Processes of Timed Petri Nets [1]

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Abstract. Processes of timed Petri nets are represented by labelled partial orders with some extra features. These features reflect the execution times of processes and allow to combine processes sequentially and in parallel, which leads to some algebras. The processes can be represented either without specifying when particular situations appear (free processes), or together with the respective appearance times (timed processes). The processes of the latter type determine the possible firing sequences of the respective nets.

1 Motivation and introduction

Purpose

Petri nets are a widely accepted model of concurrent systems. Originally they were invented for modelling those aspects of system behaviours which can be expressed in terms of causality and choice. Recently a growing interest can be observed in modelling real-time systems, which implies a need of a representation of the lapse of time. To meet this need various models has been proposed known as timed Petri nets.

In this paper we consider the model in which usual Place/Transition nets are given together with execution times of their transitions. Our purpose is to characterize the behaviours of the corresponding timed nets.

The choice of the model with time-consuming transitions is motivated by the fact that this model admits a uniform treatment of both simple transitions and complex aggregates

[1]This work has been partially supported by grant 8T11C 029 08. It has been published in Theoret. Comput. Sci. 243 (2000) 1-34.

of transitions.

Our characterization of the behaviours of timed nets is expected to enjoy the following properties:

- adequacy: all essential features as the causality or its lack (concurrency), choice, and the lapse of time, should be fully reflected,
- economy: models of behaviours should be as succint as possible,
- aggregability: it should be possible to regard complex parts of behaviours as elementary units of activity,
- compositionality: it should be possible to obtain behaviours of complex nets by combining the behaviours of their subnets,
- compatibility with other characterizations: the characterization should allow to derive other existing characterizations,
- capability of reflecting the reactive aspects of behaviours: initial markings should represent streams of data rather than states, and processes caused by such markings should represent reactions to the respective streams.

Solution

We represent the behaviour of a timed net by an algebra of structures called concatenable weighted pomsets. These structures correspond to concatenable processes of [DMM 89] with some extra information about the lapse of time, and they can be combined with the aid of operations similar to those on concatenable processes (a sequential and a parallel composition and interchanges).

The concatenable weighted pomsets represent processes of the considered net, where a process is either an execution of a transition, or a presence of a token in a place, or a combination of such processes. A process with the lapse of time represented only by delays between participating tokens is said to be free. A process with the lapse of time represented both by delays between participating tokens and by moments at which particular participating tokens appear is said to be timed.

There are natural homomorphisms from the algebra of timed processes of a net to the algebra of its free processes, and from the algebra of free processes to an algebra whose elements reflect how much time the respective processes take. More precisely, to each free process there corresponds a table of least possible delays between its data and results (a delay table) such that the tables corresponding to the results of operations on processes can be obtained by composing properly the tables corresponding to components. The delay tables which correspond to processes generalize the concept of execution time.

An important property of free processes and their delay tables is that they do not depend on when the respective tokens appear and that one can compute how a free process proceeds in time for any given combination of appearance times of its tokens. The combination which is given plays here the role of a marking. This marking is timed in the sense that not only the presences of tokens in places but also the respective appearance times are presented. As the latter need not be the same, such a timed marking should be regarded as representing a stream or a delivery of tokens rather than a temporary situation.

The possibility of computing how a free process applies to a given delivery of tokens allows us to find the corresponding timed process and to see if such a process cannot be excluded by another process due to an earlier enabling of a transition. The timed processes which cannot be excluded are exactly those which can be realized according to the standard semantics of timed nets (cf. [GV 87]). They determine the possible firing sequences as defined by such a semantics.

Relation to other works

Timed Petri nets considered in this paper correspond to those used in [Ram 74] for evaluation of performance of concurrent systems. They can also be regarded as weighted basic nets in the sense of [B 88] with the length of each transition equal to the respective execution time, and with the interpretation according to which transitions fire as soon as they are enabled.

Executions of such nets can be represented as usual, that is as strings of subsequent states and transitions, called firing sequences, where each state has an appearance time and it determines when the transitions which have already started will be completed (cf. [GV 87]). However, the representation of behaviours of timed nets in terms of their firing sequences does not enjoy the expected properties of adequacy, economy, aggregability, and compositionality, and so we replace it by a representation in terms of free and timed processes.

Our idea of defining free and timed processes of a timed net as combinations of executions of transitions and presences of tokens in places follows that in [DMM 89], where processes of an usual Petri net are defined as morphisms of a monoidal category which is freely generated by the set of transitions and the set of places.

In the context of timed nets a similar idea has been exploited in [BG 92], where processes of a timed net are represented by assigning to processes of the underlying usual net the respective execution times. In this representation the execution time of a process is expressed by a number, which is not adequate enough when one has to do with processes consisting of independent components. For example, the execution time of a process which consists of two independent components α and β cannot be regarded as a number since it depends on when each of the components starts. In our approach we avoid this shortcoming by describing the lapse of time corresponding to a process of a timed net within the representing concatenable weighted pomset. In particular, for each process we have a delay table whose items represent delays between appearances of initial tokens and appearances of resulting tokens.

Delay tables of processes are essentially matrices over an idempotent semiring similar to those used in [FM 95] to represent the lapse of time in computations of concurrent programs with time-consuming actions. As delay tables of processes obtained by composing sequentially or in parallel given processes can be obtained by multiplying as matrices or by putting together the delay tables of component processes, the lapse of time in computations of concurrent programs with time-consuming actions can be found in a compositional way, that is without engaging an operational semantics as in [GRS 94].

The present paper exploits some ideas of [Wi 80] and [Wi 92]. It is an improved and extended synthesis of [Wi 93], [Wi 94/1], and [Wi 94/2].

2 Concatenable weighted pomsets

Processes of timed nets and delay tables will be represented by partially ordered multisets (pomsets in the terminology of [Pra 86]) with some extra arrangements of minimal and maximal elements (similar to those in concatenable processes of [DMM 89]), and with some extra features (weights).

Given a partially ordered set (poset) $\mathcal{X} = (X, \leq)$, define a *cut* of $\mathcal{X}$ as a maximal antichain which has an element in each maximal chain, denote by X_{min} the set of minimal elements of $\mathcal{X}$ and by X_{max} the set of maximal elements of $\mathcal{X}$, for $Y \subseteq X$ denote by $\downarrow Y$ the set of $x \in X$ such that $x \leq y$ for some $y \in Y$ and by $\uparrow Y$ the set of $x \in X$ such that $y \leq x$ for some $y \in Y$, and for $x \leq y$ denote by $[x, y]$ the subposet of $\mathcal{X}$ that consists of all $z \in X$ such that $x \leq z \leq y$. Define a K-*dense* poset as a poset in which each maximal antichain is a cut. Denote by R the semiring of real numbers and infinities $-\infty$, $+\infty$ with the operation $(x, y) \mapsto max(x, y)$ playing the role of addition and the operation $(x, y) \mapsto x + y$, where $(-\infty) + (+\infty)$ is defined as $-\infty$, playing the role of multiplication.

Let V be a set of labels.

2.1. Definition. A *concatenable weighted pomset* (or a *cw-pomset*) over V is an isomorphism class α of structures $\mathcal{A} = (X, \leq, d, e, s, t)$, where:

(1) $(X, \leq)$ is a finite *underlying poset*,

(2) $d : X \times X \to R$ is a *weight function* such that $d(x, y) = -\infty$ if $x \leq y$ does not hold, $d(x, x) = 0$, and $d(x, y) = max(d(x, z) + d(z, y) : z \in Z)$ for each cut Z of $[x, y]$ if $x \leq y$,

(3) $e : X \to V$ is a *labelling function*,

(4) $s = (s(v) : v \in V)$ is an *arrangement of minimal elements*, where each $s(v)$ is an enumeration of the set of minimal elements with the label v,

(5) $t = (t(v) : v \in V)$ is an *arrangement of maximal elements*, where each $t(v)$ is an enumeration of the set of maximal elements with the label v.

Each such a structure is called an *instance* of α, we write α as $[\mathcal{A}]$, and we use subscripts, $X_{\mathcal{A}}$, $\leq_{\mathcal{A}}$, $d_{\mathcal{A}}$, $e_{\mathcal{A}}$, $s_{\mathcal{A}}$, $t_{\mathcal{A}}$, when necessary. $\square$

In this definition by an enumeration of a set we mean a sequence of elements of this set in which each element occurs exactly once, and by an isomorphism from $\mathcal{A}$ to $\mathcal{A}' = (X', \leq', d', e', s', t')$ we mean a bijection $b : X \to X'$ such that $x \leq y$ iff $b(x) \leq' b(y)$, $d'(b(x), b(y)) = d(x, y)$, $e'(b(x)) = e(x)$, $s'(v) = b(s(v))$, and $t'(v) = b(t(v))$, for all $x, y \in X$ and $v \in V$, where $b(x_1...x_n)$ denotes $b(x_1)...b(x_n)$. The condition of finiteness of the underlying poset is imposed in order to avoid technical problems with infinite partial orders which are of no use in the applications considered in this paper. The last condition in (2) is equivalent to assuming that for each pair (x, y) such that $x \leq y$ the weight $d(x, y)$ is the maximum of sums of weights along maximal chains from x to y. However, the formulation we use here is more general since it applies also to the case when weights are elements of an arbitrary semiring. The arrangements in (4) and (5) are needed for equipping minimal and maximal elements with identifiers which do not depend on concrete instances of the considered cw-pomset, where the identifier of an element x consists of the label $e(x)$ and of the number indicating the position of this element in the respective sequence $s(e(x))$ or $t(e(x))$. Such identifiers allow one to concatenate cw-pomsets by identifying maximal elements of one cw-pomset with minimal elements of another.

The restriction of $\mathcal{A}$ to X_{min} with t replaced by s and that to X_{max} with s replaced by t are instances of cw-pomsets. These cw-pomsets do not depend on the choice of instance of α. We write them respectively as $\partial_0(\alpha)$ and $\partial_1(\alpha)$ and call them respectively the *source* and the *target* of α. If the underlying poset $(X, \leq)$ is K-dense then also $\mathcal{A}$ and α are said to be K-dense. If $X = X_{min} \cup X_{max}$ then we call α a *table.* If $X = X_{min} = X_{max}$, and thus the order $\leq$ reduces to the identity, then we call α a *symmetry.* If also $t = s$ then $\alpha = \partial_0(\alpha) = \partial_1(\alpha)$ and α becomes a *trivial symmetry*, and it can be identified with a multiset $ms(\alpha)$ of labels, namely with the multiset in which the multiplicity of each $v \in V$ is given by the cardinality of $e^{-1}(v) \cap X$. By $cwp(V)$, $dcwp(V)$, $tab(V)$, $sym(V)$, and $tris(V)$, we denote respectively the set of cw-pomsets, the set of K-dense cw-pomsets, the set of tables, the set of symmetries, and the set of trivial symmetries over V.

Examples of cw-pomsets are shown in figures 2.1 and 2.2. In these examples A, B, C, D are labels. The arrangements of minimal elements and the arrangements of maximal elements are represented by endowing the labels of minimal elements with subscripts and the labels of maximal elements with superscripts, where each subscript (resp.: superscript) denotes the position of the corresponding element in the respective enumeration.

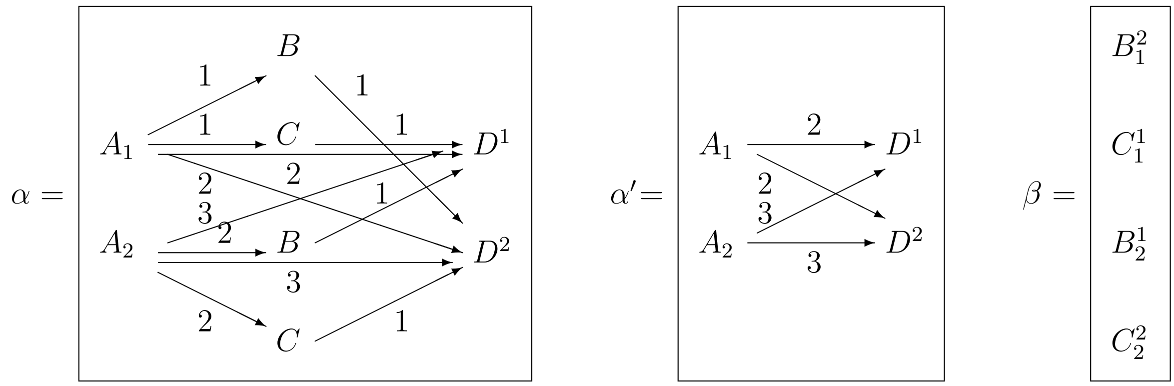

Figure 2.1

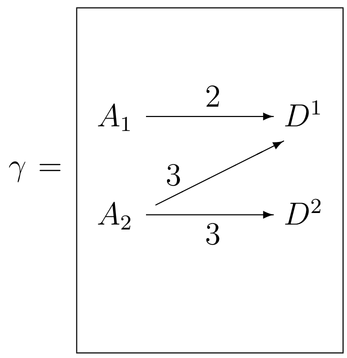

Figure 2.2

All the cw-pomsets in figure 2.1 are K-dense. The cw-pomset α' is a table. It is obtained from α by ignoring elements which are neither minimal nor maximal. The cw-pomset β is both a table and a symmetry. Instances of tables α' and β can be represented in a matrix-like form as shown in figure 2.3.

The cw-pomset γ in figure 2.2 is not K-dense since the occurrences of A_1 and D^2 in its instance constitute a maximal antichain which is not a cut.

$$\alpha' = \begin{array}{c|cc} & D^1 & D^2 \\ \hline A_1 & 2 & 2 \\ A_2 & 3 & 3 \end{array} \qquad \beta = \begin{array}{c|cccc} & B^1 & B^2 & C^1 & C^2 \\ \hline B_1 & -\infty & 0 & -\infty & -\infty \\ B_2 & 0 & -\infty & -\infty & -\infty \\ C_1 & -\infty & -\infty & 0 & -\infty \\ C_2 & -\infty & -\infty & -\infty & 0 \end{array}$$

Figure 2.3

When defining cw-pomsets we assume only such their properties which are needed for defining suitable operations on cw-pomsets and the respective algebras. This allows us to simplify the presentation by defining delay tables as cw-pomsets. However, only K-dense cw-pomsets will represent processes of timed nets. Such cw-pomsets enjoy a number of interesting properties.

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of a cw-pomset. By a *cut* of $\mathcal{A}$ we mean a cut of the underlying poset $(X, \leq)$ and by $cuts(\mathcal{A})$ we denote the set of cuts of $\mathcal{A}$. Given $Y, Y' \in cuts(\mathcal{A})$, we write $Y \sqsubseteq Y'$ if $\downarrow Y \subseteq \downarrow Y'$.

2.2. Proposition.; If $\mathcal{A}$ is K-dense then the relation $\sqsubseteq$ is a partial order on the set $cuts(\mathcal{A})$ such that $cuts(\mathcal{A})$ with this order is a lattice. $\square$

Proof: For $Y, Y' \in cuts(\mathcal{A})$ we define $Y \sqcup Y'$ as the set of all $x \in X$ of the form $max(TY, TY')$, where T is a maximal chain and TY, TY' denote the unique elements of this chain in Y and in Y', respectively. Then $Y \sqcup Y'$ cannot contain two different x, y such that $x \leq y$ (since each maximal chain containing x and y may have at most one member in $Y \sqcup Y'$) and each $x \in X$ must be comparable with $max(TY, TY') \in Y \sqcup Y'$ for each maximal chain T which contains x. Thus $Y \sqcup Y'$ is a maximal antichain. From the definition it follows that $Y \sqcup Y'$ has an element in each maximal chain and thus it is a cut. It is also obvious that $Y \sqcup Y'$ is the least upper bound of Y and Y', as required. Similarly for the greatest lower bounds. $\square$

2.3. Proposition. If $\mathcal{A}$ is K-dense then for each $Y \in cuts(\mathcal{A})$ the order $\leq$ is the transitive closure of the union of its restrictions to the subsets $\downarrow Y$ and $\uparrow Y$. $\square$

Proof: Let $\leq_1$ and $\leq_2$ be the restrictions of $\leq$ to $\downarrow Y$ and $\uparrow Y$, respectively. The fact that $\leq$ contains $(\leq_1 \cup \leq_2)^*$, the reflexive and transitive closure of $\leq_1 \cup \leq_2$, is immediate. Conversely, if $x \leq y$ then $x \leq_1 y$ whenever $x, y \in\downarrow Y$, $x \leq_2 y$ whenever $x, y \in\uparrow Y$, and $x \leq_1 z \leq_2 y$ with some $z \in Y$ whenever $x \in\downarrow Y$ and $y \in\uparrow Y$ since then there exists a maximal chain which contains x and y and this chain has an element z in Y. $\square$

2.4. Proposition. If $\mathcal{A}$ is K-dense then for each $Z \in cuts(\mathcal{A})$, each $x \in\downarrow Z$, and each $y \in\uparrow Z$ such that $x \leq y$, we have

$$d(x, y) = max(d(x, z) + d(z, y) : z \in Z). \square$$

A proof follows immediately from the conditions in (2) of 2.1.

2.5. Proposition. If $\mathcal{A}$ is K-dense then for all $x, y \in X$ such that $x \leq y$ the weight $d(x, y)$ is the maximum of sums $d(x, x_1) + ... + d(x_n, y)$ over all maximal chains $x \leq x_1 \leq ... \leq x_n \leq y$ from x to y. $\square$

Proof: The proposition can be proved by induction on the number of elements in $[x, y]$. Suppose that the required property holds for the number of elements not exceeding n and consider x, y such that the cardinality of $[x, y]$ is $n + 1$. Choose any $z \in [x, y]$ which is an immediate predecessor of y. Choose in $[x, y]$ a maximal antichain Z that contains z. As $[x, y]$ is K-dense, Z is a cut of $[x, y]$ and thus $d(x, y) = max(d(x, t) + d(t, y) : t \in Z)$ with $d(x, y) = d(x, u) + d(u, y)$ for some $u \in Z$. As $[x, u]$ has at most n elements, $d(x, u)$ is the maximum of sums $d(x, x_1) + ... + d(x_k, u)$ over all maximal chains $x \leq x_1 \leq ... \leq x_k \leq u$. Consequently, $d(x, y) = d(x, x_1) + ... + d(x_k, u) + d(u, y)$ for a maximal chain $x \leq x_1 \leq ... \leq x_k \leq u \leq y$. On the other hand, each maximal chain from x to y is of the form $x \leq x_1 \leq ... \leq x_k \leq t \leq y$ for some $t \in Z$ and we have $d(x, x_1) + ... + d(x_k, t) + d(t, y) \leq d(x, y)$. Hence $d(x, y)$ is the maximum of sums $d(x, x_1) + ... + d(x_k, t) + d(t, y)$ over maximal chains from x to y. $\square$

2.6. Proposition. If $\mathcal{A}$ is K-dense and $f : X \to R$ and $g : X \to R$ are functions such that $f(x) = g(x)$ for all $x \in X_{min}$, $f(y) = max(f(x) + d(x, y) : x \leq y, x \neq y)$ for all $y \in X - X_{min}$, and $g(y) = max(g(x) + d(x, y) : x$ immediately precedes $y)$ for all $y \in X - X_{min}$, then $f = g$. $\square$

Proof: By induction on the number of predecessors of an element it can be shown that $g(x) \leq f(x)$ for all $x \in X$. In order to prove that also $f(x) \leq g(x)$ for all $x \in X$ suppose that $g(y) < f(y)$ for some $y \in X$. Without a loss of generality we may assume that y is a minimal element such that $g(y) < f(y)$. This implies that $g(x) = f(x)$ for all $x \leq y$ such that $x \neq y$. From the properties of f it follows that $f(y) = f(t) + d(t, y)$ for some $t \leq y$ such that $t \neq y$. As $f(t) = g(t)$, we obtain $f(y) = g(t) + d(t, y)$. By 2.5 there exists a maximal chain $t \leq x_1 \leq ... \leq x_n \leq y$ from t to y such that $d(t, y) = d(t, x_1) + ... + d(x_n, y)$. From the properties of g we obtain $g(t) + d(t, x_1) \leq g(x_1)$,...,$g(x_n) + d(x_n, y) \leq g(y)$, which implies $g(t) + d(t, y) \leq g(y)$. Consequently, $f(y) \leq g(y)$, which contradicts to our assumption. $\square$

2.7. Proposition. For each K-dense finite poset $\mathcal{X} = (X, \leq)$ and each function $d : X^2 \to R$ such that $d(x, y) = -\infty$ if $x \leq y$ does not hold, $d(x, x) = 0$, and $d(x, y)$ is the maximum of sums $d(x, x_1) + ... + d(x_n, y)$ over all maximal chains $x \leq x_1 \leq ... \leq x_n \leq y$ from x to y if $x \leq y$, there exists an instance $\mathcal{A}$ of a cw-pomset with the underlying poset $\mathcal{X}$ and the weight function d. $\square$

Proof: It suffices to show that for every x, y such that $x \leq y$ and for each cut Z of the poset $[x, y]$ we have $d(x, y) = max(d(x, z) + d(z, y) : z \in Z)$. The proof can be carried out by an easy induction on the cardinality of $[x, y]$ noting that each maximal chain from x to y must consist of a maximal chain from x to some $z \in Z$ and of a maximal chain from z to y. $\square$

3 Operations

The set $cwp(V)$ of cw-pomsets can be made an algebra by equipping it with suitable operations. In this paper we consider operations of taking sources and targets of cw-pomsets, operations of composing cw-pomsets sequentially and in parallel, and so called interchanges (the latter similar to those in [DMM 89]).

The operations of taking sources and targets of cw-pomsets are defined as $\partial_0 : \alpha \mapsto \partial_0(\alpha)$ and $\partial_1 : \alpha \mapsto \partial_1(\alpha)$.

The sequential composition of cw-pomsets is defined by specifying how the result of composing two cw-pomsets is related to these cw-pomsets, if it exists, and by showing that the respective relation defines a partial binary operation on cw-pomsets. Such an indirect definition is more convenient than a direct definition by construction since it implies easier the properties of the defined operation.

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of a cw-pomset.

3.1. Proposition. For each $Y \in cuts(\mathcal{A})$, and each arrangement of Y into a family $r = (r(v) : v \in V)$ of enumerations of the sets $e^{-1}(v) \cap Y$, the restriction of $\mathcal{A}$ to $\downarrow Y$ with r playing the role of arrangement of maximal elements, and that to $\uparrow Y$ with r playing the role of arrangement of minimal elements, are instances of cw-pomsets. We write these instances as $head_{Y,r}(\mathcal{A})$ and $tail_{Y,r}(\mathcal{A})$, respectively. $\square$

A proof reduces to a simple verification.

The cw-pomset $[\mathcal{A}]$ is said to *consist* of the cw-pomset $[head_{Y,r}(\mathcal{A})]$ *followed* by the cw-pomset $[tail_{Y,r}(\mathcal{A})]$.

Note that each cw-pomset α can be represented in the form $[head_{X_{max},t}(\mathcal{A})]$ and in the form $[tail_{X_{min},s}(\mathcal{A})]$, where $\mathcal{A} = (X, \leq, d, e, s, t)$ is any instance of α.

3.2. Proposition. For every two cw-pomsets α and β with $\partial_0(\beta) = \partial_1(\alpha)$ there exists a unique cw-pomset $\alpha;\beta$ which consists of α followed by β. This cw-pomset is K-dense whenever α and β are K-dense. It is a symmetry whenever α and β are symmetries. $\square$

Proof: As $\partial_0(\beta) = \partial_1(\alpha)$ and instances of α and β may be chosen arbitrarily up to isomorphism, we may choose an instance $\mathcal{A} = (X_\mathcal{A}, \leq_\mathcal{A}, d_\mathcal{A}, e_\mathcal{A}, s_\mathcal{A}, t_\mathcal{A})$ of α and an instance $\mathcal{B} = (X_\mathcal{B}, \leq_\mathcal{B}, d_\mathcal{B}, e_\mathcal{B}, s_\mathcal{B}, t_\mathcal{B})$ of β such that $(t_\mathcal{A}(v))(i) = (s_\mathcal{B}(v))(i)$ for all v and i for which either side is defined and such that these are the only common elements of $X_\mathcal{A}$ and $X_\mathcal{B}$. Then we define

$$X_\mathcal{C} = X_\mathcal{A} \cup X_\mathcal{B}$$

$$U = X_\mathcal{A} \cap X_\mathcal{B}$$

$$x \leq_\mathcal{C} y \text{ whenever } x \leq_\mathcal{A} y \text{ or } x \leq_\mathcal{B} y \text{ or } x \leq_\mathcal{A} z \leq_\mathcal{B} y \text{ for some } z \in U$$

$$d_\mathcal{C}(x,y) = \begin{cases} d_\mathcal{A}(x,y) & \text{for } x,y \in X_\mathcal{A} \\ d_\mathcal{B}(x,y) & \text{for } x,y \in X_\mathcal{B} \\ max(d_\mathcal{A}(x,u) + d_\mathcal{B}(u,y) : u \in U) & \text{for } x \in X_\mathcal{A}, y \in X_\mathcal{B} \\ -\infty & \text{for the remaining } x,y \in X_\mathcal{C} \end{cases}$$

$$e_\mathcal{C}(x) = \begin{cases} e_\mathcal{A}(x) & \text{for } x \in X_\mathcal{A} \\ e_\mathcal{B}(x) & \text{for } x \in X_\mathcal{B} \end{cases}$$

$$s_\mathcal{C} = s_\mathcal{A}$$

$$t_\mathcal{C} = t_\mathcal{B}.$$

In order to prove that $\mathcal{C} = (X_\mathcal{C}, \leq_\mathcal{C}, d_\mathcal{C}, e_\mathcal{C}, s_\mathcal{C}, t_\mathcal{C})$ is an instance of a cw-pomset it suffices to consider $x \in X_\mathcal{A}$ and $y \in X_\mathcal{B}$ such that $x \leq_\mathcal{C} y$, to take a cut Z of $[x,y]$, and to show that $d_\mathcal{C}(x,y) = max(d_\mathcal{A}(x,z) + d_\mathcal{B}(z,y) : z \in Z)$. To this end we exploit the fact that $U \cap [x,y]$ is a cut of $[x,y]$, define U_1 as the set of $u \in U \cap [x,y]$ such that $z \leq_\mathcal{C} u$ for some $z \in Z$ such that $z \neq u$, U_2 as the set of $u \in U \cap [x,y]$ such that $u \leq_\mathcal{C} z$ for some $z \in Z$, Z_1 as the set of $z \in Z$ such that $z \leq u$ for some $u \in U$ such that $u \neq z$, Z_2 as the set of $z \in Z$ such that $u \leq z$ for some $u \in U$, and make use of the following equalities:

$$d_\mathcal{C}(x,y) = max(d_\mathcal{A}(x,u) + d_\mathcal{B}(u,y) : u \in U \cap [x,y])$$

$$= max(d_\mathcal{A}(x,u_1) + d_\mathcal{B}(u_1,y) : u_1 \in U_1) + max(d_\mathcal{A}(x,u_2) + d_\mathcal{B}(u_2,y) : u_2 \in U_2)$$

$$= max(max(d_\mathcal{A}(x,z_1) + d_\mathcal{A}(z_1,u_1) : z_1 \in Z_1) + d_\mathcal{B}(u_1,y) : u_1 \in U_1)$$

$$+max(d_{\mathcal{A}}(x,u_2)+max(d_{\mathcal{B}}(u_2,z_2)+d_{\mathcal{B}}(z_2,y):z_2\in Z_2):u_2\in U_2)$$

$$=max(d_{\mathcal{A}}(x,z_1)+d_{\mathcal{A}}(z_1,u_1)+d_{\mathcal{B}}(u_1,y):z_1\in Z_1,u_1\in U_1)$$

$$+max(d_{\mathcal{A}}(x,u_2)+d_{\mathcal{B}}(u_2,z_2)+d_{\mathcal{B}}(z_2,y):z_2\in Z_2),u_2\in U_2)$$

$$=max(d_{\mathcal{A}}(x,z_1)+max(d_{\mathcal{A}}(z_1,u_1)+d_{\mathcal{B}}(u_1,y):u_1\in U_1):z_1\in Z_1)$$

$$+max(max(d_{\mathcal{A}}(x,u_2)+d_{\mathcal{B}}(u_2,z_2):u_2\in U_2)+d_{\mathcal{B}}(z_2,y):z_2\in Z_2)$$

$$=max(d_{\mathcal{C}}(x,z_1)+d_{\mathcal{C}}(z_1,y):z_1\in Z_1)+max(d_{\mathcal{C}}(x,z_2)+d_{\mathcal{C}}(z_2,y):z_2\in Z_2)$$

$$=max(d_{\mathcal{C}}(x,z)+d_{\mathcal{C}}(z,y):z\in Z).$$

In order to prove that $\mathcal{C}$ is an instance of $\alpha;\beta$ it suffices to note that U is a cut of $\mathcal{C}$ and apply 2.3 and 2.4.

In order to prove that $\mathcal{C}$ is K-dense if $\mathcal{A}$ and $\mathcal{B}$ are K-dense we have to prove that in this case $Z\cap T$ is nonempty for each maximal antichain Z and each maximal chain T. To this end we prove first that $P=(Z-X_{\mathcal{B}})\cup(\downarrow Z\cap U)$ and $Q=(Z-X_{\mathcal{A}})\cup(\uparrow Z\cap U)$ are maximal antichains.

It is clear that P is an antichain. Suppose that P is not a maximal antichain. Then there exists x, say in $X_{\mathcal{A}}$, which is incomparable with the elements of P. For such x there exists $z\in Z$ which is comparable with x and such z must belong to $Z-X_{\mathcal{A}}$. By the definition of $\leq_{\mathcal{C}}$ there exists $u\in U$ such that $x\leq_{\mathcal{C}}u\leq_{\mathcal{C}}z$ and it must belong to $\downarrow Z\cap U$ since otherwise it would be an element of $\uparrow Z\cap U$ and z would be comparable with an element of $\downarrow Z\cap U$. Consequently, x is comparable with an element of $\downarrow Z\cap U$, which contradicts to our assumption. For similar reasons we cannot have any $x\in X_{\mathcal{B}}$ which would be incomparable with the elements of P. Thus P is a maximal antichain. Similarly, Q is a maximal antichain.

Now, $T\cap X_{\mathcal{A}}$ is a maximal chain of $\mathcal{A}$ and $T\cap X_{\mathcal{B}}$ is a maximal chain of $\mathcal{B}$. Thus $T\cap X_{\mathcal{A}}\cap P\neq\emptyset$ and $T\cap X_{\mathcal{B}}\cap Q\neq\emptyset$. Let $T\cap X_{\mathcal{A}}\cap P\neq\emptyset$. If $(T\cap X_{\mathcal{A}})\cap(Z-X_{\mathcal{B}})$ is empty then $(T\cap X_{\mathcal{A}})\cap(\downarrow Z\cap U)$ is nonempty and hence $(T\cap X_{\mathcal{A}})\cap Z\neq\emptyset$ or $(T\cap X_{\mathcal{A}})\cap(\uparrow Z)=\emptyset$. In the first case we have $T\cap Z\neq\emptyset$. In the second case we have $(T\cap X_{\mathcal{B}})\cap Q=(T\cap X_{\mathcal{B}})\cap((Z-X_{\mathcal{A}})\cup(\uparrow Z\cap U))$ with $(T\cap X_{\mathcal{B}})\cap(\uparrow Z\cap U)=\emptyset$, so that $(T \cap X_{\mathcal{B}}) \cap (Z - X_{\mathcal{A}}) \neq \emptyset$, i.e. $T \cap Z \neq \emptyset$, as required. Similarly for $T \cap X_{\mathcal{B}} \cap Q \neq \emptyset$. Thus $\mathcal{C}$ is K-dense.

Finally, it is obvious that $\alpha;\beta$ is a symmetry if α and β are symmetries. This ends the proof. $\square$

The operation $(\alpha,\beta) \mapsto \alpha;\beta$ is called the *sequential composition* of cw-pomsets. Examples of application of this operation are shown in figures 3.1 and 3.2.

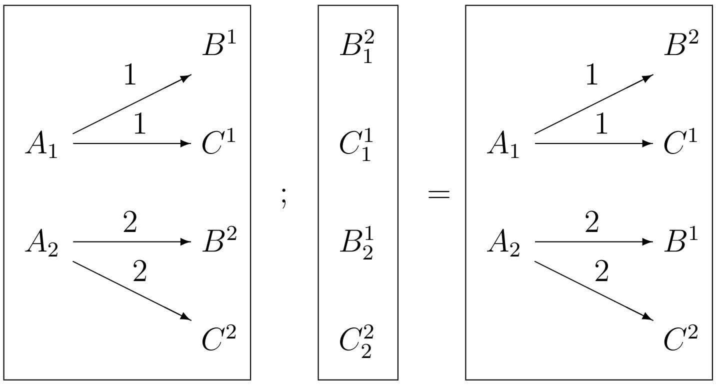

Figure 3.1

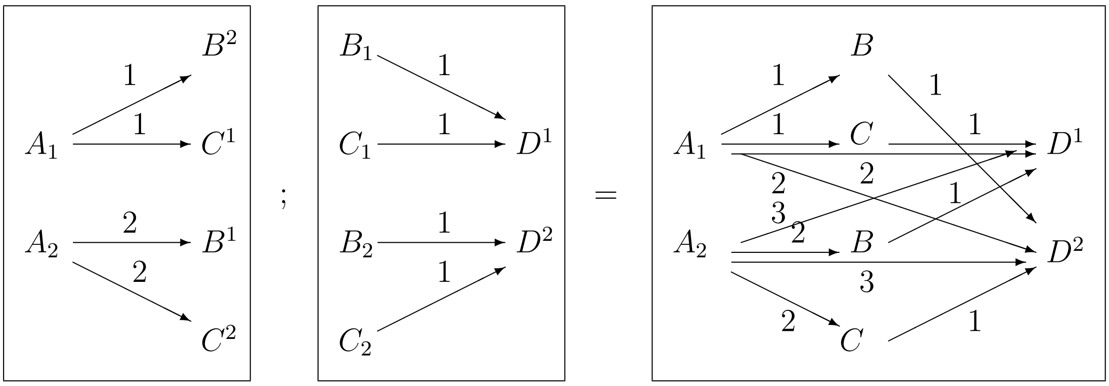

Figure 3.2

3.3. Proposition. The sequential composition is defined for all pairs (α,β) of cw-pomsets with $\partial_0(\beta) = \partial_1(\alpha)$, it is associative and such that $\partial_0(\alpha;\beta) = \partial_0(\alpha)$ and $\partial_1(\alpha;\beta) = \partial_1(\beta)$ and $\partial_0(\alpha);\alpha = \alpha;\partial_1(\alpha) = \alpha$ for all cw-pomsets α,β. $\square$

A proof follows immediately from the fact that $\alpha;\beta$ consists of α followed by β.

Similarly to the sequential composition, the parallel composition of cw-pomsets is defined by specifying how the result of composing in parallel two cw-pomsets is related to these cw-pomsets, and by showing that the respective relation defines a binary operation on cw-pomsets.

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of a cw-pomset.

By a *splitting* of $\mathcal{A}$ we mean a partition $p = (X', X'')$ of X into two disjoint subsets X', X'' which are *independent* in the sense that x', x'' are incomparable whenever $x' \in X'$ and $x'' \in X''$, each $s(v)$ is $(s(v)|X')(s(v)|X'')$, the concatenation of the restrictions of $s(v)$ to X' and X'', and each $t(v)$ is $(t(v)|X')(t(v)|X'')$, the concatenation of the restrictions of $t(v)$ to X' and X''. By $splittings(\mathcal{A})$ we denote the set of splittings of $\mathcal{A}$.

3.4. Proposition. For each $p = (X', X'') \in splittings(\mathcal{A})$ the restrictions of $\mathcal{A}$ to X' and X'' with arrangements of minimal elements given respectively by $s|X' = (s(v)|X' : v \in V)$ and $s|X'' = (s(v)|X'' : v \in V)$, and arrangements of maximal elements given respectively by $t|X' = (t(v)|X' : v \in V)$ and $t|X'' = (t(v)|X'' : v \in V)$, are instances of cw-pomsets. We write them respectively as $left_p(\mathcal{A})$ and $right_p(\mathcal{A})$. $\Box$

A proof is straightforward.

The cw-pomset $[\mathcal{A}]$ is said to *consist* of the cw-pomset $[left_p(\mathcal{A})]$ *accompanied* by the cw-pomset $[right_p(\mathcal{A})]$.

Note that each cw-pomset α can be represented in the form $[left_{(X,\emptyset)}(\mathcal{A})]$ and in the form $[right_{(\emptyset,X)}(\mathcal{A})]$, where $\mathcal{A} = (X, \leq, d, e, s, t)$ is any instance of α.

3.5. Proposition. For every two cw-pomsets α and β there exists a unique cw-pomset $\alpha \otimes \beta$ which consists of α accompanied by β. This cw-pomset is K-dense whenever α and β are K-dense, and it is a symmetry whenever α and β are symmetries. $\Box$

Proof: As instances of α and β may be chosen arbitrarily up to isomorphism, we may choose an instance $\mathcal{A} = (X_\mathcal{A}, \leq_\mathcal{A}, d_\mathcal{A}, e_\mathcal{A}, s_\mathcal{A}, t_\mathcal{A})$ of α and an instance $\mathcal{B} = (X_\mathcal{B}, \leq_\mathcal{B}, d_\mathcal{B}, e_\mathcal{B}, s_\mathcal{B}, t_\mathcal{B})$ of β such that $X_\mathcal{A}$ and $X_\mathcal{B}$ are disjoint. Then we define

$$X_\mathcal{C} = X_\mathcal{A} \cup X_\mathcal{B}$$

$$p = (X_\mathcal{A}, X_\mathcal{B})$$

$$x \leq_\mathcal{C} y \text{ whenever } x \leq_\mathcal{A} y \text{ or } x \leq_\mathcal{B} y$$
$$d_\mathcal{C}(x,y) = \begin{cases} d_\mathcal{A}(x,y) & \text{for } x, y \in X_\mathcal{A} \\ d_\mathcal{B}(x,y) & \text{for } x, y \in X_\mathcal{B} \\ -\infty & \text{for the remaining } x, y \in X_\mathcal{C} \end{cases}$$
$$e_\mathcal{C}(x) = \begin{cases} e_\mathcal{A}(x) & \text{for } x \in X_\mathcal{A} \\ e_\mathcal{B}(x) & \text{for } x \in X_\mathcal{B} \end{cases}$$
$$(s_\mathcal{C})(v) = ((s_\mathcal{A})(v))((s_\mathcal{B})(v)) \text{ for all } v \in V$$

$$(t_\mathcal{C})(v) = ((t_\mathcal{A})(v))((t_\mathcal{B})(v)) \text{ for all } v \in V.$$

It is straightforward to verify that the structure $\mathcal{C} = (X_\mathcal{C}, \leq_\mathcal{C}, d_\mathcal{C}, e_\mathcal{C}, s_\mathcal{C}, t_\mathcal{C})$ is an instance of $\alpha \otimes \beta$, as required. □

The operation $(\alpha, \beta) \mapsto \alpha \otimes \beta$ is called the *parallel composition* of cw-pomsets. An example of application of this operation is shown in figure 3.3.

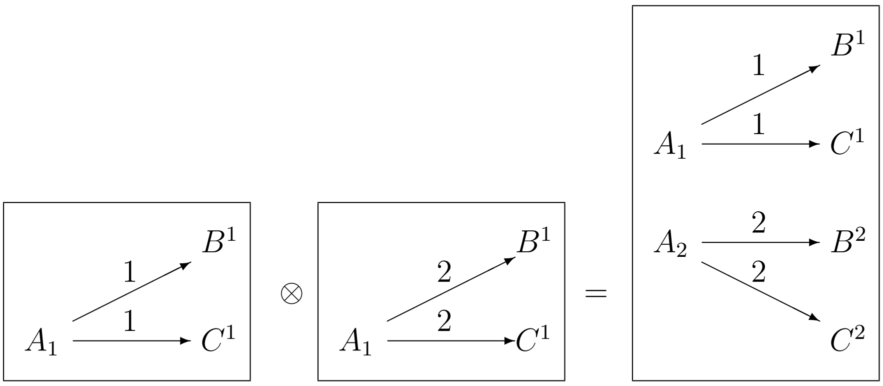

Figure 3.3

3.6. Proposition. The parallel composition is defined for all pairs (α, β) of cw-pomsets, it is associative, and has a neutral element *nil*, where *nil* is the unique cw-pomset with the empty instance. □

A proof follows immediately from the fact that $\alpha \otimes \beta$ consists of α accompanied by β.

3.7. Proposition. The parallel composition is *functorial* in the sense that

$$\alpha;\beta \otimes \gamma;\delta = (\alpha \otimes \gamma);(\beta \otimes \delta)$$

whenever $\alpha;\beta$ and $\gamma;\delta$ are defined. $\square$

Proof: Let $\mathcal{C} = (X, \leq, d, e, s, t)$ be an instance of $\alpha;\beta \otimes \gamma;\delta$. Then $\alpha;\beta = [left_p(\mathcal{C})]$ and $\gamma;\delta = [right_p(\mathcal{C})]$ for some $p = (X', X'') \in splittings(\mathcal{C})$, $\alpha = [head_{Y',r'}(left_p(\mathcal{C}))]$ and $\beta = [tail_{Y',r'}(left_p(\mathcal{C}))]$ for some Y' and r', and $\gamma = [head_{Y'',r''}(right_p(\mathcal{C}))]$ and $\delta = [tail_{Y'',r''}(right_p(\mathcal{C}))]$ for some Y'' and r''. Consequently, the restriction of $\mathcal{C}$ to $\downarrow Y' \cup \downarrow Y''$ with s playing the role of arrangement of minimal elements and $r = (r'(v)r''(v) : v \in V)$ playing the role of arrangement of maximal elements is an instance $\mathcal{A}_0$ of $\alpha \otimes \gamma$ and that to $\uparrow Y' \cup \uparrow Y''$ with r playing the role of arrangement of minimal elements and t playing the role of arrangement of maximal elements is an instance $\mathcal{C}_1$ of $\beta \otimes \delta$. As $Y = Y' \cup Y''$ is a cut and $\downarrow Y = \downarrow Y' \cup \downarrow Y''$, $\uparrow Y = \uparrow Y' \cup \uparrow Y''$, $\mathcal{C}$ is an instance of $(\alpha \otimes \gamma);(\beta \otimes \delta)$, as required. $\square$

The interchanges are operations which produce symmetries by combining trivial symmetries. They can be defined as follows.

Let $a_1, ..., a_n$ be trivial symmetries and let p be a permutation of the sequence $1, ..., n$.

3.8. Proposition. There exists a unique symmetry $I_p(a_1, ..., a_n)$ such that each instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of this symmetry can be partitioned into instances $\mathcal{A}_i = (X_i, \leq_i, d_i, e_i, s_i, t_i)$ of the respective a_i, where X is a disjoint union of all X_i, $\leq$ is a disjoint union of all $\leq_i$, d is a disjoint union of all d_i, e is a disjoint union of all e_i, each $s(v)$ is $s_1(v)...s_n(v)$, the concatenation of $s_1(v), ..., s_n(v)$, and each $t(v)$ is $s_{p(1)}(v)...s_{p(n)}(v)$, the concatenation of $s_{p(1)}(v), ..., s_{p(n)}(v)$. $\square$

For a proof it suffices to choose disjoint instances of the respective trivial symmetries and construct $\mathcal{A}$ by combining these instances such that the requirements of the proposition are satisfied.

The operation $(a_1, ..., a_n) \mapsto I_p(a_1, ..., a_n)$ is called the *interchange* of trivial symmetries according to p. By $*$ and I_* we denote respectively the permutation $1 \mapsto 2, 2 \mapsto 1$ and the corresponding interchange.

An example of application of this operation is shown in figure 3.4.

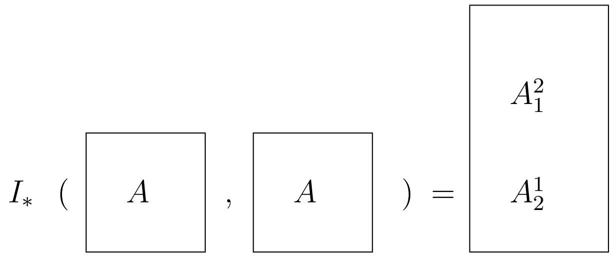

Figure 3.4

3.9. Proposition. The interchanges enjoy the following properties:

$$I_p(a_1, ..., a_n); I_{p^{-1}}(a_{p(1)}, ..., a_{p(n)}) = a_1 \otimes ... \otimes a_n$$

$$(I_*(a_1, a_2) \otimes a_3); (a_2 \otimes I_*(a_1, a_3)) = I_*(a_1, a_2 \otimes a_3). \ \square$$

A proof follows immediately from the definition.

The introduced operations are related as follows.

3.10. Proposition. The parallel composition is *coherent* in the sense that

$$I_p(u_1, ..., u_n); \alpha_{p(1)} \otimes ... \otimes \alpha_{p(n)} = \alpha_1 \otimes ... \otimes \alpha_n; I_p(v_1, ..., v_n)$$

for all $\alpha_1, ..., \alpha_n \in cwp(V)$ with $\partial_0(\alpha_i) = u_i$ and $\partial_1(\alpha_i) = v_i$, and for each permutation p of the sequence $1, ..., n$. $\square$

Proof: Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of $\alpha_1 \otimes ... \otimes \alpha_n$. For $i = 1, ..., n$ there exist instances $\mathcal{A}_i = (X_i, \leq_i, d_i, e_i, s_i, t_i)$ of the respective α_i such that all X_i are mutually disjoint, and $X_1 \cup ... \cup X_n = X$. Then $\mathcal{A}$ with t replaced by $t' = (t_{\mathcal{A}_{p(1)}}(v)...t_{\mathcal{A}_{p(n)}}(v) : v \in V)$ is an instance of $I_p(u_1, ..., u_n); \alpha_{p(1)} \otimes ... \otimes \alpha_{p(n)}$ and an instance of $\alpha_1 \otimes ... \otimes \alpha_n; I_p(v_1, ..., v_n)$ as well, which implies the required equality. $\square$

3.11. Proposition. The subset of K-dense cw-pomsets and the subset of symmetries are closed w.r. to the compositions and interchanges. $\square$

A proof is straightforward.

The stated properties of operations on cw-pomsets can be summarized in a brief way in the language of category theory.

3.12. Theorem. The (partial) algebra

$$CWP(V) = (cwp(V), \partial_0, \partial_1, ; , \otimes, nil, I_*)$$

is a symmetric strict monoidal category (the *monoidal category of cw-pomsets* over V) with cw-pomsets playing the role of morphisms, trivial symmetries playing the role of object identities, and I_* playing the role of a natural transformation from $(\alpha, \beta) \mapsto \alpha \otimes \beta$ to $(\alpha, \beta) \mapsto \beta \otimes \alpha$. It contains $DCWP(V)$, the subalgebra of K-dense cw-pomsets, and $SYM(V)$, the subalgebra of symmetries. $\square$

It is the matter of convenience rather than of merit to characterize the algebras $CWP(V)$, $DCWP(V)$, and $SYM(V)$, as monoidal categories. In the present paper we do not study these algebras from the point of view of category theory. The only fact which is important for our considerations is that they are some algebras and that their operations enjoy some specific properties.

In the rest of this section we describe how the subalgebra $DCWP(V)$ of K-dense cw-pomsets is situated in $CWP(V)$. The respective relation can be expressed with the aid of *atomic* cw-pomsets.

By atomic cw-pomsets we mean cw-pomsets of one of the following two types:

(1) *one-element* cw-pomsets, one for each $v \in V$, namely the one-element cw-pomset with v being the label of the only element of its instance,

(2) *prime* cw-pomsets $\pi = [\mathcal{P}]$ for some $\mathcal{P} = (X_\mathcal{P}, \leq_\mathcal{P}, d_\mathcal{P}, e_\mathcal{P}, s_\mathcal{P}, t_\mathcal{P})$ such that $X_\mathcal{P} = (X_\mathcal{P})_{min} \cup (X_\mathcal{P})_{max}$, where $(X_\mathcal{P})_{min}$ and $(X_\mathcal{P})_{max}$ are nonempty and disjoint and each $x \in (X_\mathcal{P})_{min}$ is comparable with each $y \in (X_\mathcal{P})_{max}$.

3.13. Proposition. Each K-dense cw-pomset α can be obtained from atomic cw-pomsets by applying interchanges and compositions. All the expressions representing α as a result of applying interchanges and compositions to atomic processes have the same number, written as $|\alpha|(\pi)$, of occurrences of each prime process π. $\square$

Proof: We start with recalling that each permutation is a superposition of transpositions of elements which are neighbours. Consequently, by applying interchanges and the parallel composition to one-element cw-pomsets we obtain all the possible symmetries over V.

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of α and let $Y_0 \sqsubseteq Y_1 \sqsubseteq ... \sqsubseteq Y_{n-1} \sqsubseteq Y_n$ be a maximal chain of maximal antichains of $\mathcal{A}$. Due to the maximality of this chain each $x \in Y_i - Y_{i+1}$ is comparable with each $y \in Y_{i+1} - Y_i$ (since otherwise between Y_i and Y_{i+1} there would be a maximal antichain containing x and y and it would be different from Y_i and Y_{i+1}).

There exists a symmetry $\sigma_1 = [\mathcal{S}_1]$ rearranging the minimal elements of X such that for each $v \in V$ the elements of $e^{-1}(v) \cap Y_0 \cap Y_1$ precede those of $e^{-1}(v) \cap (Y_0 - Y_1)$ and the orders of elements in $e^{-1}(v) \cap Y_0 \cap Y_1$ are consistent with an enumeration $y_{11}y_{12}...y_{1i_1}$ of entire $Y_0 \cap Y_1$. Besides, there exists an arrangement t_1 of elements of Y_1 which is identical with the arrangement of maximal elements of $\mathcal{S}_1$ in $Y_0 \cap Y_1$ and such that for each $v \in V$ the elements of $e^{-1}(v) \cap Y_0 \cap Y_1$ precede those of $e^{-1}(v) \cap (Y_1 - Y_0)$.

The restriction of $\mathcal{A}$ to $\uparrow Y_0 \cap \downarrow Y_1$ with the arrangement of minimal elements given by the arrangement of maximal elements of $\mathcal{S}_1$ and the arrangement of maximal elements given by t_1 is an instance of a cw-pomset $\alpha_1 = [\mathcal{A}_1]$.

Now, α_1 can be represented in the form

$$\alpha_1 = u_{11} \otimes ... \otimes u_{1i_1} \otimes \pi_1$$

where $u_{11}, ..., u_{1i_1}, \pi_1$ correspond to the respective restrictions of $\mathcal{A}$ to the subsets $\{y_{11}\}$, ..., $\{y_{1i_1}\}$, $(Y_0 - Y_1) \cup (Y_1 - Y_0)$. Thus we obtain a decomposition of α_1 into the one-element cw-pomsets u_{11},...,u_{1i_1} and the prime cw-pomset π_1.

Similarly, for Y_1,Y_2 we can define a symmetry rearranging the maximal elements of $\mathcal{A}_1$, the corresponding restriction $\mathcal{A}_2$ of $\mathcal{A}$, and a representation of $\alpha_2 = \mathcal{A}_2$ in the form

$$\alpha_2 = u_{21} \otimes ... \otimes u_{2i_2} \otimes \pi_2$$

and so on, until reaching

$$\alpha_n = u_{n1} \otimes ... \otimes u_{ni_n} \otimes \pi_n.$$

Finally, we define σ_{n+1} as the symmetry which rearranges the maximal elements of $\mathcal{A}_n$ to t.

Thus we obtain a sequence

$$\sigma_1, \alpha_1, \sigma_2, \alpha_2, ..., \sigma_n, \alpha_n, \sigma_{n+1}$$

such that $\sigma_1; \alpha_1; \sigma_2; \alpha_2; ...; \sigma_n; \alpha_n; \sigma_{n+1}$ is defined and equal to α, as required. Moreover, the subsets of X to which the prime cw-pomsets π_1,...,π_n correspond are determined uniquely by $\mathcal{A}$ and thus they do not depend on the particular choice of the maximal chain $Y_0 \sqsubseteq Y_1 \sqsubseteq ... \sqsubseteq Y_{n-1} \sqsubseteq Y_n$. Consequently, the number of copies of each prime process π_i which is needed in order to construct α depends only on α. $\square$

The one-element and prime cw-pomsets u_{ij} and π_i in this proof are called *components* of α.

The correspondence $\pi \mapsto |\alpha|(\pi)$, written as $|\alpha|$, may be regarded as the multiset of prime cw-pomsets which is needed to construct a cw-pomset α.

By *atomic*(V), *one_element*(V), *prime*(V) we denote respectively the set of atomic, one-element, and prime cw-pomsets over V. For each subset P of cw-pomsets over V by *closure*(P) we denote the least subset of *cwp*(V) that contains P and is closed w.r.

to interchanges and compositions. With these notions we can summarize our results as follows.

3.14. Theorem. The subalgebras $DCWP(V)$ and $SYM(V)$ of the monoidal category $CWP(V)$ are generated respectively by the set $atomic(V)$ of atomic cw-pomsets and the subset $one\_element(V)$ of one-element cw-pomsets in the sense that

$$dcwp(V) = closure(atomic(V))$$

$$sym(V) = closure(one\_element(V)) \quad \square$$

4 Tables

Tables of delays between data and results of processes (delay tables) will be represented by tables as defined in section 2, that is by cw-pomsets consisting only of minimal and maximal elements. Instances of such tables can be regarded as matrix-like objects with a special indexing of rows and columns as shown in figure 2.3.

As tables are cw-pomsets, the operations on cw-pomsets can be applied to tables. However, only in the case of the operations of taking the origin and the target, the parallel composition, and the interchanges, the respective results are tables, whereas this is not necessarily the case for the standard sequential composition of cw-pomsets. Consequently, a specific sequential composition must be defined for tables. To this end it suffices to use the standard sequential composition of cw-pomsets and to reduce the resulting cw-pomsets to tables. More precisely, for each cw-pomset α we define $table(\alpha)$ as the table whose instances are obtained from instances of α by ignoring elements which are neither minimal nor maximal. Then for arbitrary tables α and β such that $\partial_1(\alpha) = \partial_0(\beta)$ we define $\alpha;' \beta$, the *sequential composition of tables*, as $table(\alpha; \beta)$.

Thus we come to the following operations on tables:

$$\partial'_0(\alpha) = \partial_0(\alpha)$$

$$\partial'_1(\alpha) = \partial_1(\alpha)$$

$$\alpha;' \beta = table(\alpha; \beta)$$

$$\alpha \otimes' \beta = \alpha \otimes \beta$$

$$nil' = nil$$

$$I'_p(a_1, ..., a_n) = I_p(a_1, ..., a_n).$$

When endowed with these operations the set $tab(V)$ of tables over V forms a partial algebra.

4.1. Theorem. The (partial) algebra

$$TAB(V) = (tab(V), \partial'_0, \partial'_1, ;', \otimes', nil', I'_*)$$

is a symmetric strict monoidal category (the *monoidal category of tables* over V) with tables playing the role of morphisms, trivial table symmetries playing the role of object identities, and I'_* playing the role of a natural transformation from $(\alpha, \beta) \mapsto \alpha \otimes' \beta$ to $(\alpha, \beta) \mapsto \beta \otimes' \alpha$. This structure contains $SYM(V)$ as a subalgebra. The correspondence

$$\alpha \mapsto table(\alpha) : CWP(V) \to TAB(V)$$

is a homomorphism. The restriction of this homomorphism to the subalgebra of symmetries is the identity. $\square$

A proof reduces to a simple verification.

There is a close relation between the monoidal category of tables and an algebra of matrices.

An instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of a table α may be viewed as the matrix

$$(d(x, y) : x \in X_{min}, y \in X_{max})$$

where the rows and the columns are labelled as specified by e and they are arranged according to s and t, respectively. We represent such matrices as shown in figure 2.3.

Matrices corresponding to isomorphic instances of tables may be regarded as equivalent and tables themselves may be regarded as equivalence classes of such matrices.

The sequential composition of tables can be represented by an operation similar to matrix multiplication: a matrix $\mathcal{C}$ representing the sequential composition $\alpha;' \beta$ of tables

α and β can be obtained from a matrix $\mathcal{A}$ which represents α and a matrix $\mathcal{B}$ which represents β, where $(X_{\mathcal{A}})_{max} = (X_{\mathcal{B}})_{min} = U$, by defining:

$$d_{\mathcal{C}}(x,y) = max(d_{\mathcal{A}}(x,u) + d_{\mathcal{B}}(u,y) : u \in U).$$

An example of such a multiplication is shown in figure 4.1.

$$\begin{array}{c|cccc} & B^1 & B^2 & C^1 & C^2 \\ \hline A_1 & 1 & -\infty & 1 & -\infty \\ A_2 & -\infty & 2 & -\infty & 2 \end{array} \;\; ;' \;\; \begin{array}{c|cc} & D^1 & D^2 \\ \hline B_1 & 1 & -\infty \\ B_2 & -\infty & 1 \\ C_1 & 1 & -\infty \\ C_2 & -\infty & 1 \end{array} \;\; = \;\; \begin{array}{c|cc} & D^1 & D^2 \\ \hline A_1 & 2 & -\infty \\ A_2 & -\infty & 3 \end{array}$$

Figure 4.1

The parallel composition of tables can be represented by an operation similar to building a matrix from blocks: a matrix $\mathcal{C}$ representing the parallel composition $\alpha \otimes' \beta$ of tables α and β can be obtained from a matrix $\mathcal{A}$ which represents α and a matrix $\mathcal{B}$ which represents β, where $X_{\mathcal{A}} \cap X_{\mathcal{B}} = \emptyset$, by defining:

$$d_{\mathcal{C}}(x,y) = \begin{cases} d_{\mathcal{A}}(x,y) & \text{for } x,y \in X_{\mathcal{A}} \\ d_{\mathcal{B}}(x,y) & \text{for } x,y \in X_{\mathcal{B}} \\ -\infty & \text{for the remaining } x,y \in X_{\mathcal{C}} \end{cases}$$

An example of such an operation is shown in figure 4.2.

$$\begin{array}{c|cc} & B^1 & C^1 \\ \hline A_1 & 1 & 1 \end{array} \;\; \otimes' \;\; \begin{array}{c|cc} & B^1 & C^1 \\ \hline A_1 & 2 & 2 \end{array} \;\; = \;\; \begin{array}{c|cccc} & B^1 & B^2 & C^1 & C^2 \\ \hline A_1 & 1 & -\infty & 1 & -\infty \\ A_2 & -\infty & 2 & -\infty & 2 \end{array}$$

Figure 4.2

5 Processes of timed nets and their delay tables

Let $N = (Pl, Tr, pre, post, D)$ be a timed place/transition Petri net in the sense of [GV 87] with a set Pl of places of infinite capacities, a set Tr of transitions, input and output functions $pre, post : Tr \to Pl^+$, where Pl^+ denotes the set of multisets of places, and with a duration function $D : Tr \to [0, +\infty)$. The multiset $pre(\tau)$ represents a collection of tokens, $pre(\tau, p)$ tokens in each place p, which must be consumed in order to execute a transition τ. The multiset $post(\tau)$ represents a collection of tokens, $post(\tau, p)$ tokens in each place p, which is produced by executing τ. The non-negative real number $D(\tau)$ represents the duration of each execution of τ. In order to be able to represent processes of N by cw-pomsets we assume that $pre(\tau) \neq 0$, $post(\tau) \neq 0$, $D(\tau) \neq 0$ for all transitions τ, and that $pre(\tau)$, $post(\tau)$, $D(\tau)$ determine τ uniquely.

A distribution of tokens in places is represented by a marking $\mu \in Pl^+$, where $\mu(p)$, the multiplicity of p in μ, represents the number of tokens in p. If many executions of transitions are possible for the current marking but there is too few tokens to start all these executions then a conflict which thus arises is resolved in an indeterministic manner. We assume that it takes no time to resolve conflicts: when an execution of a transition can start, it starts immediately, or it is disabled immediately. Finally, we admit many concurrent nonconflicting executions of the same transition.

The behaviour of N can be described by characterizing the possible *processes* of N, where a process is either an execution of a transition, or a presence of a token in a place, or a combination of such processes. The processes are considered together with the lapse of time. Each such a process may be represented as a cw-pomset α, where each instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α represents a concrete process execution, elements of X represent the tokens which take part in this execution, the partial order $\leq$ specifies the causal succession of tokens, the weight function d specifies the delays with which tokens appear after their causal predecessors, the labelling function e characterizes each of the tokens, and s and t are respectively arrangements of the tokens which the process receives from its environment and an arrangement of the tokens which the process delivers to its environment. It may be given either without specifying when its tokens appear and then called a *free process*, or together with the respective appearance times and then called a *timed process*. In the first case the labelling e specifies for each token x only the place in which x appears. In the second case e consists of a *proper part*, e_{proper}, and of a *timing*,

e_{time}, that is $e(x) = (e_{proper}(x), e_{time}(x))$, where $e_{proper}(x)$ specifies the place in which x appears and $e_{time}(x)$ specifies the appearance time of x. In this case one is in a position to say whether the respective process is only *potential* and it is excluded by another process due to an earlier enabling of a transition, or it is *actual* and can really happen (cf. the next section).

Free processes of N are defined as follows.

For each place $p \in Pl$ we have the free process of presence of a token in p. This process, $fp(p)$, is defined as the one-element cw-pomset with the label p.

For each transition $\tau \in Tr$ we have the free process of executing τ. This process, $fp(\tau)$, is defined as the prime cw-pomset whose each instance $\mathcal{A}$ satisfies the following conditions: the cardinality of each set $e_{\mathcal{A}}^{-1}(p) \cap (X_{\mathcal{A}})_{min}$ is $pre(\tau, p)$, the cardinality of each set $e_{\mathcal{A}}^{-1}(p) \cap (X_{\mathcal{A}})_{max}$ is $post(\tau, p)$, and $d_{\mathcal{A}}(x, y) = D(\tau)$ for all $x \in (X_{\mathcal{A}})_{min}$ and $y \in (X_{\mathcal{A}})_{max}$.

Processes which are combinations of free processes of the above two types are defined as cw-pomsets which can be obtained from the respective atomic cw-pomsets of the forms $fp(p)$ and $fp(\tau)$ with the aid of compositions and interchanges. Thus we obtain a set $fproc(N)$ of cw-pomsets representing all possible free processes of N.

An example of a free process of the net in figure 5.1, where $D(\varphi) = D(\psi) = D(\tau) = 1$ and $D(\sigma) = D(\nu) = 2$, is shown in figure 5.2.

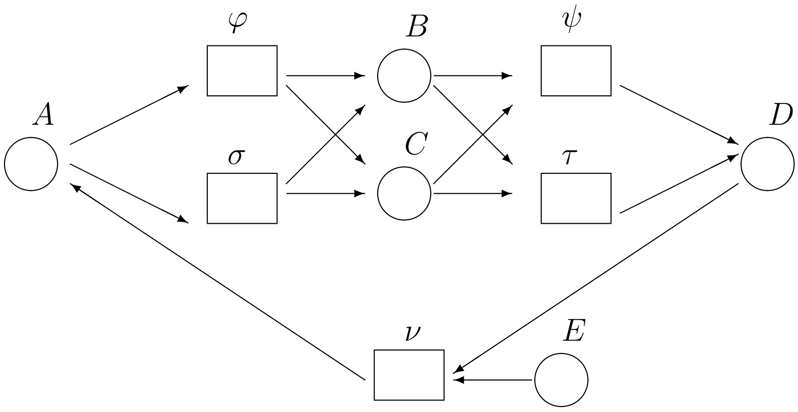

Figure 5.1

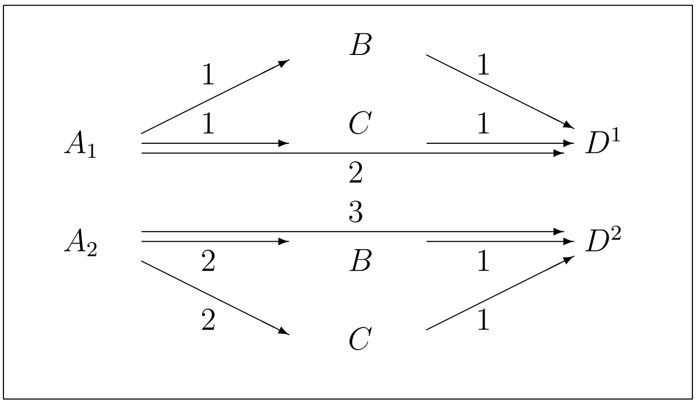

Figure 5.2

The set $fproc(N)$ of free processes of N is closed with respect to the considered operations on cw-pomsets. When equipped with the respective restrictions of these operations, it becomes a subalgebra $FPROC(N)$ of the monoidal category $DCWP(Pl)$. We call this subalgebra the *algebra of free processes* of N.

For each free process $\alpha \in fproc(N)$ we have $table(\alpha)$, called the *delay table* of α. Moreover, it is straightforward to verify the following property of this correspondence between free processes of N and their delay tables.

5.1. Theorem. The correspondence $\alpha \mapsto table(\alpha)$ is a homomorphism from $FPROC(N)$ to $TAB(Pl)$. $\Box$

Timed processes of N are defined as follows.

Given a delivery of tokens to places of N, we consider the appearance times of tokens delivered to each place $p \in Pl$ as arranged into an arbitrary (not necessarily monotonic) sequence $\vartheta(p)$ and define formally such a delivery as the family $\vartheta = (\vartheta(p) : p \in Pl)$. If only a single token is delivered to a place $p \in Pl$ at instant u then we identify the respective family ϑ with the single element of its only nonempty sequence, that is with u.

For each place $p \in Pl$ and each delivery of a token to p at instant u we have the timed process of presence of the delivered token in p starting from u. This process, $tp(p, u)$, is defined as the one-element cw-pomset with the label (p, u).

For each transition $\tau \in Tr$ and each delivery $\vartheta = (\vartheta(p) : p \in Pl)$ of tokens to places of N, where the length of $\vartheta(p)$ is $pre(\tau, p)$, we have a timed process of executing τ with a collection of delivered tokens, say $X_{in} = \{x(p, i) : p \in Pl, 1 \leq i \leq pre(\tau, p)\}$, and a collection $X_{out} = \{y(q, j) : q \in Pl, 1 \leq j \leq post(\tau, p)\}$ of produced tokens, where each $x(p, i)$ appears at instant $\xi(p, i) = (\vartheta(p))(i)$ and each $y(q, j)$ appears at instant $\eta(q, j) = max((\vartheta(p))(k) : p \in Pl, 1 \leq k \leq pre(\tau, p)) + D(\tau)$. This process, $tp(\tau, \vartheta)$, is defined as the prime cw-pomset with the instance $\mathcal{A} = (X, \leq, d, e, s, t)$, where

$$X = X_{min} \cup X_{max} \text{ with } X_{min} = X_{in} \text{ and } X_{max} = X_{out}$$

$$d(x, y) = \begin{cases} D(\tau) & \text{for } x \in X_{in} \text{ and } y \in X_{out} \\ -\infty & \text{for the remaining } x, y \in X \end{cases}$$

$$e(x) = (e_{proper}(x), e_{time}(x))$$

with

$$e_{proper}(z) = \begin{cases} p & \text{for } z = x(p, i) \in X_{in} \\ q & \text{for } z = y(q, j) \in X_{out} \end{cases}$$

$$e_{time}(z) = \begin{cases} \xi(p, i) & \text{for } z = x(p, i) \in X_{in} \\ \eta(q, j) & \text{for } z = y(q, j) \in X_{out} \end{cases}$$

and where $s(p, u)$ is the subsequence of the sequence $x(p, 1)...x(p, pre(\tau, p))$ consisting of those $x(p, i)$ for which $\xi(p, i) = u$, and $t(q, w)$ is the subsequence of the sequence $y(q, 1)...y(q, post(\tau, q))$ consisting of those $y(q, j)$ for which $\eta(q, j) = w$. (Note that all $\eta(q, j)$ are equal, which implies that either $t(q, w)$ is entire sequence $y(q, 1)...y(q, post(\tau, q))$ or $t(q, w)$ is empty.)

Processes which are combinations of timed processes of the above two types are defined as cw-pomsets which can be obtained from the respective atomic cw-pomsets of the forms $tp(p, u)$ or $tp(\tau, \vartheta)$ with the aid of compositions and interchanges. Thus we obtain a set $tproc(N)$ of cw-pomsets representing all timed processes of N.

An example of a timed process of the net in figure 5.1 is shown in figure 5.3.

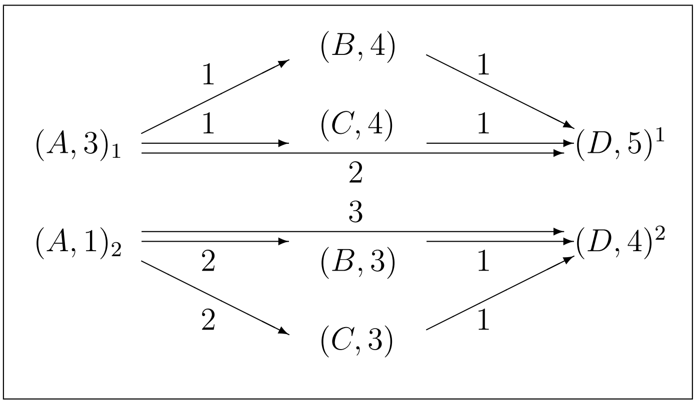

Figure 5.3

The set $tproc(N)$ of timed processes of N is closed with respect to the considered operations on cw-pomsets. When equipped with the respective restrictions of these operations, it becomes a subalgebra $TPROC(N)$ of the monoidal category $CWP(Pl \times R)$. We call this subalgebra the *algebra of timed processes* of N.

Timed processes enjoy the following property.

5.2. Proposition. The value $e_{time}(y)$ of the timing of an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of a timed process for $y \in X - X_{min}$ is given by the following formula:

$$e_{time}(y) = max(e_{time}(x) + d(x, y) : x \leq y, x \neq y). \ \Box$$

Proof: The property formulated in the proposition holds for timed processes representing presences of tokens in places and executions of transitions, and it is preserved under the parallel composition and interchanges. Thus it suffices to show that it is preserved under the sequential composition. To this end suppose that the proposition holds for instances $\mathcal{A}$ and $\mathcal{B}$ of processes α and β such that $\alpha;\beta$ is defined and has an instance $\mathcal{C}$ with $head_{Y,r}(\mathcal{C}) = \mathcal{A}$ and $tail_{Y,r}(\mathcal{C}) = \mathcal{B}$. According to 2.6 it suffices to show that for all $y \in X_{\mathcal{C}} - (X_{\mathcal{C}})_{min}$ we have the formula:

$$e_{time}(y) = max(e_{time}(x) + d(x,y) : x \text{ immediately precedes } y).$$

To this end it suffices to notice that if $y \in X_{\mathcal{A}}$ then the formula follows from the assumed property of $\mathcal{A}$, and that if $y \in X_{\mathcal{B}} - X_{\mathcal{A}}$ then the immediate predecessors of y are in $X_{\mathcal{B}}$ and, consequently, the formula follows from the assumed property of $\mathcal{B}$. Thus the formula holds for all $y \in X_{\mathcal{C}}$. $\square$

For each timed process $\alpha \in tproc(N)$ we have a free process, $free(\alpha) \in fproc(N)$, namely the free process whose instance can be obtained from any instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α by reducing the labelling function $e : X \to Pl \times R$ to its proper part $e_{proper} : X \to Pl$.

5.3. Proposition. For each free process α and each delivery $\vartheta = (\vartheta(p) : p \in Pl)$ of tokens to places of N, where the length of each $\vartheta(p)$ coincides with the multiplicity of p in $\partial_0(\alpha)$, there exists a unique timed process $\beta \in tproc(N)$, written also as $timed(\vartheta, \alpha)$ and called the result of *applying* α to ϑ, such that $free(\beta) = \alpha$ and the multiplicity of each pair (p, w), where $p \in Pl$ and w is an instant, coincides with the multiplicity of this pair in $\partial_0(\beta)$. Moreover, each timed process $\beta \in tproc(N)$ is of the form $timed(\vartheta, \alpha)$ for some ϑ and α as above. $\square$

Proof: Consider an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α. Replace the labelling function e by e', where $e'(x) = (e'_{proper}(x), e'_{time}(x))$ with $e'_{proper}(x) = e(x)$ and $e'_{time}(x) = (\vartheta(p))(i)$ for a minimal x with $e(x) = p$ and $(s(p))(i) = x$, and $e'_{time}(y) = max(e'_{time}(x) + d(x,y) : x \leq y, x \neq y)$ for each y which is not minimal. It is straightforward to verify that the structure thus obtained is an instance of a timed process as required. For the second part of the proposition it suffices to define ϑ as consisting of the values of timing for the minimal elements of an instance of β and to define α as $free(\beta)$. $\square$

5.4. Theorem. The correspondence $\alpha \mapsto free(\alpha) : TPROC(N) \to FPROC(N)$ is a homomorphism and it is surjective. $\square$

A proof follows directly from the respective definitions and from 5.3.

From 5.3 it follows that, being relatively small, the algebra of free processes of N determines uniquely the much larger algebra of timed processes of N. Nevertheless, we cannot avoid dealing with timed processes since they are needed in order to formulate important concepts and problems.

Firstly, we are interested in concrete executions of timed nets and these can be represented only as timed processes. Secondly, timed processes are only potential since some of them can be excluded by other timed processes due to an earlier enabling of transitions. For example, the process in figure 5.4 is only potential since it can be excluded by the process in figure 5.3 due to the fact that the transition starting from $(B, 3)$ and $(C, 3)$ is enabled before the transition starting from $(B, 4)$ and $(C, 3)$.

Consequently, we have to confront processes in order to see which of them can really happen, and it makes sense only for timed processes. Finally, from timed processes of a net we are able to reconstruct its firing sequences similar to real-time executions in the sense of [GV 87].

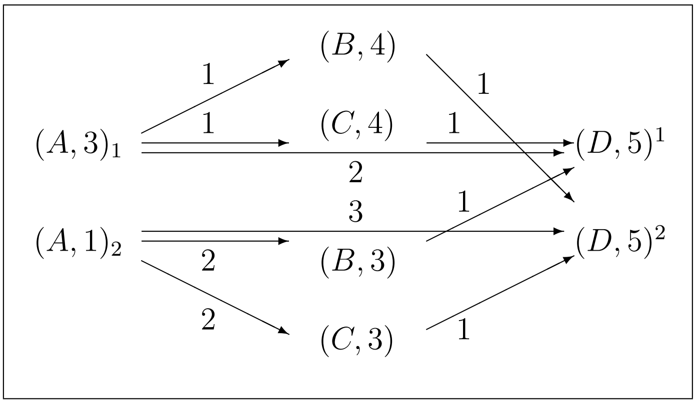

Figure 5.4

6 Realizable processes

Let N be a timed net as in the previous section. In order to characterize those timed processes of N which can really happen we have to describe formally how a timed process may exclude another due to an earlier enabling of a transition.

We start with some auxiliary notions and observations.

Each timed process $\alpha \in tproc(N)$ has a unique *beginning of activity*, $act(\alpha)$, and a unique *beginning of completion*, $cpl(\alpha)$, where

$$act(\alpha) = inf(sup(e_{time}(x) : x \leq y, x \neq y) : y \in X - X_{min})$$

$$cpl(\alpha) = sup(e_{time}(x) : x \leq y \text{ and } x \neq y \text{ for some } y \in X_{max} - X_{min})$$

for each instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α. In particular, $act(\alpha) = +\infty$ if α is a symmetry (since then $X - X_{min}$ is empty), and $cpl(\alpha) = -\infty$ if α is a symmetry (since then $X_{max} - X_{min}$ is empty). Intuitively, $act(\alpha)$ and $cpl(\alpha)$ are respectively the earliest and the latest instants at which some of the transitions represented in α start.

6.1. Proposition. Given a timed process $\alpha \in tproc(N)$, for each instant $u < +\infty$ there exists a decomposition $\alpha = \alpha_1; \alpha_2$ such that $cpl(\alpha_1) \leq u < act(\alpha_2)$. Such a decomposition, called in the sequel a *natural* decomposition at u, is unique up to a symmetry in the sense that $\partial_1(\alpha_1) = \partial_1(\alpha_1')$ and $\alpha_1' = \alpha_1; \sigma$ and $\alpha_2' = \sigma^{-1}; \alpha_2$ with a symmetry σ and its inverse σ^{-1} for each other decomposition $\alpha = \alpha_1'; \alpha_2'$ satisfying $cpl(\alpha_1') \leq u < act(\alpha_2')$. $\square$

Proof: Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be any instance of α. Define $X(u)$ as the set of $x \in X$ such that either $x \in X_{min}$, or $e_{time}(x) \leq u$, or $e_{time}(y) \leq u$ for each y being an immediate predecessor of x.

The set $X(u)$ represents the tokens which are received from the environment or are results of those transitions represented in α which start not later than at u (for example, for an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of the process in figure 5.3 and for $u = 3$ the set $X(u)$ consists of the occurrences of $(A, 3)_1$, $(B, 4)$, $(C, 4)$, $(A, 1)_2$, $(B, 3)$, $(C, 3)$, $(D, 4)^2$, as it is shown in figure 6.1).

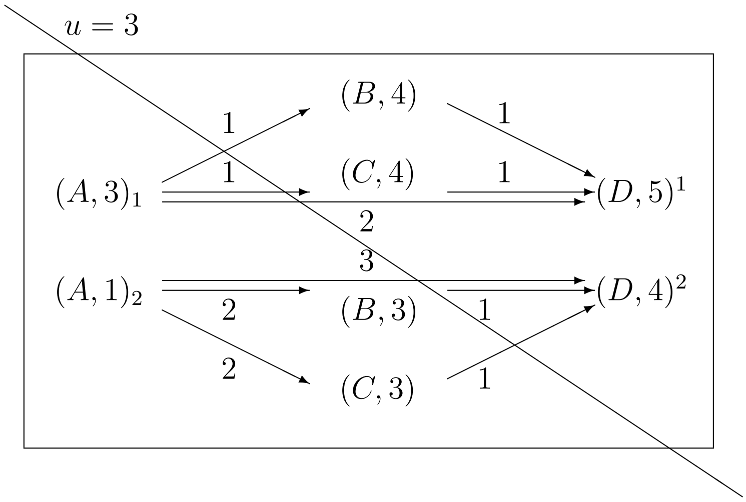

Figure 6.1

Define $Y(u)$ as the subset of those elements of $X(u)$ which are maximal in $X(u)$. Now we shall show that $Y(u)$ is a maximal antichain of $(X, \leq)$.

From the definition of $X(u)$ and the fact that $x \leq y$ implies $e_{time}(x) \leq e_{time}(y)$ it follows that $y \in X(u)$ implies $x \in X(u)$ for all x such that $x \leq y$. Hence $Y(u)$ is an antichain. In order to prove that $Y(u)$ is a maximal antichain suppose that some $z \in X$ is incomparable with all $y \in Y(u)$. Then it must be $e_{time}(z) > u$ and z cannot be in X_{min} which is contained in $X(u)$. As there exists a maximal chain from an element of X_{min} to z, we may assume that z has an immediate predecessor $y \in X(u)$. As y is comparable with z, it cannot belong to $Y(u)$ and thus it must have an immediate successor $x \in X(u)$. As x does not belong to X_{min}, we have $e_{time}(x) \leq u$ or $e_{time}(x') \leq u$ for all immediate predecessors of x and, in particular, for y. As z does not belong to $X(u)$, it must have an immediate predecessor y' with $e_{time}(y') > u$ and, due to the K-density of $\mathcal{A}$, this predecessor must be an immediate predecessor of x (cf. the proof of 3.13). We have $e_{time}(x') \geq e_{time}(y') > u$ and hence for all x' which are immediate predecessors of x, including y', there must be $e_{time}(x') \leq u$, which contradicts to $e_{time}(y') > u$. Consequently, $Y(u)$ is indeed a maximal antichain of $(X, \leq)$.

As $Y(u)$ is a maximal antichain of $(X, \leq)$, it is a cut of $\mathcal{A}$. Thus we can choose an arbitrary arrangement r of elements of $Y(u)$ and define $\alpha_1 = [head_{Y(u),r}(\mathcal{A})]$ and $\alpha_2 = [tail_{Y(u),r}(\mathcal{A})]$. It is easy to verify that $\alpha = \alpha_1; \alpha_2$ is a decomposition as required.

□

From 6.1 it follows that for each timed process $\alpha \in tproc(N)$ and for each instant $u < +\infty$ we have a set $\alpha|u$ of processes α_1 such that $\alpha = \alpha_1; \alpha_2$ with $cpl(\alpha_1) \leq u < act(\alpha_2)$ for a unique α_2, and that $\partial_1(\alpha'_1) = \partial_1(\alpha_1)$ and $\alpha'_1 = \alpha_1; \sigma$ with a symmetry σ whenever $\alpha_1, \alpha'_1 \in \alpha|u$.

The phenomenon of exclusion of a timed process by another such a process due to an earlier enabling of a prime component can be described with the aid of concepts of dominance and realizability.

Given two timed processes α and β, we say that β *dominates* α if there exist an instant $w \leq cpl(\alpha)$ and decompositions $\alpha = \alpha_1; \alpha_2$ and $\beta = \beta_1; \beta_2$ such that $\alpha_1 = \beta_1 \in \alpha|w = \beta|w$ and either α_2 is not a symmetry (that is $\alpha_2 \neq \partial_0(\alpha_2)$) and then $act(\beta_2) < act(\alpha_2)$, or α_2 is a symmetry (that is $\alpha_2 = \partial_0(\alpha_2)$) and then $act(\beta_2) \leq cpl(\alpha_1)$.

The first case corresponds to a situation when α and β develop identically up to w and then some transitions are executed both in α and in β such that some transition of β starts before all the transitions of α that are still to be executed. The second case corresponds to a situation when α and β develop identically up to w and then no more transition is executed in α while still some transitions of β could start at w.

Given any set P of timed processes, a member α of P is said to be *realizable* in this set if there is no $\beta \in P$ which dominates α. Thus P determines a subset $real(P)$ of its realizable members.

For example, the timed process in figure 5.3 is realizable in the set of timed processes of the net in figure 5.1 whereas the process in figure 5.4 is not realizable.

Note that the realizability of a timed process of N is a property which depends not only on this process but also on other processes in $tproc(N)$. In fact, checking the realizability of a process does not require considering all processes in $tproc(N)$. We are able to formulate a simple criterion of realizability according to which the checked process must be confronted only with prime processes corresponding to transitions. This can be done as follows.

6.2. Proposition. A timed process $\alpha \in \mathit{tproc}(N)$ is realizable iff for each natural decomposition $\alpha = \alpha_1; \alpha_2$ there is no prime $\pi \in \mathit{tproc}(N)$ such that:

(1) π is *enabled* after α_1 in the sense that $\partial_0(\alpha_2) = \partial_0(\pi) \otimes c$ with some c,

(2) either α_2 is not a symmetry and then $\mathit{act}(\pi) < \mathit{act}(\alpha_2)$, or α_2 is a symmetry and then $\mathit{act}(\pi) \leq \mathit{cpl}(\alpha_1)$. $\square$

Proof: Suppose that α is realizable and that (1) and (2) holds for a natural decomposition $\alpha = \alpha_1; \alpha_2$ at u and for some prime π. As π is enabled after α_1, by composing in parallel π and one-element timed processes of N we obtain $\alpha_2' \in \mathit{tproc}(N)$ such that $\partial_1(\alpha_1) = \partial_0(\alpha_2')$. Thus we construct a timed process $\beta = \alpha_1; \alpha_2' \in \mathit{tproc}(N)$. Consequently, for $w = \mathit{act}(\pi)$, we obtain decompositions $\alpha = \alpha_1; \alpha_2$ and $\beta = \alpha_1; \alpha_2'$ such that $\alpha_1 \in \alpha|w = \beta|w$ and either α_2 is not a symmetry and then $\mathit{act}(\alpha_2') = \mathit{act}(\pi) < \mathit{act}(\alpha_2)$ or α_2 is a symmetry and then $\mathit{act}(\alpha_2') \leq \mathit{cpl}(\alpha_1)$. This means that β dominates α, which contradicts to the assumed realizability of α.

Suppose that α is dominated by β with $w \leq \mathit{cpl}(\alpha)$ and decompositions $\alpha = \alpha_1; \alpha_2$ and $\beta = \beta_1; \beta_2$ such that α_2 is not a symmetry, $\alpha_1 = \beta_1 \in \alpha|w = \beta|w$ and $\mathit{act}(\beta_2) < \mathit{act}(\alpha_2)$). Then $\alpha = \alpha_1; \alpha_2$ is a natural decomposition at w and, due to $\mathit{act}(\beta_2) < \mathit{act}(\alpha_2)$, β_2 has a prime component π such that $\partial_0(\alpha_2) = \partial_0(\beta_2) = \partial_0(\pi) \otimes c$ with some c and $\mathit{act}(\pi) < \mathit{act}(\alpha_2)$. Thus the decomposition $\alpha = \alpha_1; \alpha_2$ is natural and (1) and (2) holds for α_1, α_2, π. Similarly for the case with α_2 being a symmetry and $\mathit{act}(\beta_2) \leq \mathit{cpl}(\alpha_1)$ $\square$

Another criterion of realizability of timed processes of a net can be formulated in terms of process instances.

6.3. Proposition. A timed process $\alpha \in \mathit{tproc}(N)$ is realizable iff it has an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ such that each antichain $Y \subseteq X$, where the restriction of $\mathcal{A}$ to Y (with an arrangement of elements) is an instance of the source of a prime timed process $\pi \in \mathit{tproc}(N)$ with $\mathit{act}(\pi) \leq \mathit{cpl}(\alpha)$, contains a minimal element of a subset $X' \subseteq X$, where the restriction of $\mathcal{A}$ to X' (with some arrangements of minimal and maximal elements) is an instance of a prime component ϱ of α with $act(\varrho) \leq act(\pi)$. The existence of an instance $\mathcal{A}$ with such a property implies that all instances of α enjoy this property. $\square$

Proof: Necessity. Suppose that α is realizable. Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be any instance of α. Let $Y \subseteq X$ be an antichain such that the restriction of $\mathcal{A}$ to Y is an instance of the source of a prime timed process $\pi \in tproc(N)$ with $act(\pi) \leq cpl(\alpha)$. Let $\alpha = \alpha_1; \alpha_2$ be a natural decomposition of α at $u = act(\pi)$. Let Z be a cut of $\mathcal{A}$ and s an arrangement of Z such that $head_{Z,s}(\mathcal{A})$ is an instance of α_1 and $tail_{Z,s}(\mathcal{A})$ is an instance of α_2.

Suppose that $X' \cap Y = \emptyset$ for $X' \subseteq X$ such that the restriction of $\mathcal{A}$ to X' is an instance of a prime component ϱ of α with $act(\varrho) \leq act(\pi)$. Then each instance of a prime component of α that is contained in $\mathcal{A}$ and has a minimal element in Y must be contained in $tail_{Z,s}(\mathcal{A})$. Consequently, Y must be contained in Z, which implies that π is enabled after α_1 in the sense of (1) of 6.2. On the other hand, $cpl(\alpha_1) \leq u = act(\pi) < act(\alpha_2)$ by the definition of the decomposition $\alpha = \alpha_1; \alpha_2$. In the case of $\alpha_2 \neq \partial_0(\alpha 2)$ by 6.2 this implies that α is not realizable. In the case of α_2 being a symmetry we have $cpl(\alpha) = cpl(\alpha_1)$ and α cannot be realizable as well. Thus Y must contain a minimal element of $X' \subseteq X$, as required.

Sufficiency. Suppose that the respective instance $\mathcal{A} = (X, \leq, d, e, s, t)$ exists. Consider a natural decomposition $\alpha = \alpha_1; \alpha_2$ at u, where $cpl(\alpha_1) \leq u < act(\alpha_2)$. Let Z be a cut of $\mathcal{A}$ and s an arrangement of Z such that $head_{Z,s}(\mathcal{A})$ is an instance of α_1 and $tail_{Z,s}(\mathcal{A})$ is an instance of α_2.

Assuming that α_2 is a symmetry and that there exists a prime timed process π as in 6.2 we obtain that Z must contain Y such that the restriction of $\mathcal{A}$ to Y is an instance of the source of π, and that in $\mathcal{A}$ there is no instance of a prime component of α with a minimal element in Y, which contradicts to our assumption on $\mathcal{A}$.

Assuming that α_2 is not a symmetry and that there exists a prime timed process π as in 6.2 we obtain again that Z contains Y such that the restriction of $\mathcal{A}$ to Y is an instance of the source of π, and that in $\mathcal{A}$ there is no instance of a prime component ϱ of α with a minimal element in Y and with $act(\varrho) \leq act(\pi)$, which contradicts to our assumption on $\mathcal{A}$.

Thus α must be realizable. $\square$

The criterion of realizability in 6.3 can be simplified with the aid of two auxiliary notions.

Let $\mathcal{A} = (X, \leq, d, e, s, t)$ be an instance of a timed process $\alpha \in tproc(N)$. Given an antichain $Y \subseteq X$ and a transition $\tau \in Tr$, we say that Y *enables* τ in α if the restriction of $\mathcal{A}$ to Y is an instance of the source of a prime timed process $\pi = tp(\tau, \vartheta)$ with $act(\pi) \leq cpl(\alpha)$, and we say that Y *originates* τ in α if Y is the set of minimal elements of a subset $X' \subseteq X$ such that the restriction of $\mathcal{A}$ to X' is an instance of a prime timed process $tp(\tau, \vartheta)$.

With these notions 6.3 can be reformulated as follows.

6.4. Proposition. A timed process $\alpha \in tproc(N)$ is realizable iff the following inequality is satisfied for each antichain Y of an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of α such that Y enables a transition $\pi \in Tr$ and for some antichain $Z(Y)$ of $\mathcal{A}$ such that $Y \cap Z(Y) \neq \emptyset$ and $Z(Y)$ originates a transition $\tau' \in Tr$:

$$max(e_{time}(z) : z \in Z(Y)) \leq max(e_{time}(y) : y \in Y). \ \square$$

The criteria of realizability in 6.3 and 6.4 can be illustrated on the example of the timed process in figure 5.3. In order to check the realizability of this process in the set of timed processes of the net in figure 5.1 it suffices to consider those antichains of an instance of this process which might originate some executions of transitions, but do not originate them due to conflicts with transitions which are represented in the considered process, and to see if the respective executions have been eliminated by the represented ones due to an earlier enabling. There are two such antichains: the antichain consisting of the occurrences of $(B, 3)$ and $(C, 4)$, say Y_1, and the antichain consisting of the occurrences of $(C, 3)$ and $(B, 4)$, say Y_2. On the other hand, the antichain consisting of the occurrences of $(B, 3)$ and $(C, 3)$, say Z, originates the execution of a transition which is represented in the considered process and we have

$$max(e_{time}(z) : z \in Z) = 3 < max(e_{time}(y) : y \in Y_1) = 4$$

and

$$max(e_{time}(z) : z \in Z) = 3 < max(e_{time}(y) : y \in Y_2) = 4.$$

So, the considered process of the net in figure 5.1 satisfies the criterion of realizability in 6.4.

Taking into account 6.4 and the fact that, due to (2) of 2.1, 2.5, 5.2, and 5.3, the appearance time of each token y of an instance $\mathcal{A} = (X, \leq, d, e, s, t)$ of a timed process β which is the result of applying a free process α to a delivery ϑ of tokens is given by the formula:

$$e_{time}(y) = max(e_{time}(x) + d(x, y) : x \leq y, x \in X_{min}),$$

we obtain, for each antichain Y of $\mathcal{A}$ that enables a transition in α, the following inequality

$$min(max(max(e_{time}(x) + d(x, z) : x \leq z, x \in X_{min}) : z \in Z) : Z \in \zeta(Y)) \leq$$

$$max(max(e_{time}(x) + d(x, y) : x \leq y, x \in X_{min}) : y \in Y),$$

where $\zeta(Y)$ denotes the set of those antichains of $\mathcal{A}$ which originate in α a transition of N and have some elements in Y. Consequently, the result of applying to α a delivery ϑ of tokens is a realizable timed process of N iff all ϑ_x defined as $(\vartheta(p))(i)$ with p and i such that $(s(p))(i) = x$ satisfy the system of inequalities

$$min(max(max(\vartheta_x + d(x, z) : x \leq z, x \in X_{min}) : z \in Z) : Z \in \zeta(Y)) \leq$$

$$max(max(\vartheta_x + d(x, y) : x \leq y, x \in X_{min}) : y \in Y)$$

with Y ranging over all antichains of $\mathcal{A}$ which enable transitions in α.

By indexing elements $x \in X_{min}$ and antichains Z and Y and by replacing them by the respective indices k, j, i, and by transforming the above inequalities, we obtain the following result.

6.5. Corollary. To each free process $\alpha \in fproc(N)$ there corresponds a system of inequalities of the form:

$$min(max(a_{1jk} + \vartheta_k : 1 \leq k \leq l) : 1 \leq j \leq m) \leq max(b_{1k} + \vartheta_k : 1 \leq k \leq l)$$

$$...$$

$$min(max(a_{njk} + \vartheta_k) : 1 \leq k \leq l) : 1 \leq j \leq m) \leq max(b_{nk} + \vartheta_k : 1 \leq k \leq l),$$

where a_{ijk} and b_{ik} are constants from the semiring R of real numbers and infinities, such that the result of applying α to a delivery ϑ of tokens is a realizable timed process of N iff the appearance times ϑ_k of the delivered tokens satisfy this system of inequalities. □

For example, for the free process in figure 5.2 we obtain the following inequalities for p, q denoting the appearance times of tokens in A:

$$p + 2 \leq max(p + 2, q + 1)$$
$$q + 1 \leq max(p + 2, q + 1).$$

As these inequalities are satisfied for all p, q, all timed processes which are results of applying the free process in figure 5.2 to a delivery of two tokens to A are realizable. Note that for the free process of the net in figure 5.1 in which the token produced in B by executing φ and that produced in C by executing σ are used to execute τ, whereas the token produced in C by executing φ and that produced in B by executing σ are used to execute ψ, we obtain the converse inequalities which are satisfied only by some p, q.

7 Processes and firing sequences

With the concept of realizability we are able to say which timed processes of the considered net N are not only potential, but also actual, and we are able to describe how they define firing sequences of N.

From 6.1 it follows that for each timed process $\alpha \in tproc(N)$ and for each instant $u < +\infty$ we have a set $\alpha|u$ of processes α_1 such that $\alpha = \alpha_1; \alpha_2$ with $cpl(\alpha_1) \leq u < act(\alpha_2)$ for a unique α_2, and that $\partial_1(\alpha'_1) = \partial_1(\alpha_1)$ and $\alpha'_1 = \alpha_1; \sigma$ with a symmetry σ whenever $\alpha_1, \alpha'_1 \in \alpha|u$. In particular, $|free(\alpha'_1)| = |free(\alpha_1)|$ and $\partial_1(\alpha_1) = \partial_1(\alpha'_1)$ whenever $\alpha_1, \alpha'_1 \in \alpha|u$.

Thus we obtain a multiset $\Theta_{\alpha,u}$ of transitions of N and a multiset $\mu_{\alpha,u}$ of presences of tokens in places of N, where the multiplicity $\Theta_{\alpha,u}(\tau)$ of each transition τ in $\Theta_{\alpha,u}$ is defined as the multiplicity of the respective prime free process $fp(\tau)$ in the multiset $|free(\alpha_1)|$, and the multiplicity $\mu_{\alpha,u}(p,w)$ of each presence of a token appearing at an instant w in a place p is defined as the multiplicity of the pair (p,w) in $ms(\partial_1(\alpha_1))$, the multiset corresponding to $\partial_1(\alpha_1)$, for some $\alpha_1 \in \alpha|u$. It is clear that the multisets $\Theta_{\alpha,u}$ and $\mu_{\alpha,u}$ do not depend on the choice of α_1 in $\alpha|u$. Intuitively, $\Theta_{\alpha,u}$ is the multiset consisting of those transitions represented in α which start not later than at u, and $\mu_{\alpha,u}$ is the multiset consisting of those tokens represented in α which are not consumed before u or at u. The multiset $\mu_{\alpha,u}$ may be regarded as a *timed marking* whose each item (p,w) represents a token which appears in the place p at the instant w, and whose value $\mu_{\alpha,u}(p,w)$ for such an item represents the multiplicity of this item in $\mu_{\alpha,u}$. Note that it represents not only the tokens existing immediately after u, but also the tokens which are produced due to transitions going on at u or are delivered by the environment after u.

From these observations it follows that to each $\alpha \in tproc(N)$ there corresponds a sequence $-\infty = u_0 < u_1 < ... < u_n < u_{n+1} = +\infty$ such that $\alpha|u$, $\mu_{\alpha,u}$, $\Theta_{\alpha,u}$ are constant and respectively equal to some α_i,μ_i,Θ_i on each interval $[u_i, u_{i+1})$. In this manner to α there corresponds a sequence $fs(\alpha) = \mu_0[\Theta_1\rangle\mu_1...[\Theta_n\rangle\mu_n$ which may be regarded as a candidate for a possible firing sequence of N. For example, for α being the timed process in figure 5.3 we obtain

$$fs(\alpha) = \mu_0[\sigma\rangle\mu_1[\varphi\rangle\mu_2[\tau\rangle\mu_3[\psi\rangle\mu_4,$$

where

$$\mu_0 = (A,1) + (A,3),$$

$$\mu_1 = (A,3) + (B,3) + (C,3),$$

$$\mu_2 = (B,3) + (C,3) + (B,4) + (C,4),$$

$$\mu_3 = (B,4) + (C,4) + (D,4),$$

$$\mu_4 = (D,4) + (D,5),$$

and where $\alpha_1 x_1 + ... + \alpha_m x_m$ denotes the multiset with multiplicities $\alpha_1, ..., \alpha_m$ of $x_1, ..., x_m$, respectively.

Whether indeed $fs(\alpha)$ can be regarded as a possible firing sequence depends on whether the process α is actual, that is realizable in the set of timed processes of N.

Thus we obtain a set $real(tproc(N))$ of realizable timed processes of N such that only members of this set can be regarded as actual timed processes of N, and firing sequences of N can be defined as $fs(\alpha)$ for realizable α. This is justified by the following fact.

7.1. Theorem. If $fs(\alpha) = \mu_0[\Theta_1)\mu_1...[\Theta_n)\mu_n$ for some $\alpha \in real(tproc(N))$ then for each $i = 1, ..., n$ there exists a time instant u_i such that

(1) u_i is the earliest instant of time such that, for some τ and all $p \in Pl$,

$$\Sigma(\mu_{i-1}(p,u) : u \leq u_i) \geq pre(\tau, p),$$

(2) Θ_i is a maximal multiset of transitions such that, for all $p \in Pl$,

$$\Sigma(\mu_{i-1}(p,u) : u \leq u_i) \geq \Sigma(\Theta_i(\tau)pre(\tau, p) : \tau \in Tr),$$

(3) for all $u > u_i$ and all $p \in Pl$ we have

$$\mu_i(p,u) = \mu_{i-1}(p,u) + \Sigma(\Theta_i(\tau)post(\tau,p) : \tau \in Tr, u_i + D(\tau) = u)$$

and

$$\Sigma(\mu_i(p,u) : u \leq u_i) = \Sigma(\mu_{i-1}(p,u) : u \leq u_i) - \Sigma(\Theta_i(\tau)pre(\tau,p) : \tau \in Tr).$$

Conversely, each sequence $\mu_0[\Theta_1)\mu_1...[\Theta_n)\mu_n$, where $\mu_0, \mu_1, ..., \mu_n$ are timed markings and $\Theta_1, ..., \Theta_n$ are multisets of transitions, such that for each $i = 1, ..., n$ there exists a time instant u_i such that the conditions (1) - (3) are satisfied is of the form $fs(\alpha)$ for some $\alpha \in real(tproc(N))$. $\square$

Proof: For a proof of the first part it suffices to consider the case $i = 1$ and then to repeat the reasoning for $i = 2, ..., n$.

Let $\mathcal{A}$, u, $X(u)$, and $Y(u)$ be defined as in the proof of 6.1. For all u up to a certain value we have $Y(u) = X_{min}$, $\mu_{\alpha,u} = \mu_0$, and $\Theta_{\alpha,u} = 0$. Let u_1 be the earliest instant of time such that $\Theta_{\alpha,u_1} \neq 0$. Then each prime free process corresponding to a transition τ

with $\Theta_{\alpha,u_1}(\tau) = k(\tau) > 0$ has in $X(u_1)$ exactly $k(\tau)$ instances which correspond to the restrictions of $X(u_1)$ to some mutually disjoint subsets $X_{\tau,1}(u_1), ..., X_{\tau,k(\tau)}(u_1)$. Thus we have a family J of instances of timed processes corresponding to transitions of N such that the sets of elements of these instances are mutually disjoint and the minimal elements of these instances belong to X_{min} and satisfy the condition $e_{time}(x) \leq u_1$ with equality for at least one minimal element of each instance. Moreover, the realizability of α implies that only for the members of J the minimal elements belong to X_{min} and have appearance times not exceeding u_1. Consequently, u_1 is the earliest instant of time satisfying (1), and Θ_{α,u_1} corresponds to a maximal multiset of transitions satisfying (2).

The property (3) follows easily from the fact that μ_1 is obtained from μ_0 by replacing X_{min} by $Y(u_1)$ and by taking the respective multisets of presences of tokens with given appearance times in places.

The second part of theorem can be proved by constructing a realizable timed process α such that $fs(\alpha) = \mu_0[\Theta_1\rangle\mu_1...[\Theta_n\rangle\mu_n$. It suffices to describe the first step of such a construction and repeat it for the next steps.

Let $\{\tau \in Tr : \Theta_1(\tau) > 0\} = \{\tau_1, ..., \tau_m\}$. Due to (1), (2), and (3), μ_0 and μ_1 can be represented by $\vartheta_0 = (\vartheta_0(p) : p \in Pl)$ and $\vartheta_1 = (\vartheta_1(p) : p \in Pl)$, respectively, where ϑ_0 and ϑ_1 are deliveries of tokens with

$$\vartheta_0(p_j) = u'((\tau_1, 1), (p_j, 1))...u'((\tau_1, 1), (p_j, pre(\tau_1, p_j)))...$$

$$...u'((\tau_1, \Theta_1(\tau_1)), (p_j, 1))...u'((\tau_1, \Theta_1(\tau_1)), (p_j, pre(\tau_1, p_j)))...$$

$$...u'((\tau_m, 1), (p_j, 1))...u'((\tau_m, 1), (p_j, pre(\tau_m, p_j)))...$$

$$...u'((\tau_m, \Theta_1(\tau_m)), (p_j, 1))...u'((\tau_m, \Theta_1(\tau_m)), (p_j, pre(\tau_m, p_j)))\vartheta(p_j)$$

such that:

(a) for each (τ_i, k) we have $u'((\tau_i, k), (p_j, l)) \leq u_1$ with the equality for some j' and l',

(b) $\vartheta(p_j) = u(p_j, 1)...u(p_j, r_j)$,

(c) for each $\tau \in Tr$ there exists j such that the number the items of $\vartheta(p_j)$ which do not exceed u_1 is less than $pre(\tau, p_j)$, and with

$$\vartheta_1(p_j) = u''((\tau_1, 1), (p_j, 1))...u''((\tau_1, 1), (p_j, post(\tau_1, p_j)))...$$

$$...u''((\tau_1, \Theta_1(\tau_1)), (p_j, 1))...u''((\tau_1, \Theta_1(\tau_1)), (p_j, post(\tau_1, p_j)))...$$

$$...u''((\tau_m, 1), (p_j, 1))...u''((\tau_m, 1), (p_j, post(\tau_m, p_j)))...$$

$$...u''((\tau_m, \Theta_1(\tau_m)), (p_j, 1))...u''((\tau_m, \Theta_1(\tau_m)), (p_j, post(\tau_m, p_j)))\vartheta(p_j)$$

such that:

(d) for each (τ_i, k) we have $u''((\tau_i, k), (p_j, l)) = u_1 + D(\tau_i)$,

(e) $\vartheta(p_j) = u(p_j, 1)...u(p_j, r_j)$ as for $\vartheta_0(p_j)$.

To each (τ_i, k) we assign sets

$$X'(\tau_i, k) = \{x'((\tau_i, k), (p_1, 1)), ..., x'((\tau_i, k), (p_1, pre(\tau_i, p_1))), ...$$

$$..., x'((\tau_i, k), (p_n, 1)), ..., x'((\tau_i, k), (p_n, pre(\tau_i, p_n)))\}$$

$$X''(\tau_i, k) = \{x''((\tau_i, k), (p_1, 1)), ..., x''((\tau_i, k), (p_1, post(\tau_i, p_1))), ...$$

$$..., x''((\tau_i, k), (p_n, 1)), ..., x''((\tau_i, k), (p_n, post(\tau_i, p_n)))\}$$

such that $X'(\tau_i, k) \cap X''(\tau_i, k) = \emptyset$ and all $X(\tau_i, k) = X'(\tau_i, k) \cup X''(\tau_i, k)$ are mutually disjoint. Next we choose mutually disjoint sets

$$Y(p_j) = \{y(p_j, 1), ..., y(p_j, r_j)\}$$

such that all of them are disjoint with all $X(\tau_i, k)$ and define X as the union of all $X(\tau_i, k)$ and $Y(p_j)$. By assuming

$$x'((\tau_i, k), (p_j, l)) \leq x''((\tau_i, k), (p_g, h))$$

$$d(x'((\tau_i, k), (p_j, l)), x''((\tau_i, k), (p_g, h))) = D(\tau_i)$$

$$e_{proper}(x'((\tau_i, k), (p_j, l))) = p_j$$

$$e_{proper}(x''((\tau_i, k), (p_g, h)) = p_g$$

$$e_{proper}y(p_j, l) = p_j$$

$$e_{time}(x'((\tau_i, k), (p_j, l)) = u'((\tau_i, k), (p_j, l))$$

$$e_{time}(x''((\tau_i, k), (p_g, h)) = u''((\tau_i, k), (p_g, h))$$

$$e_{time}(y(p_j, l) = u(p_j, l))$$

and by choosing some arrangements of minimal and maximal elements we obtain an instance of a realizable timed process α_1. From the construction of this process and from (3) it follows that $\partial_1(\alpha_1)$ defines the timed marking μ_1. Similarly, for the subsequent steps we obtain the respective $\alpha_2, ..., \alpha_n$. By choosing properly arrangements of minimal elements and arrangements of maximal elements of instances of these processes we obtain $\alpha = \alpha_1; ...; \alpha_n$, as required. □

8 Closing remarks

The representation of the behaviours of timed Petri nets in terms of processes and their delay tables seems to be conceptually simple due to its algebraic nature. In this representation nets can be viewed as sets of atomic generators of their behaviours considered as subalgebras of a monoidal category. Processes which constitute such behaviours are represented together with all essential information about the causal order, concurrency, and the lapse of time such that their execution times are represented in a natural way in the form of delay tables rather than of single numbers. The mechanism of choice for execution of a particular timed process is reflected by the concepts of dominance and realizability. Due to these features the proposed representation of the behaviours of timed Petri nets seems to be adequate and convenient for analyzing concurrent systems with time-consuming actions and their performance.

The results of section 7 show that the representation of the behaviour of a timed net in terms of timed processes is compatible with the characterization of this behaviour in terms of firing sequences.

However, each process represents usually many firing sequences (economy) and it may be regarded as an elementary unit of activity, that is without referring to its internal structure (aggregability).

As the sets of generators of algebras of free and timed processes of compound nets are unions of the sets of generators of component subnets, the algebras of free and timed processes of such compound nets can be obtained from the algebras of the respective processes of component subnets (compositionality).

Considering timed markings which represent streams of delivered tokens allows to consider timed nets as reactive systems whose behaviours are influenced by streams of input data.

In general, realizable timed processes of a net do not form a subalgebra of the algebra of timed processes of this net. This prevents from characterizing the set of such processes in a purely algebraic way and thus causes some difficulties in possible practical applications of our approach. Similar problems arise when considering nets whose behaviours are restricted by limitating capacities or in other ways. Thus our algebraic description of net behaviours should be considered only as a general framework for elaborating specific methods of characterizating such behaviours for concrete classes of nets.

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