An Approach to Cardinality of First Order Metasets

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Abstract. Metaset is a new approach to sets with partial membership relation. Metasets are designed to represent and process vague, imprecise data, similarly to fuzzy sets. They enable expressing fractional certainty of membership, equality, and other relations. Even though the general idea stems from and is firmly suited in the classical set theory, it is directed towards efficient computer implementations and applications.
In this paper we introduce the concept of cardinality for metasets and we investigate its basic properties. For simplicity we focus on the subclass of first order metasets however, the discussed ideas remain valid in general. We also present additional results obtained for finite first order metasets which are relevant for computer applications.

Keywords: metaset, partial membership, set theory, cardinality

1 Introduction

Metaset is the new concept of set with partial membership relation. It was inspired by the method of forcing [2] in the classical Zermelo-Fraenkel Set Theory (ZFC) [4,3]. Nonetheless it is directed towards artificial intelligence applications and efficient computer implementations. Its scope of practical usage is similar to fuzzy sets [12], intuitionistic fuzzy sets [1] or rough sets [5]. These traditional approaches to partial membership find successful applications in science and industry nowadays. Unfortunately, they are not well suited for computer implementations. They also have other drawbacks like the growth of fuzziness by multiple algebraic operations on fuzzy sets. Therefore, we tried to develop another idea of set with fractional members, which would be closer to ZFC, void of faults of currently used techniques and which would allow for efficient computer implementations.

The results obtained so far indicate success. We defined the basic set-theoretic relations for metasets, which may be satisfied to variety of degrees other than truth or falsity [7]. Algebraic operations for metasets satisfy the axioms of Boolean algebra [9]. The metasets language enables expressing uncertainty [11,10], particularly of membership, in a wider scope than intuitionistic fuzzy sets [8].

Experimental computer application for character recognition based on metaset approach [6] seems to correctly reflect human perception of simple images.

In this paper we introduce the notion of cardinality for metasets. Instead of analyzing the nature of cardinality and then trying to implement it within the metaset world we just transferred this notion directly from classical crisp sets onto metasets. We use the technique of interpretations for that purpose, as we usually do when defining new relations or operations for metasets (e.g., membership or equality).

For the sake of simplicity we focus in this paper on cardinality of first order metasets. The presented results remain valid in general (see Sec. 7). In computer applications we always deal with finite objects, therefore we also investigate additional results obtained for finite first order metasets. It turns out that objects representing cardinalities of such metasets are quite close to fuzzy numbers.

2 Metasets

A metaset is a classical crisp set with the specific structure which reflects membership degrees of its members.[1] The degrees are expressed as nodes (or rather sets of nodes) of the binary tree $\mathbb{T}$. In fact, they are elements of some Boolean algebra and they can be evaluated as real numbers.

For simplicity, in this paper we deal with first order metasets only.[2] A metaset of this type is a relation between some set and the set of nodes of $\mathbb{T}$. Thus, the mentioned structure which we use to encode the degrees of membership is based on ordered pairs. The first element of each pair is the member and the second element is a node of the binary tree, which contributes to the membership degree of the first element.

Definition 1. *A set which is either the empty set $\emptyset$ or which has the form:*

$$\tau = \{ \langle \sigma, p \rangle : \sigma \text{ is a set, } p \in \mathbb{T} \}$$

is called a first order metaset.

The binary tree $\mathbb{T}$ is the set of all finite binary sequences, i.e., functions whose domains are finite ordinals, valued in 2:[3]

$$\mathbb{T} = \bigcup_{n \in \mathbb{N}} 2^n \ . \tag{1}$$

The ordering $\leq$ in the tree $\mathbb{T}$ (see Fig. 1) is the reverse inclusion of functions: for $p, q \in \mathbb{T}$ such, that $p \colon n \mapsto 2$ and $q \colon m \mapsto 2$, we have $p \leq q$ whenever $p \supseteq q$, i.e., $n \geq m$ and $p_{\restriction m} = q$. The root $\mathbb{1}$ is the largest element of $\mathbb{T}$ in this ordering: it is included in each function and for all $p \in \mathbb{T}$ we have $p \leq \mathbb{1}$.

[1] We use the term "degree of membership" rather informally here and throughout the whole paper. For the precise discussion of evaluating degrees of membership and other relations the reader is referred to [11,10].

[2] See [7] for the introduction to metasets in general.

[3] For $n \in \mathbb{N}$, let $2^n = \{ f \colon n \mapsto 2 \}$ denote the set of all functions with the domain n and the range $2 = \{ 0, 1 \}$ – they are binary sequences of the length n.

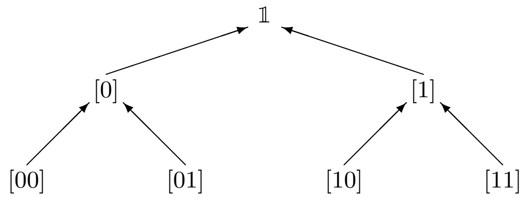

Fig. 1. The levels $\mathbb{T}_0$–$\mathbb{T}_2$ of the binary tree $\mathbb{T}$ and the ordering of nodes. Arrows point at the larger element.

We denote binary sequences which are elements of $\mathbb{T}$ using square brackets, for example: [00], [101]. If $p \in \mathbb{T}$, then we denote its children with $p \cdot 0$ and $p \cdot 1$. A *level* in $\mathbb{T}$ is the set of all finite binary sequences with the same length. The set 2^n consisting of sequences of the length n is the level n, denoted by $\mathbb{T}_n$. The level 0 consists of the empty sequence $\mathbb{1}$ only. A *branch* in $\mathbb{T}$ is an infinite binary sequence, i.e., a function $\mathbb{N} \mapsto 2$. We will write $p \in \mathcal{C}$ to mark, that the binary sequence $p \in \mathbb{T}$ is a prefix of the branch $\mathcal{C}$. A branch intersects all levels in $\mathbb{T}$, and each of them only once.

Ordering of nodes in $\mathbb{T}$ is consistent with the ordering of membership degrees they correspond to. The root node $\mathbb{1}$ represents the highest, full membership similar to classical set membership. The first element σ of an ordered pair $\langle \sigma, p \rangle$ contained in a first order metaset τ is called a *potential element* of τ. A potential element may be simultaneously paired with multiple different nodes which contribute to the overall membership degree of the potential element. Nodes on levels with greater numbers contribute less membership information than those which are closer to the root $\mathbb{1}$.

For the given metaset τ, the set of its potential elements:

$$\mathrm{dom}(\tau) = \{\, \sigma \colon \exists_{p \in \mathbb{T}} \;\; \langle \sigma, p \rangle \in \tau \,\} \tag{2}$$

is called the *domain* of the metaset τ, and the set:

$$\mathrm{ran}(\tau) = \left\{\, p \colon \exists_{\sigma \in \mathrm{dom}(\tau)} \;\; \langle \sigma, p \rangle \in \tau \,\right\} \tag{3}$$

is called the *range* of the metaset τ. The class of first order metasets is denoted by $\mathfrak{M}^1$. Thus,

$$\tau \in \mathfrak{M}^1 \quad \text{iff} \quad \tau \subset \mathrm{dom}(\tau) \times \mathrm{ran}(\tau) \subset X \times \mathbb{T} \;, \tag{4}$$

where X is some set.

A metaset is *finite* when it is finite as a set of ordered pairs. Consequently, its domain and range are finite. The class of finite first ordered metasets is denoted by $\mathfrak{MF}^1$. Thus,

$$\tau \in \mathfrak{MF}^1 \quad \text{iff} \quad |\mathrm{dom}(\tau)| < \aleph_0 \;\wedge\; |\mathrm{ran}(\tau)| < \aleph_0 \;. \tag{5}$$

This class is particularly important for computer applications where we deal with finite objects exclusively.

If σ is a first order metaset such, that $\operatorname{ran}(\sigma) = \{ \mathbb{1} \}$, then we call it a *canonical metaset*. We denote the class of canonical first order metasets with the symbol $\mathfrak{M}^{\mathfrak{c}}$. Such metasets resemble classical crisp sets; they have similar properties. In fact, there is a natural one-to-one correspondence between canonical metasets and crisp sets. Thus,

$$\sigma \in \mathfrak{M}^{\mathfrak{c}} \quad \text{iff} \quad \sigma = X \times \{ \mathbb{1} \} \ , \tag{6}$$

where X is some set (clearly, $X = \operatorname{dom}(\sigma)$).

3 Interpretations of Metasets

An interpretation of a first order metaset is a crisp set. It is produced out of the given metaset with a branch of the binary tree. Different branches determine different interpretations of the given metaset. All of them taken together make up a collection of sets with specific internal dependencies, which represents the source metaset by means of its crisp views.

Properties of crisp sets which are interpretations of the given first order metaset determine the properties of the metaset itself. In particular we use interpretations to define set-theoretic relations for metasets.

Definition 2. *Let τ be a first order metaset and let $\mathcal{C}$ be a branch. The set*

$$\tau_{\mathcal{C}} = \{ \sigma \in \operatorname{dom}(\tau) \colon \ \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \}$$

is called the interpretation of the first order metaset τ given by the branch $\mathcal{C}$.

An interpretation of the empty metaset is the empty set, independently of the branch. Generally, interpretations of canonical metasets are independent of the chosen branch.

Proposition 1. *If $\sigma \in \mathfrak{M}^{\mathfrak{c}}$, then $\sigma_{\mathcal{C}} = \operatorname{dom}(\sigma)$, for any branch $\mathcal{C}$.*

The process of producing an interpretation of a first order metaset consists in two stages. In the first stage we remove all the ordered pairs whose second elements are nodes which do not belong to the branch $\mathcal{C}$. The second stage replaces the remaining pairs – whose second elements lie on the branch $\mathcal{C}$ – with their first elements. As the result we obtain a crisp set contained in the domain of the metaset.

Example 1. Let $p \in \mathbb{T}$ and let $\tau = \{ \langle \emptyset, p \rangle \}$. If $\mathcal{C}$ is a branch, then

$$\begin{aligned} p \in \mathcal{C} &\to \tau_{\mathcal{C}} = \{ \emptyset \} \ , \\ p \notin \mathcal{C} &\to \tau_{\mathcal{C}} = \emptyset \ . \end{aligned}$$

Depending on the branch the metaset τ acquires one of two different interpretations: $\{ \emptyset \}$ or $\emptyset$. Note, that $\operatorname{dom}(\tau) = \{ \emptyset \}$.

As we see, a first order metaset may have multiple different interpretations – each branch in the tree determines one. Usually, most of them are pairwise equal, so the number of different interpretations is much less than the number of branches. Finite first order metasets always have a finite number of different interpretations. For such metasets we consider the greatest level number of all the levels whose elements may affect interpretations.

Definition 3. *Let $\tau \in \mathfrak{MF}^{\mathbf{1}}$. The natural number*

$$\mathfrak{l}_\tau = \begin{cases} \max \{\, |p| \colon p \in \operatorname{ran}(\tau) \,\} & \textit{if } \tau \neq \emptyset \, , \\ 0 & \textit{if } \tau = \emptyset \, . \end{cases}$$

is called the deciding level for τ.

Since $p \in \mathbb{T}$ is a function, then $|p|$ is its cardinality – the number of ordered pairs which is just the length of the binary sequence p. It is also equal to the level number to which it belongs: $p \in \mathbb{T}_{|p|}$. Thus, $\mathfrak{l}_\tau$ is the length of the longest sequence in $\operatorname{ran}(\tau)$. The following lemma claims that nodes on levels below $\mathfrak{l}_\tau$ (with greater level numbers) do not affect interpretations of τ.

Lemma 1. *Let τ be a finite first order metaset and let $\mathcal{C}'$ and $\mathcal{C}''$ be branches. If initial segments of size $\mathfrak{l}_\tau$ of $\mathcal{C}'$ and $\mathcal{C}''$ are equal, then they produce equal interpretations:*

$$\forall_{n \leq \mathfrak{l}_\tau} \, \mathcal{C}'(n) = \mathcal{C}''(n) \ \rightarrow \ \tau_{\mathcal{C}'} = \tau_{\mathcal{C}''} \, .$$

Proof. Since there are no nodes on levels below $\mathfrak{l}_\tau$ in $\operatorname{ran}(\tau)$, and by the assumption, we obtain $\{\, \langle \sigma, p \rangle \in \tau \colon p \in \mathcal{C}' \,\} = \{\, \langle \sigma, p \rangle \in \tau \colon p \in \mathcal{C}'' \,\}$. Therefore, $\tau_{\mathcal{C}'} = \{\, \sigma \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C}' \,\} = \{\, \sigma \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C}'' \,\} = \tau_{\mathcal{C}''}$.

Note, that for a canonical $\sigma \in \mathfrak{M}^{\mathfrak{c}}$ we always have $\mathfrak{l}_\sigma = 0$.

4 Set-theoretic Relations for Metasets

We briefly sketch the methodology behind the definitions of standard set-theoretic relations for metasets within the scope necessary for the introduction of cardinality. For the detailed discussion of the relations or their evaluation the reader is referred to [9] or [11].

We use interpretations for transferring relations from crisp sets onto metasets.

Definition 4. *We say that the metaset σ belongs to the metaset τ under the condition $p \in \mathbb{T}$, whenever for each branch $\mathcal{C}$ containing p holds $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$. We use the notation $\sigma \epsilon_p \tau$.*

Formally, we define an infinite number of membership relations: each $p \in \mathbb{T}$ specifies another relation ϵ_p. Any two metasets may be simultaneously in multiple membership relations qualified by different nodes: $\sigma \epsilon_p \tau \wedge \sigma \epsilon_q \tau$. Membership under the root condition $\mathbb{1}$ resembles the full, unconditional membership of crisp sets, since it is independent of branches. In such case we skip the subscript $\mathbb{1}$ and we just write $\sigma \mathrel{\epsilon} \tau$ instead of $\sigma \mathrel{\epsilon_{\mathbb{1}}} \tau$.

The conditional membership reflects the idea that a metaset μ belongs to a metaset τ whenever some conditions are fulfilled. The conditions are represented by nodes of $\mathbb{T}$. In applications they refer to a modeled reality, e.g.: *the man X is big since X is tall* (i.e., X belongs to a metaset of big people under the condition tall), or *the man X is big since X is tall and fat* (i.e., X belongs to a metaset of big people under two conditions: tall and fat).

There are two substantial properties of this technique exposed by the following two lemmas. Although we show them for the membership relation they also hold for other relations.

Lemma 2. *Let $\tau, \sigma \in \mathfrak{M}^1$ and let $p, q \in \mathbb{T}$. If $\sigma \mathrel{\epsilon_p} \tau$ and $q \leq p$, then $\sigma \mathrel{\epsilon_q} \tau$.*

Proof. If $\mathcal{C}$ is a branch containing q then also $p \in \mathcal{C}$. Therefore $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$.

Lemma 3. *Let $\tau, \sigma \in \mathfrak{M}^1$ and let $p \in \mathbb{T}$. If $\forall_{q<p}\, \sigma \mathrel{\epsilon_q} \tau$, then $\sigma \mathrel{\epsilon_p} \tau$.*

Proof. If $\mathcal{C} \ni p$, then it also contains some $q < p$. Therefore, $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$.

In other words: $\sigma \mathrel{\epsilon_p} \tau$ is equivalent to $\sigma \mathrel{\epsilon_{p \cdot 0}} \tau \wedge \sigma \mathrel{\epsilon_{p \cdot 1}} \tau$, i.e., being a member under the condition p is equivalent to being a member under both conditions $p \cdot 0$ and $p \cdot 1$, which are the direct descendants of p. Indeed, by lemma 2 we have $\sigma \mathrel{\epsilon_p} \tau \;\rightarrow\; \sigma \mathrel{\epsilon_{p \cdot 0}} \tau \wedge \sigma \mathrel{\epsilon_{p \cdot 1}} \tau$. And if $\sigma \mathrel{\epsilon_{p \cdot 0}} \tau$, then again, by lemma 2 we have $\forall_{q \leq p \cdot 0}\, \sigma \mathrel{\epsilon_q} \tau$, and similarly for $p \cdot 1$. Consequently, we have $\forall_{q<p}\, \sigma \mathrel{\epsilon_q} \tau$ and by lemma 3 we obtain $\sigma \mathrel{\epsilon_{p \cdot 0}} \tau \wedge \sigma \mathrel{\epsilon_{p \cdot 1}} \tau \;\rightarrow\; \sigma \mathrel{\epsilon_p} \tau$.

Example 2. Recall, that the ordinal number 1 is the set $\{ 0 \}$ and 0 is just the empty set $\emptyset$. Let $\tau = \{ \langle 0, [0] \rangle, \langle 1, [1] \rangle \}$ and let $\sigma = \{ \langle 0, [1] \rangle \}$. Let $\mathcal{C}^0 \ni [0]$ and $\mathcal{C}^1 \ni [1]$ be arbitrary branches containing $[0]$ and $[1]$, respectively. Interpretations are: $\tau_{\mathcal{C}^0} = \{ 0 \}$, $\tau_{\mathcal{C}^1} = \{ 1 \}$, $\sigma_{\mathcal{C}^0} = 0$ and $\sigma_{\mathcal{C}^1} = \{ 0 \} = 1$. We see that $\sigma \mathrel{\epsilon_{[0]}} \tau$ and $\sigma \mathrel{\epsilon_{[1]}} \tau$. Also, $\sigma \mathrel{\epsilon} \tau$ holds.

Note, that even though interpretations of τ and σ vary depending on the branch, the metaset membership relation is maintained

Similarly to membership we define conditional equality and subset relations for metasets.

Definition 5. *We say that the metaset σ is equal to the metaset τ under the condition $p \in \mathbb{T}$, whenever for each branch $\mathcal{C}$ containing p holds $\sigma_{\mathcal{C}} = \tau_{\mathcal{C}}$. We use the notation $\mu \approx_p \tau$.*

If $p = \mathbb{1}$, then we skip the subscript and we just write $\mu \approx \tau$. Clearly, $\mu = \tau \rightarrow \mu \approx \tau$, but the converse implication fails.

Example 3. Consider $\tau = \{ \langle 0, \mathbb{1} \rangle \}$ and $\sigma = \{ \langle 0, [0] \rangle, \langle 0, [1] \rangle \}$. Since for any branch $\mathcal{C}$ we have $\tau_{\mathcal{C}} = \{ \emptyset \} = \sigma_{\mathcal{C}}$, then $\tau \approx \sigma$ however, $\tau \neq \sigma$.

Definition 6. *We say that the metaset σ is a subset of the metaset τ under the condition $p \in \mathbb{T}$, whenever for each branch $\mathcal{C}$ containing p holds $\sigma_{\mathcal{C}} \subset \tau_{\mathcal{C}}$. We use the notation $\mu \Subset_p \tau$.*

Again, if $p = \mathbb{1}$, then we just write $\mu \Subset \tau$ instead of $\mu \Subset_{\mathbb{1}} \tau$. Note, that if $\sigma, \tau \in \mathfrak{M}^1$ and $p \in \mathbb{T}$, then

$$\sigma \approx_p \tau \quad \leftrightarrow \quad \sigma \Subset_p \tau \wedge \tau \Subset_p \sigma \ . \tag{7}$$

There are many other properties of set-theoretic relations for metasets which are similar to well known properties for classical sets. We do not discuss them here since they are beyond the scope of this paper. As an example consider the metaset version of extensionality: If $\sigma, \tau \in \mathfrak{M}^1$ and $p \in \mathbb{T}$, then

$$\sigma \approx_p \tau \quad \leftrightarrow \quad \forall_\mu \forall_{q \leq p} \left(\mu \epsilon_q \sigma \leftrightarrow \mu \epsilon_q \tau\right) \ . \tag{8}$$

To prove the above refer to interpretations.

5 Cardinality of First Order Metasets

Cardinality of a crisp set is an ordinal number – the “number of elements” of the set. A metaset may be interpreted as a family of crisp sets (Sec. 3). Therefore, cardinality of a metaset is a family of ordinal numbers. Since each branch in the tree $\mathbb{T}$ determines an interpretation, then this family is indexed with infinite binary sequences, i.e. all the branches in $\mathbb{T}$.

Let **On** denote the class of ordinal numbers and let τ be a first order metaset. We define the cardinality of τ to be a function from the set of all infinite binary sequences into **On**.

Definition 7. *Let $\tau \in \mathfrak{M}^1$. The cardinality of τ, denoted with $\overline{\overline{\tau}}$, is a function $\overline{\overline{\tau}} \colon 2^{\mathbb{N}} \mapsto$ **On** such, that for each branch $\mathcal{C}$ in $\mathbb{T}$ holds:*

$$\overline{\overline{\tau}}\,(\mathcal{C}) = |\tau_{\mathcal{C}}| \ .$$

The symbol $|\tau_{\mathcal{C}}|$ denotes the cardinality of the set $\tau_{\mathcal{C}}$.

As we see, the cardinality of τ at the branch $\mathcal{C}$ is the cardinality of the interpretation of τ given by the branch $\mathcal{C}$. The cardinality of the empty metaset is the constant function $2^{\mathbb{N}} \mapsto \{\, \emptyset \,\}$ and generally, the cardinality of a canonical metaset τ is the constant function $2^{\mathbb{N}} \mapsto \{\, |\mathrm{dom}(\tau)| \,\}$.

To prove that the proposed approach to cardinality is correct we should show that metasets with equal cardinalities are equinumerous, i.e., there exists a one-to-one mapping between them, and vice versa: equinumerosity implies equality of cardinalities. Due to limited scope of this paper we cannot present the proof that indeed such property holds for finite first order metasets.[4] Anyway, we shall try to convince the reader, that metaset cardinality has properties similar to the concept of cardinality for crisp sets. One of the most basic of them says that equal sets have the same cardinality. Translated into metaset language it says, that conditionally equal first order metasets have the same cardinality.

[4] It will be published in another paper soon

Theorem 1. *If $\tau, \sigma \in \mathfrak{M}^1$ and $\tau \approx \sigma$, then $\overline{\overline{\tau}} = \overline{\overline{\sigma}}$.*

Proof. The assumption $\tau \approx \sigma$ implies that for any branch $\mathcal{C} \in \mathbb{T}$ holds $\tau_{\mathcal{C}} = \sigma_{\mathcal{C}}$. Therefore, also $|\tau_{\mathcal{C}}| = |\sigma_{\mathcal{C}}|$.

Since $\tau = \sigma \to \tau \approx \sigma$, then metasets which are equal sets have identical cardinality. If $\tau \approx_p \sigma$, then cardinalities generally are not equal. However, they are equal as functions restricted to a subset of $2^{\mathbb{N}}$ consisting of all the sequences containing p.

Example 4. Let $\tau = \{ \langle \emptyset, [0] \rangle, \langle \emptyset, [1] \rangle \}$ and $\sigma = \{ \langle \emptyset, [0] \rangle \}$. If $\mathcal{C}^0$ is a branch containing $[0]$ and $\mathcal{C}^1 \ni [1]$, then $\tau_{\mathcal{C}^0} = \{ \emptyset \} = \sigma_{\mathcal{C}^0}$ and $\tau_{\mathcal{C}^1} = \{ \emptyset \}$, whereas $\sigma_{\mathcal{C}^1} = \emptyset$. Therefore, $|\tau_{\mathcal{C}^0}| = |\sigma_{\mathcal{C}^0}|$ and $|\tau_{\mathcal{C}^1}| \neq |\sigma_{\mathcal{C}^1}|$. We also see that $\tau \approx_{[0]} \sigma$ holds, whereas both $\tau \approx_{[1]} \sigma$ and $\tau \approx \sigma$ fail.

Note also, that for $\eta = \{ \langle \emptyset, \mathbb{1} \rangle \}$ holds $\overline{\overline{\eta}} = \overline{\overline{\tau}}$, since $\eta \approx \tau$.

We now introduce the partial ordering of metaset cardinalities.

Definition 8. *Let $\tau, \sigma \in \mathfrak{M}^1$. If for each branch $\mathcal{C} \in \mathbb{T}$ holds $|\tau_{\mathcal{C}}| \leq |\sigma_{\mathcal{C}}|$, then we say that the cardinality of τ is less than or equal than the cardinality of σ. We use standard notation $\overline{\overline{\tau}} \leq \overline{\overline{\sigma}}$.*

The element $\overline{\overline{\emptyset}}$ is the least one in this ordering. Note, that if τ, η are a first order metasets such, that η is canonical and $\mathrm{dom}(\tau) = \mathrm{dom}(\eta)$, then $\overline{\overline{\tau}} \leq \overline{\overline{\eta}}$ since $\overline{\overline{\tau}}(\mathcal{C}) \leq |\mathrm{dom}(\tau)| = \overline{\overline{\eta}}(\mathcal{C})$, for any branch $\mathcal{C}$.

Proposition 2. *The relation $\leq$ satisfies axioms of partial ordering: it is reflexive, antisymmetric and transitive.*

Proof. Reflexivity means $\overline{\overline{\tau}} \leq \overline{\overline{\tau}}$ for any $\tau \in \mathfrak{M}^1$ and it is satisfied since $|\tau_{\mathcal{C}}| \leq |\tau_{\mathcal{C}}|$ holds for any branch $\mathcal{C}$. Antisymmetry ($\overline{\overline{\tau}} \leq \overline{\overline{\sigma}} \wedge \overline{\overline{\sigma}} \leq \overline{\overline{\tau}} \to \overline{\overline{\tau}} = \overline{\overline{\sigma}}$) and transitivity ($\overline{\overline{\tau}} \leq \overline{\overline{\sigma}} \wedge \overline{\overline{\sigma}} \leq \overline{\overline{\eta}} \to \overline{\overline{\tau}} \leq \overline{\overline{\eta}}$) are satisfied similarly by referring to analogous properties for cardinalities of interpretations.

If $\overline{\overline{\tau}} \leq \overline{\overline{\sigma}}$, then – roughly speaking – it means that σ is always, under all conditions, independently of branches, "larger" than or equal to τ. Otherwise, if $\overline{\overline{\tau}} \not\leq \overline{\overline{\sigma}}$, then under some condition there is more of τ (it is "bigger") than σ.

The ordering of metaset cardinalities is consistent with metaset inclusion.

Theorem 2. *If $\tau, \sigma \in \mathfrak{M}^1$ are such, that $\tau \Subset \sigma$, then $\overline{\overline{\tau}} \leq \overline{\overline{\sigma}}$.*

Proof. By the assumption, for any branch $\mathcal{C}$ holds $\tau_{\mathcal{C}} \subset \sigma_{\mathcal{C}}$. Therefore, also $|\tau_{\mathcal{C}}| \leq |\sigma_{\mathcal{C}}|$, what implies the thesis.

We do not define nor discuss algebraic operations for metasets here (see [9]), however it is worth noting, that the algebraic operations are also consistent with the definition of cardinality. In particular, since (by the definition) the metaset union $\tau \uplus \eta$ of τ and σ coincides with their set-theoretic union $\tau \cup \eta$, i.e., $\tau \uplus \eta = \tau \cup \eta$, then the cardinality of the union makes up an upper bound for the cardinality of operands: $\overline{\overline{\tau}} \leq \overline{\overline{\tau \uplus \eta}}$. Similarly for the intersection $\tau \Cap \eta$ (defined in [9]): since $\tau \Cap \eta \Subset \tau$, then $\overline{\overline{\tau \Cap \eta}} \leq \overline{\overline{\tau}}$.

6 Cardinality in $\mathfrak{MF}^1$

In computer applications we always deal with finite sets. For finite first order metasets the concept of cardinality may be simplified so that it is easily representable as a step function[5] on the unit interval, valued in natural numbers. Such representation facilitates application of metasets to real-life problems.

A branch $\mathcal{C}$ is a binary sequence $\{\,\mathcal{C}(i)\,\}_{i\in N}$ which determines a real number $x \in [0\ldots 1]$ by the following formula: $x = 0.\mathcal{C}(0)\mathcal{C}(1)\ldots$. There exist pairs of branches which determine equal real numbers. For instance, if $\mathcal{C}^0 = 011\ldots$ and $\mathcal{C}^1 = 100\ldots$, then $0.011\ldots = 0.5 = 0.100\ldots$. Generally, different branches $\mathcal{C}' \neq \mathcal{C}''$ determine different interpretations: $\tau_{\mathcal{C}'} \neq \tau_{\mathcal{C}''}$, what may imply different cardinalities $|\tau_{\mathcal{C}'}| \neq |\tau_{\mathcal{C}''}|$ even when $\mathcal{C}'$ and $\mathcal{C}''$ determine the same real value x. For finite first order metasets we may ignore this ambiguity as follows.

The lemma 1 says, that for $\tau \in \mathfrak{MF}^1$ and for branches $\mathcal{C}'$ and $\mathcal{C}''$ that are equal up to the deciding level $\mathfrak{l}_\tau$ the interpretations are equal: $\tau_{\mathcal{C}'} = \tau_{\mathcal{C}''}$. Therefore, also $|\tau_{\mathcal{C}'}| = |\tau_{\mathcal{C}''}|$. Consequently, we may assign to each $p \in \mathbb{T}_{\mathfrak{l}_\tau}$ the unique cardinality $|\tau_{\mathcal{C}}|$ which is given by any branch $\mathcal{C}$ containing p.

Each $p \in \mathbb{T}$ determines an interval $I_p \subset [0\ldots 1)$ of the length $2^{-|p|}$ defined as $I_p = \big[l_p,\ l_p + 2^{-|p|}\big)$, where $l_p \in [0\ldots 1)$ and

$$l_p = \begin{cases} \sum_{i=0}^{i=|p|-1} \; p(i)\cdot 2^{-(i+1)} & \text{for } p \neq \mathbb{1}, \\ 0 & \text{for } p = \mathbb{1}. \end{cases} \tag{9}$$

For instance, $I_{\mathbb{1}} = [0\ldots 1)$, $I_{[0]} = [0\ldots {}^1\!/\!_2)$ and $I_{[1]} = [{}^1\!/\!_2\ldots 1)$.

Thus, to the given $\tau \in \mathfrak{MF}^1$ and $x \in [0\ldots 1)$ we may assign a unique natural number $|\tau_{\mathcal{C}^x}|$, where $\mathcal{C}^x$ is a branch such, that $x = 0.\mathcal{C}^x(0)\mathcal{C}^x(1)\ldots$. We also know that $\mathcal{C}^x$ contains the unique $p \in \mathbb{T}_{\mathfrak{l}_\tau}$ for which $x \in I_p$.

Definition 9. *Let $\tau \in \mathfrak{MF}^1$ and let $\mathfrak{l}_\tau$ be the deciding level for τ. We define the cardinality spectrum for τ as the function $\overline{\overline{\tau}}\colon [0\ldots 1) \mapsto \mathbb{N}$ such, that $\overline{\overline{\tau}}(x) = |\tau_{\mathcal{C}}|$, where $\mathcal{C}$ is an arbitrary branch satisfying the condition:*

$$\sum_{i=0}^{i=\mathfrak{l}_\tau-1} \mathcal{C}(i)\cdot 2^{-(i+1)} \leq x < 2^{-\mathfrak{l}_\tau} + \sum_{i=0}^{i=\mathfrak{l}_\tau-1} \mathcal{C}(i)\cdot 2^{-(i+1)}$$

when $\mathfrak{l}_\tau > 0$ or $\mathcal{C}$ is an arbitrary branch when $\mathfrak{l}_\tau = 0$.

In other words, $\overline{\overline{\tau}}(x)$ is the unique cardinality $|\tau_{\mathcal{C}}|$ given by any branch containing $p \in \mathbb{T}_{\mathfrak{l}_\tau}$, where p is a prefix of (is contained in) $\mathcal{C}$ and $x \in I_p$. By the lemma 1 the value of $|\tau_{\mathcal{C}}|$ is constant on I_p, i.e., it is constant for all the branches containing p.

Proposition 3. *If $\tau \in \mathfrak{MF}^1$, then its cardinality spectrum is a step function.*

[5] A step function is a piecewise constant function having only finitely many pieces.

Thus, for a $\tau \in \mathfrak{MF}^1$ we may split the unit interval $[0 \ldots 1)$ into $2^{\mathfrak{l}_\tau}$ disjoint subintervals of equal size $2^{-\mathfrak{l}_\tau}$. The cardinality spectrum for τ is constant and it is equal to some natural number on each of these intervals.

Example 5. Let $\tau = \{ \langle \emptyset, [0] \rangle \}$. The cardinality spectrum for τ is shown on the Figure 2. As we see, for $x \in [0 \ldots 0.5)$ we have $\overline{\overline{\tau}}(x) = 1$, whereas for $x \in [0.5 \ldots 1)$ we have $\overline{\overline{\tau}}(x) = 0$.

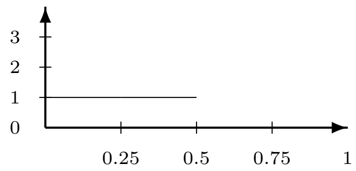

Fig. 2. The cardinality spectrum for $\tau = \{ \langle \emptyset, [0] \rangle \}$ (Ex. 5)

As a real-life application illustrating the examples 5 and 6 let us consider the number of tiny beans or other particles in a large basket. Calculations made by different experts give different results due to errors in calculations or changes in content over time. We represent them all in a single metaset. Its interpretations correspond to different results obtained by the experts.

Example 6. Let $\tau = \{ \langle \mu, [0] \rangle , \langle \eta, [00] \rangle , \langle \sigma, [11] \rangle \}$, where μ, η, σ are arbitrary different sets. The cardinality spectrum for τ is shown on the Figure 3. The metaset τ contains 0, 1 or 2 elements depending on the interpretation.

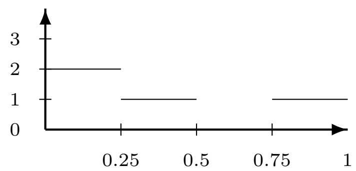

Fig. 3. The cardinality spectrum for $\tau = \{ \langle \mu, [0] \rangle , \langle \eta, [00] \rangle , \langle \sigma, [11] \rangle \}$ (Ex. 6)

In $\mathfrak{MF}^1$ the ordering of cardinalities is consistent with the functional ordering of cardinality spectrums which is imposed by the ordering of natural numbers. Namely, if $\tau, \sigma \in \mathfrak{MF}^1$, then

$$\overline{\overline{\tau}} \leq \overline{\overline{\sigma}} \;\leftrightarrow\; \forall_{x \in [0 \ldots 1)} \, \overline{\overline{\tau}}(x) \leq \overline{\overline{\sigma}}(x) \; . \tag{10}$$

This justifies the slight abuse of notation $\overline{\overline{\tau}}$ for cardinality and cardinality spectrum.

7 Generalization and Further Results

We focused on the class of first order metasets in this paper, however the presented results hold for metasets in general. It means, that in the definitions of cardinality (Def. 7) and cardinality spectrum (Def. 9) we may drop the assumption, that the metasets in concern are first order ones.

For completeness, we cite below the general definitions of metaset and interpretation (see [7,11] for a brief discussion of metasets in general).

Definition 10. *A set which is either the empty set $\emptyset$ or which has the form:*

$$\tau = \{\, \langle \sigma, p \rangle : \sigma \text{ is a metaset, } p \in \mathbb{T} \,\}$$

is called a metaset.

Formally, this is a definition by induction on the well founded relation $\in$ (see [4, Ch. VII, §2] for justification of such type of definitions). The general definition of interpretation for metasets is recursive too.

Definition 11. *Let τ be a metaset and let $\mathcal{C} \subset \mathbb{T}$ be a branch. The set*

$$\tau_{\mathcal{C}} = \{\, \sigma_{\mathcal{C}} \colon\ \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \,\}$$

is called the interpretation of the metaset τ given by the branch $\mathcal{C}$.

The discussion of cardinality naturally leads to the idea of cardinal numbers for metasets, i.e., objects representing cardinalities of metasets which are also subject to some arithmetical operations. Such project is undergoing and the results will be published soon. In fact, for first order metasets the algebraic operations on "cardinal metanumbers" are natural consequence of algebraic operations for metasets [9]. The result resembles fuzzy numbers, however the operations are defined differently.

Cardinality is associated with the notion of equinumerosity. In classical set theory for any two sets that have equal cardinality there exists a one-to-one mapping between them and we say in such case that these sets are equinumerous. A notion of equinumerosity similar to the classical one is also defined for metasets. It is worth stressing that two finite first order metasets have equal cardinality if and only if they are equinumerous – just like in the ZFC. The method for establishing equinumerosity in such case is constructible, meaning it is an algorithm which may be easily implemented in a programming language. These results will be published soon.

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