W., *An Introduction to Probability Theory and its Applications, Volume II*, John Wiley and Sons, Inc. (1966)

[10] Milner, R., *A Calculus of Communicating Systems*, Springer LNCS 92 (1980)

[11] Milner, R., *Calculi of interaction*, Acta Informatica 33 (1996) 707-737

[12] Montanari, U., Rossi, F., *Contextual Nets*, Acta Informatica 32 (1995) 545-596

[13] Nerode, A., Kohn, W., *Models for Hybrid Systems: Automata, Topologies, Controllability, Observability*, Springer LNCS 736 (1993) 317-356

[14] Petri, C., A., *Non-Sequential Processes*, Interner Bericht ISF-77-5, Gesellschaft fuer Mathematik und Datenverarbeitung, 5205 St. Augustin, Germany (1977)

[15] Pluenecke, H., *K-density, N-density and finiteness properties*, APN 84, Springer LNCS 188 (1985) 392-412

[16] Rozenberg, G., Thiagarajan, P. S., *Petri Nets: Basic Notions, Structure, Behaviour*, in J. W. de Bakker, W. P. de Roever and G. Rozenberg (Eds.): Current Trends in Concurrency, Springer LNCS 224 (1986) 585-668

[17] Winkowski, J., *Towards a Framework for Modelling Systems with Rich Structures of States and Processes*, Fundamenta Informaticae 68 (2005), 175-206, http://www.ipipan.waw.pl/∼wink/winkowski.htm

[18] Winkowski, J., *An Axiomatic Characterization of Algebras of Processes of Petri Nets*, Fundamenta Informaticae 72 (2006), 407-420, http://www.ipipan.waw.pl/∼wink/winkowski.htm

[19] Winkowski, J., *Behaviour Algebras*, Fundamenta Informaticae 75 (2007), 537-560 http://www.ipipan.waw.pl/∼wink/winkowski.htm