AN ALGEBRAIC CHARACTERIZATION OF INDEPENDENCE OF PETRI NET PROCESSES

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Józef Winkowski
Instytut Podstaw Informatyki PAN
01-237 Warszawa, Ordona 21, Poland
e-mail: wink@ipipan.waw.pl

Abstract

The paper is concerned with processes of Petri nets. A partial operation on such processes is defined that allows one to concatenate processes whenever one process is a continuation of another. It is shown that for any Petri net as defined in the paper its set of processes equipped with this operation forms a category in which independence of processes can be characterized in a natural, purely algebraic way.

Key words: distributed systems, Petri nets, processes, concatenation, category, independence.

1 Introduction

A typical way of modelling computational systems is to specify their *states* and *transitions* between states, the latter possibly labelled in order to reflect the kind of *events* they represent. The respective models are called *transition systems*. They abstract away from the internal structure of states and transitions and formally are graphs. Their paths represent runs of the modelled systems.

However, this way of modelling may be insufficient to reflect the behaviour of systems in a world-wide network, that is systems which are distributed in the sense that their states and activities consist of more or less independent components. Better suited for modelling distributed systems are *Petri nets*, *asynchronous transition systems* and *transition systems with independence*.

Petri nets in their original form (cf. [Petri 62] and [Petri 80]) are essentially transition systems whose states are given by sets of *conditions*, and whose transitions correspond to events which depend on and affect only some conditions. Events with disjoint sets of related conditions are *independent* and may occur concurrently. Partially ordered structures which can be obtained by unfolding Petri nets, called *processes* (cf. [Petri 77]), represent runs of the modelled systems.

Asynchronous transition systems (cf. [Sh 85] and [Bedn 88]) reflect the independence of events, but, similarly to usual transition systems, they abstract away from the internal structure of states and transitions. Equivalence classes of their paths that are obtained by reducing paths to the corresponding sequences of events and ignoring the order of independent events, called *traces* (cf. [Maz 88]), represent runs of the modelled systems.

Finally, transition systems with independence (cf. [WN 95]) are models which reflect the independence of transitions without deriving it from the independence of events. Equivalence classes of their paths that are obtained by ignoring ways in which independent transitions change states represent runs of the modelled systems.

In this paper we exploit the fact that the sets of processes of Petri nets can be equipped with operations such that they form categories (cf. [Wink 80] and [Wink 82]), and we study algebraic properties of such categories. In particular, we extend the notion of independence of transitions on arbitrary processes, and we show that the extended notion can be characterized with the aid of purely algebraic means. Thus we show that the categories of processes of Petri nets are members of an axiomatically defined class of categories, and that in categories of this class it is possible to define independence. Moreover, the categories of the class thus obtained have sets of atomic generators and by reduction to these sets they become structures close to transition systems with independence.

2 Petri nets and their processes

We start with some preliminaries.

For a relation $R \subseteq X \times Y$ we write $(x, y) \in R$ as xRy, we define the inverse as the relation $R^{-1} = \{(y, x) : xRy\}$, and for $A \subseteq X$ and $B \subseteq Y$ we define AR and RB as the sets $\{y : aRy$ for some $a \in A\}$ and $\{x : xRb$ for some $b \in B\}$, respectively, and we write $\{a\}R$ and $R\{b\}$ as aR and Rb, respectively. For relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ we define the composition of R and S as the relation $RS = \{(x, z) : xRy$ and ySz for some $y \in Y\}$. In particular, for mappings $f : X \to Y$ and $g : Y \to Z$ we define the composition of f and g as the mapping $fg : x \mapsto g(f(x))$. For a relation $R \subseteq X \times X$ by R^+ and R^* we denote respectively the transitive and the reflexive and transitive closure of R. Finally, by a bicartecian square in a category we mean a diagram in this category given by $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ such that $(v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ is a pushout of $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w)$ and $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w)$ is a pullback of $(v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ (see Figure 1).

$$
\begin{tikzcd}
v \arrow[r, "\pi_2'"] & u' \\
u \arrow[u, "\pi_1"] \arrow[r, "\pi_2"'] & w \arrow[u, "\pi_1'"']
\end{tikzcd}
$$

Figure 1

A Petri net is a model of a concurrent system. Formally, such a net is defined as a bipartite directed graph (cf. [Petri 62], [Petri 80], [Re 85], [RT 86]).

2.1. Definition. A *Petri net*, or briefly a *net*, is a triple $N = (B, E, F)$, where

(1) B is a nonempty set of *conditions*,

(2) E is a set of *events* such that $B \cap E = \emptyset$,

(3) $F \subseteq (B \times E) \cup (E \times B)$ is a *flow relation* such that, for every event $e \in E$, the sets Fe and eF, called the set of *preconditions* and the set of *postconditions* of e, are nonempty. □

We use subscripts, B_N, E_N, F_N, when necessary.

Global states of the system represented by N are characterized by subsets of the set of conditions, called *markings*, A marking specifies those conditions which hold. Given a marking M and an event e, we say that e is *enabled* at M if $Fe \subseteq M$ and $eF - Fe$ is disjoint with M. In such a case we define $M' = (M - Fe) \cup eF$, say that e changes M to M', and write $M \overset{e}{\rightarrow} M'$.

By representing explicitly formal negations of conditions of a net we obtain a new model of the represented system - also in the form of a net.

2.2. Definition. Given a net $N = (B, E, F)$, a *safe completion* of N is the net $sc(N) = (\overline{B}, \overline{E}, \overline{F})$, where

(1) $\overline{B}$ is the set of conditions $b \in B$ and their negations $\overline{b}$,

(2) $\overline{E} = E$,

(3) $\overline{F}$ is the relation defined as follows:

$x\overline{F}y$ iff xFy or $x = \overline{b}$ for some $b \in yF - Fy$ or $y = \overline{b}$ for some $b \in Fx - xF$. □

Given $x \in \overline{B}$, by $|x|$ we denote that $b \in B$ for which $x = b$ or $x = \overline{b}$.

Global states of the represented system are such markings M of the net $sc(N)$ which are *consistent* in the sense that they do not contain subsets consisting of a condition $b \in B$ and its negation $\overline{b}$, and *complete* in the sense that for each condition $b \in B$ they contain either this condition or its negation $\overline{b}$.

A process of a net is a model of a run or of a segment of a run of the represented system. Formally, such a process is defined as a particular net together with a particular mapping to the safe completion of the original net (cf. [Petri 77], [RT 86], [BD 87], [DMM 89], [Eng 91]).

2.3. Definition. A *causal net* is a net $N = (B, E, F)$ such that

(1) F^*, the reflexive and transitive closure of the flow relation F, is a partial order $\leq$,

(2) for every $b \in B$ there exists at most one $e \in E$ satisfying eFb, and at most one $e' \in E$ satisfying bFe'.

We say that N is *finitary* if the set E is finite. □

2.4. Definition. A *homomorphism* from a net $N' = (B', E', F')$ to a net $N = (B, E, F)$ is a mapping $h : B' \cup E' \to B \cup E$ such that

(1) $h(B') \subseteq B$ and $h(E') \subseteq E$,

(2) for every $e' \in E'$, the restriction of h to $F'e'$ is a bijection between $F'e'$ and $Fh(e')$, and similarly for $e'F'$ and $h(e')F$. $\square$

As usual, an *isomorphism* is defined as a bijective homomorphism whose inverse is also a homomorphism.

2.5. Definition. A *concrete process* of a net N is a pair $P = (N', h)$, where N' is a causal net and h is a homomorphism from N' to $sc(N)$, the safe completion of N, such that, for all $b', b'' \in B_{N'}$, the relation $|h(b')| = |h(b'')|$ implies $b' \leq_{N'} b''$ or $b'' \leq_{N'} b'$. We say that such a process P is *global* if $|h(B_{N'})| = B_N$. We say that P is *finitary* if the underlying causal net N' is finitary. $\square$

Each $b' \in B_{N'}$ represents a *holding* of the condition $h(b')$ of $sc(N)$. Each $e' \in E_{sc(N)}$ represents an *occurrence* of the event $h(e')$ of $sc(N)$. According to (1) of 2.3, the reflexive and transitive closure of the flow relation $F_{N'}$ is a partial order $\leq_{N'}$. Maximal antichains of the partially ordered set $(B_{N'} \cup E_{N'}, \leq_{N'})$ that are contained in $B_{N'}$ are called *cuts* of P. They represent potential states of development of the represented process. They are partially ordered by the following relation
$c_1 \sqsubseteq c_2$ iff for every $b' \in c_1$ there exists $b'' \in c_2$ such that $b' \leq_{N'} b''$.
To each cut c there corresponds the set $h(c)$ of conditions of $sc(N)$. Due to the required in 2.5 property of the homomorphism h, for each condition b of N there is in c at most one b' such that $|h(b')| = b$. In particular, $h(c)$ is a consistent marking of $sc(N)$.

Let $P = (N', h)$ be a finitary concrete process of a net N.

The following properties of P are variants of well known facts from net theory and follow easily from definitions.

2.6. Proposition.; The partial order $\leq_{N'}$ is *K-dense*, that is each maximal chain of the partially ordered set $(B_{N'} \cup E_{N'}, \leq_{N'})$ has an element in each maximal antichain. $\square$

2.7. Proposition. The set of minimal elements of $(B_{N'} \cup E_{N'}, \leq_{N'})$, written as *origin*$(P)$, is a cut. Similarly, the set of maximal elements, written as *end*(P), is a cut. $\square$

2.8. Proposition. The set of cuts of P, written as *cuts*(P), is finite. $\square$

2.9. Proposition. The partially ordered set $(cuts(P), \sqsubseteq)$ is a lattice with the least element *origin*(P) and the greatest element *end*(P). $\square$

Given two cuts c and d, by $c \sqcup d$ and $c \sqcap d$ we denote respectively the least upper bound and the greatest lower bound of c and d.

2.10. Proposition. If a cut c'' of P is an immediate successor of another cut c' of P in the sense that $c' \sqsubseteq c''$ and $c' \sqsubseteq c \sqsubseteq c''$ implies $c = c'$ or $c = c''$, then there exists a unique event $e \in E_{N'}$ such that e is enabled at the marking c' of N' and changes this marking to c''. $\square$

2.11. Proposition. For each cut c of P the triples $N'_1 = (B'_1, E'_1, F'_1)$ and $N'_2 = (B'_2, E'_2, F'_2)$, where
$B'_1 = \{x \in B' : x \leq_{N'} b' \text{ for some } b' \in c\}$,
$E'_1 = \{y \in E' : y \leq_{N'} b' \text{ for some } b' \in c\}$,
$F'_1 = F' \cap ((B'_1 \times E'_1) \cup (E'_1 \times B'_1))$,
$B'_2 = \{x \in B' : b' \leq_{N'} x \text{ for some } b' \in c\}$,
$E'_2 = \{y \in E' : b' \leq_{N'} y \text{ for some } b' \in c\}$,
$F'_2 = F' \cap ((B'_2 \times E'_2) \cup (E'_2 \times B'_2))$,

are causal nets. The pairs $P_1 = (N'_1, h_1)$ and $P_2 = (N'_2, h_2)$, where h_1 and h_2 are the restrictions of h to N'_1 and N'_2, respectively, are finitary processes of N, called respectively the *head* and the *tail* of P with respect to c, and written respectively as *head*(P, c) and *tail*(P, c). They are global if P is global. $\square$

Usually, processes are considered up to isomorphism.

2.12. Definition. Two concrete processes $P_1 = (N_1, h_1)$ and $P_2 = (N_2, h_2)$ of a net N are said to be *isomorphic* if there exists an isomorphism $f : N_1 \to N_2$ such that $fh_2 = h_1$. $\square$

2.13. Definition. An *abstract process*, or briefly a *process*, of a net N is an isomorphism class of concrete processes of N. $\square$

Given a concrete process P of a net N, the abstract process containing P is written as $[P]$. Given an abstract process π of N, each concrete process belonging to π is called an *instance* of π, we say that π is finitary if its instances are finitary, and we say that π is global if its instances are global.

Given a net N, by $processes(N)$ we denote the set of finitary global processes of N.

In the sequel when considering processes we have in mind only finitary global processes.

3 Operations on processes

Throughout the rest of the paper let us consider an arbitrary net N.

Following [Wink 80] we can define operations allowing one to construct processes of N from processes.

Processes of N without events, called *process identities*, or *identities*, can be identified with the complete consistent markings consisting of those conditions and negations of conditions which occur in the respective instances.

For each process π there exists a unique process identity, called the *source* or the *domain* of π and written as $dom(\pi)$ (resp.: a unique process identity, called the *target* or the *codomain* of π and written as $cod(\pi)$), whose instance can be obtained from an instance P of π by restricting P to the set $begin(P)$ of minimal elements (resp.: to the set $end(P)$ of maximal elements).

Thus we have two unary operations on processes: the operation $\pi \mapsto dom(\pi)$ of taking the source (the domain), and the operation $\pi \mapsto cod(\pi)$ of taking the target (the codomain). They have the following obvious properties

$dom(dom(\pi)) = cod(dom(\pi)) = dom(\pi)$,
$dom(cod(\pi)) = cod(cod(\pi)) = cod(\pi)$.

Another operation is binary and partial. It combines two processes whenever one of them is a continuation of the other. It can be defined as follows.

3.1. Definition. A process π is said to *consist* of a process π_1 *followed* by a process π_2 if its instance P has a cut c such that $head(P, c)$ is an instance of π_1 and $tail(P, c)$ is an instance of π_2. □

3.2. Proposition. For every two processes π_1 and π_2 such that $cod(\pi_1) = dom(\pi_2)$ there exists a unique process, written as $\pi_1\pi_2$, that consists of π_1 followed by π_2. □

For a proof it suffices to take disjoint instances of π_1 and π_2 and to identify maximal elements of the instance of π_1 with minimal elements of the instance of π_2 whenever they represent holdings of the same condition of $sc(N)$.

3.3. Proposition. The correspondence $(\pi_1, \pi_2) \mapsto \pi_1\pi_2$, called *concatenation*, or *sequential composition*, is an associative partial operation such that

$dom(\pi_1\pi_2) = dom(\pi_1)$,
$cod(\pi_1\pi_2) = cod(\pi_2)$,
$dom(\pi)\pi = \pi cod(\pi) = \pi$.

Moreover, it satisfies the following cancellation laws

$\pi\sigma = \pi\sigma'$ implies $\sigma = \sigma'$,
$\tau\pi = \tau'\pi$ implies $\tau = \tau'$. □

3.4. Conclusion. The set $processes(N)$ and the operations

$\pi \mapsto dom(\pi)$, $\pi \mapsto cod(\pi)$, $(\pi_1, \pi_2) \mapsto \pi_1\pi_2$

form a (morphisms-only) category, written as $PROCESSES(N)$. □

By considering consistent complete markings of $sc(N)$ as objects, we can interpret $PROCESSES(N)$ as a standard category.

4 Independence of processes

Let N be a net.

For processes in the category $PROCESSES(N)$ there are two natural notions of independence corresponding to those introduced in [EK 76] for direct derivations of graph grammars.

4.1. Definition. Processes π_1 and π_2 from $processes(N)$ are said to be *parallel independent* (resp.: *sequential independent*) if $dom(\pi_1) = dom(\pi_2)$ (resp.: $cod(\pi_1) = dom(\pi_2)$) and π_1 and π_2 have respectively instances $P_1 = (N'_1, h_1)$ and $P_2 = (N'_2, h_2)$ such that

$|h_1(F_{N'_1}e_1 \cup e_1F_{N'_1})| \cap |h_2(F_{N'_2}e_2 \cup e_2F_{N'_2})| = \emptyset$

for all $e_1 \in E_{N'_1}$ and $e_2 \in E_{N'_2}$. □

The propositions which follow show that these notions can be expressed in categorical terms.

4.2. Proposition. For each pair $v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w$ of parallel independent processes there exists a unique pair $v \overset{\pi'_2}{\rightarrow} u' \overset{\pi'_1}{\leftarrow} w$ of processes such that the diagram $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ is a bicartesian square (see Figure 1). $\Box$

Proof outline.

There exist instances $P_1 = (N_1', h_1)$ and $P_2 = (N_2', h_2)$ of π_1 and π_2, respectively, such that $(B_{N_1'} \cup E_{N_1'}) \cap (B_{N_2'} \cup E_{N_2'}) = origin(P_1) = origin(P_2)$. The pair $P = (N', h')$, where $N' = (B', E', F')$, $B' = B_{N_1'} \cup B_{N_2'}$, $E' = E_{N_1'} \cup E_{N_2'}$, $F' = F_{N_1'} \cup F_{N_2'}$, is an instance of a process π, and there are cuts c_1 and c_2 of P such that $P_1 = head(P, c_1)$, $P_2 = head(P, c_2)$. For $\pi_1' = [tail(P, c_2)]$ and $\pi_2' = [tail(P, c_1)]$ we have $\pi_1\pi_2' = \pi_2\pi_1' = \pi$. Consequently, we obtain the commutative diagram
$\Delta = (w \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} v, w \overset{\pi_1'}{\rightarrow} u' \overset{\pi_2'}{\leftarrow} v)$.

Suppose that $\pi_1\rho_2 = \pi_2\rho_1 = \sigma$. Then in each instance S of σ there are cuts d_1 and d_2 such that $head(S, d_1)$ is an instance of π_1 and $head(S, d_2)$ is an instance of π_2. Consequently, $head(S, d_1 \sqcup d_2)$ is an instance of π and $tail(S, d_1 \sqcup d_2)$ is an instance of a process ρ such that $\pi\rho = \sigma$. By the cancellation law such a process is unique. Thus $(v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ is a pushout of $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w)$.

Suppose that $\xi_1\pi_2' = \xi_2\pi_1' = \tau$. Then in each instance T of τ there are cuts f_1 and f_2 such that $tail(T, f_1)$ is an instance of π_1' and $tail(T, f_2)$ is an instance of π_2'. Consequently, $tail(T, f_1 \sqcap f_2)$ is an instance of π and $head(T, f_1 \sqcap f_2)$ is an instance of a process ξ such that $\xi\pi = \tau$. By the cancellation law such a process is unique. Thus $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w)$ is a pullback of $(v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$.

Hence Δ is a bicartesian square.

The uniqueness of π_1' and π_2' follows from the fact that in $PROCESSES(N)$ only identity processes are isomorphisms. $\Box$

Note that Proposition 4.2 is related to the so-called diamond property (cf. [HR 91]).

4.3. Proposition. For each pair $u \overset{\pi_1}{\rightarrow} v \overset{\pi_2'}{\rightarrow} u'$ of sequential independent processes there exists a unique pair $u \overset{\pi_2}{\rightarrow} w \overset{\pi_1'}{\rightarrow} u'$ of processes such that the diagram $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ is a bicartesian square. $\Box$

For a proof it suffices to use arguments similar to those in the proof of 4.2.

4.4. Proposition. Given a bicartesian square $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$ and a decomposition $u \overset{\pi_1}{\rightarrow} v = u \overset{\pi_{11}}{\rightarrow} v_1 \overset{\pi_{12}}{\rightarrow} v$, there exist a unique decomposition $w \overset{\pi_1'}{\rightarrow} u' = w \overset{\pi_{11}'}{\rightarrow} w_1 \overset{\pi_{12}'}{\rightarrow} u'$, and a unique $v_1 \overset{\pi_2''}{\rightarrow} w_1$ such that
$(v_1 \overset{\pi_{11}}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v_1 \overset{\pi_2''}{\rightarrow} w_1 \overset{\pi_{11}'}{\leftarrow} w)$ and
$(v \overset{\pi_{12}}{\leftarrow} v_1 \overset{\pi_2''}{\rightarrow} w_1, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_{12}'}{\leftarrow} w_1)$ are bicartesian squares (see Figure 2). $\Box$

$$\begin{tikzcd}
v \arrow[r, "\pi_2'"] & u' \\
v_1 \arrow[u, "\pi_{12}"] \arrow[r, "\pi_2''"] & w_1 \arrow[u, "\pi_{12}'"'] \\
u \arrow[u, "\pi_{11}"] \arrow[uu, bend left, "\pi_1"] \arrow[r, "\pi_2"'] & w \arrow[u, "\pi_{11}'"'] \arrow[uu, bend right, "\pi_1'"']
\end{tikzcd}$$

Figure 2

Proof outline.

Let P be an instance of $\pi = \pi_1\pi_2' = \pi_2\pi_1'$. In P there are cuts c_1, c_2, d such that $P_1 = head(P, c_1)$, $P_2 = head(P, c_2)$, $P_1' = tail(P, c_2)$, $P_2' = tail(P, c_1)$, $P_{11} = head(head(P, c_1), d) = head(P, d)$, $P_{12} = tail(head(P, c_1), d)$, are instances of π_1, π_2, π_1', π_2', π_{11}, π_{12}, respectively. It suffices to define $\pi_2'' = [tail(head(P, c_2 \sqcup d), d)]$, $\pi_{11}' = [head(P, c_2 \sqcup d)]$, $\pi_{12}' = [tail(P, c_2 \sqcup d)]$. $\Box$

4.5. Proposition. Let Δ be the diagram $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi_2'}{\rightarrow} u' \overset{\pi_1'}{\leftarrow} w)$. If Δ is a bicartesian square, then processes $u \overset{\pi_1}{\rightarrow} v$ and $u \overset{\pi_2}{\rightarrow} w$ are parallel independent, and processes $u \overset{\pi_2}{\rightarrow} w$ and $w \overset{\pi_1'}{\rightarrow} u'$ are sequential independent. $\Box$

Proof outline.

It is obvious that the proposition holds true if π_1 and π_2 are process identities. Assume that it holds true if the total number of event occurrences in π_1 and π_2 does not exceed n.

Suppose that the total number of event occurrences in π_1 and π_2 does not exceed $(n+1)$. Then one of the processes, say π_1, has at least one event occurrence. Consequently, there is a decomposition $\pi_1 = \pi_{11}\pi_{12}$ with π_{11} having exactly one event occurrence, say occurrence of an event e. From the fact that Δ is a bicartesian square it follows that $\pi_1\pi_2' = \pi_2\pi_1'$. Let P be an instance of $\pi = \pi_1\pi_2' = \pi_2\pi_1'$. By 4.4 the bicartesian square Δ consists of two bicartesian squares Δ_1 and Δ_2, where

$\Delta_1 = (v_1 \overset{\pi_{11}}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v_1 \overset{\pi''_2}{\rightarrow} w_1 \overset{\pi'_{11}}{\leftarrow} w)$, and all the processes forming these squares have instances contained in P.

The instance of π_{11} cannot be contained in the instance of π_2 since otherwise there would be $\pi_2 = \pi_{11}\xi$ for ξ not being identity and Δ_1 could not be a bicartesian square. Consequently, there is no precondition or postcondition of e among the conditions that are preconditions or postconditions of events from π_2 since otherwise, by (2) of 2.3, P could not contain simultaneously instances of π_{11} and π_2.

The fact that the total number of event occurrences in the processes π_{12} and π''_2 does not exceed n and our assumption imply that π_{12} and π''_2 are parallel independent. Thus the processes π_{12} and π''_2 have disjoint sets of conditions playing the role of preconditions or postconditions of their events.

Consequently, the sets of preconditions and postconditions of events occurring in π_1 and π_2 are disjoint. Hence π_1 and π_2 are parallel independent.

Similarly, π_2 and π'_1 are sequential independent. $\square$

From 4.2 - 4.5 we obtain the following characterization of independence of processes.

4.6. Theorem. Processes of the pair $v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w$ are parallel independent iff there exists a unique pair $v \overset{\pi'_2}{\rightarrow} u' \overset{\pi'_1}{\leftarrow} w$ such that the diagram $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi'_2}{\rightarrow} u' \overset{\pi'_1}{\leftarrow} w)$ is a bicartesian square. $\square$

4.7. Theorem. Processes of the pair $u \overset{\pi_1}{\rightarrow} v \overset{\pi'_2}{\rightarrow} u'$ are sequential independent iff there exists a unique pair $u \overset{\pi_2}{\rightarrow} w \overset{\pi'_1}{\rightarrow} u'$ such that the diagram $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi'_2}{\rightarrow} u' \overset{\pi'_1}{\leftarrow} w)$ is a bicartesian square. $\square$

5 Categories of processes

Let N be a net.

Each process of N with one-element set of event occurrences (each *one-event process*) is *atomic*, or an *atom*, in the sense that it cannot be represented as the result of concatenating two processes which are not identities.

Following [Wink 80] we can show that each finitary process of N that is not identity can be obtained by concatenating one-event processes. More precisely, we have the following proposition.

5.1. Proposition. Each process $\pi \in$ *processes*(N) that is not identity can be represented in the form $\pi = \pi_1 ... \pi_n$, where π_1,..., π_n are one-event processes of N. $\square$

For a proof it suffices to consider a maximal chain of cuts of an instance of π.

In general, the representation of a process as the result of concatenation of atomic processes is not unique. The following proposition makes clear why this may take place.

5.2. Proposition. Let ξ_1, ξ_2, η_1, η_2 be processes from the *processes*(N) such that $\xi_1\xi_2 = \eta_2\eta_1$. Then there exist unique processes σ_1, σ_2, and a unique bicartesian square $(v \overset{\pi_1}{\leftarrow} u \overset{\pi_2}{\rightarrow} w, v \overset{\pi'_2}{\rightarrow} u' \overset{\pi'_1}{\leftarrow} w)$ such that $\xi_1 = \sigma_1\pi_1$, $\xi_2 = \pi'_2\sigma_2$, $\eta_2 = \sigma_1\pi_2$, $\eta_1 = \pi'_1\sigma_2$ (see Figure 3). $\square$

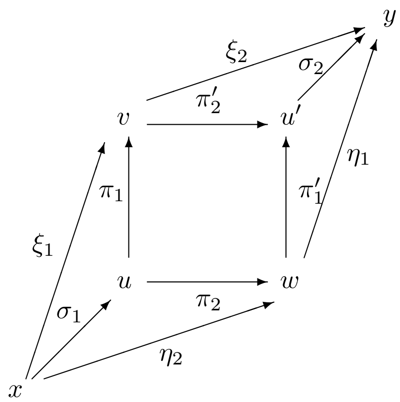

Figure 3

For a proof it suffices to consider an instance S of $\xi_1\xi_2 = \eta_2\eta_1$, its cuts c_1 and c_2 such that $\xi_1 = [head(S, c_1)]$, $\xi_2 = [tail(S, c_1)]$, $\eta_2 = [head(S, c_2)]$, $\eta_1 = [tail(S, c_2)]$, define $\sigma_1 = [head(S, c_1 \sqcap c_2)]$, $\sigma_2 = [tail(S, c_1 \sqcup c_2)$, $\pi_1 = [head(tail(S, c_1 \sqcap c_2), c_1)]$, $\pi'_2 = [head(tail(S, c_1), c_1 \sqcup c_2)]$, $\pi_2 = [head(tail(S, c_1 \sqcap c_2), c_2)]$, $\pi'_1 = head(tail(S, c_2), c_1 \sqcup c_2)]$, and exploit the fact that π_1 and π_2 are parallel independent.

Taking into account the fact that by concatenating processes which are not identities we cannot obtain identities, and combining 5.1 and 5.2 with 3.4 and 4.4, we obtain the following description of properties of the category $PROCESSES(N)$.

5.3. Theorem. $PROCESSES(N)$ is a category such that

(1) for all π and τ, if $\pi\tau$ is an identity then π and τ are identities,

(2) for all π, σ, σ', $\pi\sigma = \pi\sigma'$ implies $\sigma = \sigma'$,

(3) for all π, τ, τ', $\tau\pi = \tau'\pi$ implies $\tau = \tau'$,

(4) for every bicartesian square

$(v \stackrel{\pi_1}{\leftarrow} u \stackrel{\pi_2}{\rightarrow} w, v \stackrel{\pi_2'}{\rightarrow} u' \stackrel{\pi_1'}{\leftarrow} w)$ and every decomposition $u \stackrel{\pi_1}{\rightarrow} v = u \stackrel{\pi_{11}}{\rightarrow} v_1 \stackrel{\pi_{12}}{\rightarrow} v$, there exist a unique decomposition $w \stackrel{\pi_1'}{\rightarrow} u' = w \stackrel{\pi_{11}'}{\rightarrow} w_1 \stackrel{\pi_{12}'}{\rightarrow} u'$, and a unique $v_1 \stackrel{\pi_2''}{\rightarrow} w_1$ such that $(v_1 \stackrel{\pi_{11}}{\leftarrow} u \stackrel{\pi_2}{\rightarrow} w, v_1 \stackrel{\pi_2''}{\rightarrow} w_1 \stackrel{\pi_{11}'}{\leftarrow} w)$ and $(v \stackrel{\pi_{12}}{\leftarrow} v_1 \stackrel{\pi_2''}{\rightarrow} w_1, v \stackrel{\pi_2'}{\rightarrow} u' \stackrel{\pi_{12}'}{\leftarrow} w_1)$ are bicartesian squares,

(5) for all ξ_1, ξ_2, η_1, η_2 such that $\xi_1\xi_2 = \eta_2\eta_1$ there exist unique σ_1, σ_2, and a unique bicartesian square

$(v \stackrel{\pi_1}{\leftarrow} u \stackrel{\pi_2}{\rightarrow} w, v \stackrel{\pi_2'}{\rightarrow} u' \stackrel{\pi_1'}{\leftarrow} w)$ such that $\xi_1 = \sigma_1\pi_1$, $\xi_2 = \pi_2'\sigma_2$, $\eta_2 = \sigma_1\pi_2$, $\eta_1 = \pi_1'\sigma_2$,

(6) every π that is not an identity can be represented in the form $\pi = \pi_1 ... \pi_n$, where π_1,...,π_n are atomic. □

Note that (5) is a generalization of the Levi Lemma for strings and traces (cf. [Maz 88]).

Observe that due to 4.6 and 4.7 we can define the parallel and the sequential independence in $PROCESSES(N)$ as the existence of an appropriate bicartesian square. Moreover, the definition of independence using bicartesian squares is possible in arbitrary category satisfying (1) - (6) of 5.3. This suggests that categories of this type are interesting candidates for new models of distributed systems and justifies the following definition.

5.4. Definition. A *discrete process system*, or briefly a *process system*, is a category that enjoys the properties (1) - (6) of 5.3. Objects of such a category are called *states*. Morphisms are called *processes*. □

Process systems are models richer than other ones in the sense that they specify not only states, transitions, and independence of transitions of the modelled systems, but also their processes (runs) and how they compose. Moreover, the independence becomes a definable notion, and it can be defined not only for transitions, but also for arbitrary processes.

If we reduce discrete process systems to their states and atoms then we obtain transition systems. If we endow the transition systems thus obtained with the existing in the original process systems information on independece of atomic processes then we obtain structures close to introduced in [WN 95] transition systems with independence.

5.5. Proposition. Let Π be a discrete process system with the set S_Π of states and the set A_Π of atomic processes. Let $T(\Pi)$ be the structure $(S, Tran, I)$, where $S = S_\Pi$, $Tran$ is the set of triples (s, α, s') such that $\alpha \in A_\Pi$, $s = dom(\alpha)$, $s' = cod(\alpha)$, and I is the least binary relation in $Tran$ such that
$(s, \alpha, s_1)I(s, \beta, s_2)$ whenever α and β are parallel independent,
$(s, \alpha, s_1)I(s_1, \beta', u)$ whenever α and β' are sequential independent,
$(s, \alpha, s_1)I(w, \beta, w')$ whenever
$(s, \alpha, s_1) \prec (s_2, \alpha', u)I(w, \beta, w')$,
$(w, \beta, w')I(s_2, \alpha', u)$ whenever
$(w, \beta, w')I(s, \alpha, s_1) \prec (s_2, \alpha', u)$,
where $\prec$ is the relation defined as follows:
$(s, \alpha, s_1) \prec (s_2, \alpha', u)$ iff
$(s, \alpha, s_1)I(s, \beta, s_2)$ and $(s, \alpha, s_1)I(s_1, \beta', u)$ and $(s, \beta, s_2)I(s_2, \alpha', u)$
for some (s, β, s_2) and (s_1, β', u).
The structure $T(\Pi)$ enjoys the following properties:

(1) $(s, \alpha, s_1)I(s, \beta, s_2)$ implies the existence of unique (s_1, β', u) and (s_2, α', u) such that $(s, \alpha, s_1)I(s_1, \beta', u)$ and $(s, \beta, s_2)I(s_2, \alpha', u)$,

(2) $(s, \alpha, s_1)I(s_1, \beta', u)$ implies the existence of unique (s, β, s_2) and (s_2, α', u) such that $(s, \alpha, s_1)I(s, \beta, s_2)$ and $(s, \beta, s_2)I(s_2, \alpha', u)$,

(3) $(s, \alpha, s_1) \prec (s_2, \alpha', u)I(w, \beta, w')$ implies $(s, \alpha, s_1)I(w, \beta, w')$,

(4) $(w, \beta, w')I(s, \alpha, s_1) \prec (s_2, \alpha', u)$ implies $(w, \beta, w')I(s_2, \alpha', u)$. □

Properties (1) and (2) follow from the definition of independence in process systems as the existence of a suitable bicartesian square. (3) and (4) follow from the definition of I.

Note that properties (1) - (4) correspond to the basic axioms characterizing transition systems with independence of [WN 95]. Thus we may call $T(\Pi)$ the transition system with independence corresponding to the process system Π.

6 Recapitulation

We have described properties of categories of processes of Petri nets and we have shown that the independence of processes is equivalent to the existence of suitable bicartesian squares. Thus we have shown that categories of net processes are members of an axiomatically defined class of categories with axioms allowing to define independence of morphisms. Finally, we have shown that by reducing categories from this class to their objects and atomic morphisms, and by endowing the results of reduction with the existing information on independence, we obtain structures close to transition systems with independence of [WN 95].

By introducing morphisms between process systems one can define the respective category and study relations between this category and other models of computational systems. However, this is not the subject of the present paper.

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