Metasets, Certainty and Uncertainty

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Abstract

Metaset is a new concept of set with partial membership relation. It is directed towards computer implementations and applications. The degrees of membership for metasets are expressed as nodes of the binary tree and they may be evaluated as real numbers too. The forcing mechanism discussed in this paper is used to assign certainty degrees to sentences involving metasets, and to define basic relations like partial membership or partial equality.

We thoroughly investigate here the class of metasets with finite deep ranges which are especially suitable for computer representations because of their finite structure. It turns out, that for sentences involving such metasets it is always possible to assign the degrees of certainty that the sentence is either true or false or both at the same time. Moreover, such sentences do not allow for any hesitancy degree what implies no hesitancy in membership and other basic relations. This property does not hold for sentences involving arbitrary metasets, what is illustrated by the examples.

Keywords: metaset, set theory, partial membership, certainty degree, hesitancy degree, intuitionistic fuzzy sets

1 Introduction

The notion of set is fundamental to mathematics [4, 3]. In the classical set theory the sentence "the set x is an element of the set y" is either true or false – there are no other possibilities. Members of a classical set belong to it completely, to the highest possible degree, without any intermediate levels. This property imposes some limitations on the scope of applications of the classical set theory. Using classical sets it is difficult to express and process vague, imprecise terms like *big*, *warm*, etc. However, there is a strong and growing demand on theories allowing for expressing such terms, especially in industry applications. Therefore, new concepts of sets appeared which admit partial membership relation. The most common examples are fuzzy sets [9] and rough sets [5]. In this paper we present another approach to the problem of partial membership: the metasets.

There are many substantial differences between this approach and the above mentioned ones. Just like in the classical set theory, members of metasets are other metasets. The membership degrees for metasets, as well as the degrees to which other relations are satisfied, are expressed by means of sets of nodes of the binary tree, and they may be evaluated as real numbers. The language of metasets includes infinite number of partial membership and equality relations, as well as their negations. They allow for expressing a variety of different degrees to which a relation may hold. The technique of interpretation allows to produce a crisp set out of a metaset in multiple different ways. Consequently, a metaset may be perceived as a family of crisp sets with a specific dependencies between members of the family. Metasets allow for expressing not only membership or non-membership with different degrees, but also a hesitancy degree which is the level of uncertainty concerning the membership or non-membership – the idea known from the intuitionistic fuzzy sets field [1, 7]. One of the most important characteristics of the metaset theory is its computer orientation. Large parts of the theory are constructed so that they are easily and efficiently implementable in computer languages. This allows for productive computer applications based on metasets [6].

In this paper we investigate the forcing relation applied to metasets which, due to their specific finite structure, are easily and directly representable in computers. As it turns out, such metasets have many interesting properties. One of the most significant is the possibility of assigning a certainty degree to each sentence involving such metasets. This is not true in general and we give appropriate examples. As a consequence, there is no hesitancy of membership for such metasets: any two of them are in membership relation to the degree which complements the non-membership degree. We also show how to evaluate the membership degrees for such metasets as real values and we conclude that the membership and non-membership values sum up to unity. Again, this property does not hold in general, since the sum of membership and non-membership values for arbitrary metasets may be less than 1.

2 Preliminary Definitions and Terminology

The metaset concept is strongly based on the classical set theory. Therefore, we start with establishing some well known terms and notation concerning sets, relations and partial orders. The key role in the definition of metaset plays the concept of binary tree which we define first.

A natural number $n \in \mathbb{N}$ is a finite ordinal of form

$$n = \{ 0, \ldots, n-1 \} = n-1 \cup \{ n-1 \} \tag{1}$$

or it is the empty set $\emptyset$ corresponding to the number 0. In particular, $2 = \{ 0, 1 \}$. For $n \in \mathbb{N}$, let $2^n = \{ f \colon n \mapsto 2 \}$ denote the set of all functions with the domain n and the range 2 – they are binary sequences of the length n. There is only one function $\emptyset \mapsto 2 \colon$ it is the empty function (the empty set $\emptyset$ of ordered pairs) denoted with the symbol $\mathbb{1}$. Thus, $2^0 = \{ \emptyset \} = \{ \mathbb{1} \}$ contains only the empty function. Let $\mathbb{T}$ denote the set of all functions whose domains are finite ordinals, valued in 2:

$$\mathbb{T} = \bigcup_{n \in \mathbb{N}} 2^n \,. \tag{2}$$

We define the ordering $\leq$ in the set $\mathbb{T}$ to be the reverse inclusion of functions seen as sets. Thus, for $p, q \in \mathbb{T}$ such, that $p \colon n \mapsto 2$ and $q \colon m \mapsto 2$, we have $p \leq q$ whenever $p \supseteq q$, i.e., $n \geq m$ and $p_{\restriction m} = q$. The root $\mathbb{1}$ is the largest element of $\mathbb{T}$ in this ordering: it is included in each function and for all $p \in \mathbb{T}$ we have $p \leq \mathbb{1}$. The ordered triple $\langle \mathbb{T}, \leq, \mathbb{1} \rangle$ is the partial order called the *binary tree*. Usually, by the term binary tree we will also mean the set $\mathbb{T}$ itself.

For the given $n \in \mathbb{N}$, the set 2^n of all the $p \in \mathbb{T}$ which are functions $p \colon n \mapsto 2$ is called the *n-th level* of $\mathbb{T}$. The level 0 contains only the root $\mathbb{1}$. For the given $p \in \mathbb{T}$, the symbol $|p|$ denotes the cardinality of the set of ordered pairs p, which is equal to the ordinal being the domain of p, and at the same time it is the level number to which p belongs: $p \in 2^{|p|}$.

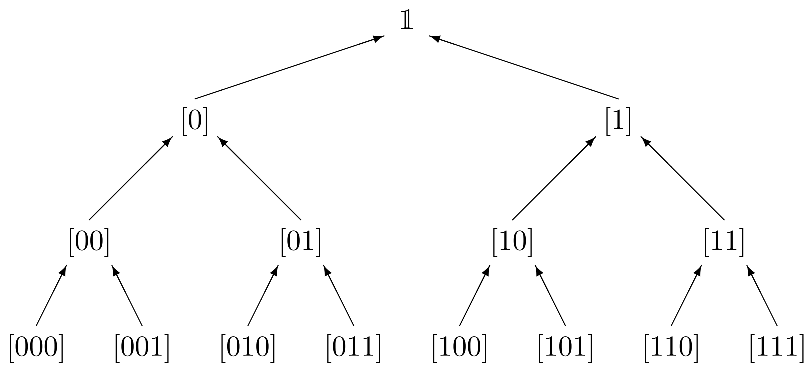

Figure 1: The binary tree $\mathbb{T}$ and the ordering of nodes (conditions). Arrows point at the larger element, i.e., the weaker condition

We represent binary sequences which are elements of $\mathbb{T}$ using square brackets surrounding consecutive elements of the sequence, as depicted on the Fig. 1. The nodes [0] and [1] are direct descendants of the root $\mathbb{1}$. The nodes [00], [01], [10], [11] form the second level, and so on. For $p \in \mathbb{T}$, we denote by $p \cdot 0$ and $p \cdot 1$ the direct descendants of p. For instance, if $p = [0]$, then $p \cdot 0 = [00]$ and $p \cdot 1 = [01]$.

Nodes of the tree $\mathbb{T}$ are sometimes called *conditions*. In applications, they are utilized to designate various circumstances affecting the degrees to which relations hold; for instance, a condition might pertain to cold or hot weather. If $p \leq q \in \mathbb{T}$, then we say that the condition p is *stronger* than the condition q, and q is *weaker* than p. A stronger condition is meant to designate a stipulation which is harder to satisfy than the one described by a weaker condition. For instance, "very cold" and "slightly cold" are stronger conditions than just "cold", since they carry more information concerning the temperature.

A set of nodes $C \subset \mathbb{T}$ is called a *chain* in $\mathbb{T}$, whenever all its elements are pairwise comparable: $\forall_{p,q \in C} \, (p \leq q \vee q \leq p)$. A set $A \subset \mathbb{T}$ is called *antichain* in $\mathbb{T}$, if it consists of mutually incomparable elements: $\forall_{p,q \in A} \, (p \neq q \to \neg \, (p \leq q) \wedge \neg \, (p \geq q))$. An example of antichain on the Fig. 1 is $\{ [00], [01], [100] \}$. A *maximal antichain* is an antichain which cannot be extended by adding new elements – it is a maximal element with respect to inclusion of antichains. Examples of maximal antichains on the Fig. 1 are $\{ [0], [1] \}$ or $\{ [00], [01], [1] \}$ or even $\{ \mathbb{1} \}$. A *branch* is a maximal chain in the tree $\mathbb{T}$. Note that p is comparable to q only, if there exists a branch containing p and q simultaneously. Similarly, p is incomparable to q whenever no branch contains both p and q. Let $R \subset \mathbb{T}$ and $p \in \mathbb{T}$. If R includes as a subset an antichain A such that $\forall_{q \in A} (q \leq p)$, then we say, that R includes an antichain *below* p.

3 Metasets

A metaset is a set whose elements – other metasets – have associated degrees of membership.[1] We formalize this idea by means of ordered pairs. Each member of a metaset – viewed as a classical set – is encapsulated in an ordered pair. The first element of the pair is the member and the second element is a node of the binary tree specifying its degree of membership.

Definition 1. *A set which is either the empty set $\emptyset$ or which has the form:*

$$\tau = \{ \langle \sigma, p \rangle : \sigma \textit{ is a metaset, } p \in \mathbb{T} \}$$

is called a metaset.

Formally, this is a definition by induction on the well founded relation $\in$. By the Axiom of Foundation in the Zermelo-Fraenkel set theory (ZFC) there are no infinite branches in the recursion as well as there are no cycles.[2] Therefore, no metaset is a member of itself. From the point of view of ZFC a metaset is a particular case of a $\mathbb{P}$-name (see also [4, Ch. VII, §2] for justification of such type of definitions).

We denote metasets with small Greek letters: τ, η, σ, etc. The class of all metasets is denoted with the letter $\mathfrak{M}$. The first element σ of an ordered pair $\langle \sigma, p \rangle$ contained in a metaset τ is called a *potential element* of τ, since it is a member of τ to a degree p which usually is less than certainty. A potential element may be simultaneously paired with multiple different conditions which taken together comprise its membership degree in the metaset. From the point of view of the set theory a metaset is a relation between a set of other metasets and a set of nodes of the binary

[1] We use the term "degree of membership" rather informally here and throughout the whole paper. We give it precise meaning by defining the term "certainty grade" in section 5.1.

[2] The Axiom of Foundation in ZFC says that every non-empty set x contains an element y which is disjoint from x:

$$\forall_{x \neq \emptyset} \exists_{y \in x} \neg \exists z \, (z \in x \land z \in y) \ .$$

tree. Therefore, we adopt the following terms and notation concerning relations. For the given metaset τ, the set of its potential elements:

$$\mathrm{dom}(\tau) = \{ \sigma \colon \exists_{p \in \mathbb{T}} \ \langle \sigma, p \rangle \in \tau \} \tag{3}$$

is called the *domain* of the metaset τ, and the set:

$$\mathrm{ran}(\tau) = \left\{ p \colon \exists_{\sigma \in \mathrm{dom}(\tau)} \ \langle \sigma, p \rangle \in \tau \right\} \tag{4}$$

is called the *range* of the metaset τ. The domain of a metaset is the domain of the relation which makes the metaset. According to this we easily see that $\tau \subset \mathrm{dom}(\tau) \times \mathrm{ran}(\tau) \subset \mathrm{dom}(\tau) \times \mathbb{T}$.

We introduce a very important class of metasets which – due to their properties – correspond to classical crisp sets. We apply the scheme analogous to the definition 1 of metaset, i.e., definition by induction on the $\in$ relation.

Definition 2. *A metaset $\check{\tau}$ is called a canonical metaset, if it is the empty set, or if it has the form:*

$$\check{\tau} = \{ \langle \check{\sigma}, \mathbb{1} \rangle : \check{\sigma} \text{ is a canonical metaset} \} \ .$$

A canonical metaset is a metaset whose domain includes only canonical metasets and whose range contains at most one element $\mathbb{1}$. For any crisp set X we may construct a canonical metaset $\check{X}$ corresponding to it, called its *canonical counterpart*, by replacing each $x \in X$ with the pair $\langle x, \mathbb{1} \rangle$ and repeating this step recursively on every level of the membership hierarchy[3] in X: we replace each member $x_i \in x$ with the pair $\langle x_i, \mathbb{1} \rangle$, and so on. Similarly, for the given canonical metaset $\check{X}$ we may construct a crisp set X by replacing each pair $\langle x, \mathbb{1} \rangle$ with x on every level of the membership hierarchy. We see, that there is a one-to-one correspondence between crisp sets and canonical metasets.

Example 1. *In the classical set theory natural (finite ordinal) numbers are defined with the formula $s(n) = n \cup \{ n \}$, where $s(n)$ is the successor of n,*

[3]By the term *membership hierarchy in X* we understand the tree of the membership relation, whose root is X. The direct members of X form the first level of this tree, the second level is made up of members of these direct members, and so on.

except for 0 *which is defined to be the empty set* $\emptyset$. *For instance:*

$$\begin{aligned} 0 &= \emptyset\,, \\ 1 &= \{\,0\,\} = \{\,\emptyset\,\}\,, \\ 2 &= \{\,0, 1\,\} = \{\,\emptyset, \{\,\emptyset\,\}\,\}\,, \\ &\vdots \\ n &= \{\,0, 1, \ldots n-1\,\} = n - 1 \cup \{\,n-1\,\}\,. \end{aligned}$$

We construct canonical metasets corresponding to the natural numbers.

$$\begin{aligned} \check{0} &= \emptyset\,, \\ \check{1} &= \{\,\langle \check{0}, \mathbb{1} \rangle\,\}\,, \\ \check{2} &= \{\langle \check{0}, \mathbb{1} \rangle, \langle \check{1}, \mathbb{1} \rangle\}\,, \\ &\vdots \\ \check{n} &= \{\,\langle \check{0}, \mathbb{1} \rangle, \ldots, \langle (n \check{-} 1), \mathbb{1} \rangle\,\} = n \check{-} 1 \cup \{\,\langle n \check{-} 1, \mathbb{1} \rangle\,\}\,. \end{aligned}$$

Left hand side of each equality defines a new symbol corresponding to the canonical counterpart of a natural number.

Another important class of metasets constitute metasets which are hereditarily finite sets.

Definition 3. *A set is called a hereditarily finite set when it is a finite set and all its members are hereditarily finite sets.*

This is a definition by induction, similar to the metaset definition. In other formulation, a set is called a hereditarily finite set, when its transitive closure (of the membership relation) is a finite set.

Metasets which are hereditarily finite sets are particularly important for computer applications, where representable entities are naturally finite.

Definition 4. *A metaset* τ *is called a hereditarily finite metaset, if its domain and range are finite sets, and each potential element is also a hereditarily finite metaset.*

We denote the class of hereditarily finite metasets with the symbol $\mathfrak{MF}$. In other words:

$$\tau \in \mathfrak{MF} \quad \text{iff} \quad |\mathrm{dom}(\tau)| < \aleph_0 \wedge |\mathrm{ran}(\tau)| < \aleph_0 \wedge \forall_{\sigma \in \mathrm{dom}(\tau)}\, \sigma \in \mathfrak{MF}\,. \tag{5}$$

Note, that elements of $\mathbb{T}$ – which are finite binary sequences – are all hereditarily finite sets. Indeed, if $p \in \mathbb{T}$, then $p \in 2^n$, for some $n \in \mathbb{N}$, i.e., p is a function from a finite ordinal n into 2, $p \colon n \mapsto 2$. Therefore, if the range of a metaset is finite, then this range is also hereditarily finite.

Although hereditarily finite metasets are the ones which we implement in computers, the results presented in this paper require slightly weaker assumptions than the hereditary finiteness. We deal here with a broader class of metasets *with finite deep range*, which we define now. One should bear in mind that all the results obtained for such metasets and presented here apply to the class $\mathfrak{MF}$ too.

Definition 5. *Let τ be a metaset and let $\operatorname{dom}^n(\tau)$ and $\operatorname{ran}^n(\tau)$ be defined as follows:*

$$\operatorname{dom}^0(\tau) = \operatorname{dom}(\tau), \qquad \operatorname{ran}^0(\tau) = \operatorname{ran}(\tau)\,,$$
$$\operatorname{dom}^{n+1}(\tau) = \bigcup_{\sigma \in \operatorname{dom}^n(\tau)} \operatorname{dom}^n(\sigma)\,, \qquad \operatorname{ran}^{n+1}(\tau) = \bigcup_{\sigma \in \operatorname{dom}^n(\tau)} \operatorname{ran}^n(\sigma)\,.$$

The set

$$\operatorname{drn}(\tau) \;=\; \bigcup_{n \in \mathbb{N}} \operatorname{ran}^n(\tau)\,.$$

is called the deep range of the metaset τ.

Thus, $\operatorname{dom}^0(\tau)$ is equal to the domain of τ, $\operatorname{dom}^1(\tau)$ is the union of the domains of potential elements of τ, $\operatorname{dom}^2(\tau)$ is the union of the domains of potential elements of potential elements of τ, and so on. The deep range of τ consists of all the conditions which occur in: the range of τ, the ranges of potential elements of τ, the ranges of potential elements of potential elements, and so on.

$$\operatorname{drn}(\tau) \;=\; \operatorname{ran}(\tau) \,\cup \bigcup_{\mu \in \operatorname{dom}(\tau)} \operatorname{ran}(\mu) \,\cup \bigcup_{\mu \in \operatorname{dom}(\tau)} \bigcup_{\nu \in \operatorname{dom}(\mu)} \operatorname{ran}(\nu) \,\cup \ldots \tag{6}$$

We denote the class of metasets with finite deep ranges with the symbol $\mathfrak{MR}$. Thus,

$$\tau \in \mathfrak{MR} \quad\leftrightarrow\quad |\operatorname{drn}(\tau)| < \aleph_0\,. \tag{7}$$

For a metaset with finite deep range, the property of having a finite range is maintained recursively on all levels of the membership hierarchy – by potential elements, their potential elements, and so on. In other words:

$$\tau \in \mathfrak{MR} \quad\leftrightarrow\quad |\operatorname{ran}(\tau)| < \aleph_0 \wedge \forall_{\sigma \in \operatorname{dom}(\tau)}\, \sigma \in \mathfrak{MR}\,. \tag{8}$$

Comparing the equations (5) and (8) we may conclude the following.

Proposition 1. *The deep range of a hereditarily finite metaset is finite:*

$$\tau \in \mathfrak{MF} \quad \to \quad \tau \in \mathfrak{MR} \,.$$

Proof. For a hereditarily finite metaset, each element of the union (6) is finite by the formula (5). Since the relation $\in$ is well founded, then there is a finite number of non-empty elements in the union (6). Thus, the deep range of a hereditarily finite metaset is a finite union of finite sets. □

The contrary does not have to be true. A canonical metaset may have an infinite domain, in which case it is not a hereditarily finite one. Its range, as well as its deep range, still contain only one element $\mathbb{1}$, so they are finite.

4 Interpretations

An interpretation of a metaset is a crisp set extracted out of the metaset by means of a branch in the binary tree. For the given metaset, each branch in $\mathbb{T}$ determines a different interpretation. All the interpretations taken together make up a collection of sets with specific internal dependencies, which represents the metaset by means of its crisp views. In practical applications these particular views are treated as various experts' opinions on some vague term represented by the metaset.

Properties of crisp sets which are interpretations of the given metaset determine the properties of the metaset itself. We use the forcing mechanism (sec. 5) for transferring relationships between sets which are interpretations onto the metasets. A good example is the definition of the membership relation which relies on membership among interpretations (sec. 5.2).

Definition 6. *Let τ be a metaset and let $\mathcal{C} \subset \mathbb{T}$ be a branch. The set*

$$\mathrm{int}(\tau, \mathcal{C}) = \{\, \mathrm{int}(\sigma, \mathcal{C}) \colon \ \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \,\}$$

is called the interpretation of the metaset τ given by the branch $\mathcal{C}$.

We usually use a shorter notation $\tau_{\mathcal{C}}$ for the interpretation $\mathrm{int}(\tau, \mathcal{C})$. Any interpretation of the empty metaset is the empty set, independently of the branch: $\check{0}_{\mathcal{C}} = \emptyset$, for each $\mathcal{C} \subset \mathbb{T}$. The process of producing the interpretation of a metaset consists in two stages. In the first stage we remove all the ordered pairs whose second elements are conditions which do not belong to the branch $\mathcal{C}$. The second stage replaces the remaining pairs – whose second elements lie on the branch $\mathcal{C}$ – with interpretations of their first elements, which are other metasets. This two-stage process is repeated recursively on all the levels of the membership hierarchy. As the result we obtain a crisp set.

Example 2. *Let $p \in \mathbb{T}$ and let $\tau = \{ \langle \emptyset, p \rangle \}$. If $\mathcal{C}$ is a branch, then*

$$\begin{aligned} p \in \mathcal{C} \quad &\to \quad \tau_{\mathcal{C}} = \{ \emptyset_{\mathcal{C}} \} = \{ \emptyset \} \ , \\ p \notin \mathcal{C} \quad &\to \quad \tau_{\mathcal{C}} = \emptyset \ . \end{aligned}$$

Depending on the branch the metaset τ acquires different interpretations.

Clearly, a metaset may have multiple different interpretations – each branch in the tree determines one. Usually, many of them are pairwise equal, so the number of different interpretations is much less than the number of branches. Hereditarily finite metasets always have a finite number of different interpretations. There are metasets whose interpretations are all equal, even when they are not hereditarily finite. For instance, interpretations of canonical metasets are always branch independent. For a canonical metaset $\check{\tau}$ all its interpretations are equal to some crisp set.

Proposition 2. *Let $\check{x}$ be a canonical metaset and let x be the crisp set such, that $\check{x}$ is the canonical counterpart of x. For any branch $\mathcal{C} \subset \mathbb{T}$:*

$$\check{x}_{\mathcal{C}} = x \ .$$

Proof. Follows directly from the definitions 2 and 6 and the fact that $\mathbb{1} \in \mathcal{C}$ for each branch $\mathcal{C}$. □

The natural correspondence between canonical metasets and crisp sets is illustrated by the following example.

Example 3. *Let $\check{0}$, $\check{1}$, $\check{2}$, ..., be canonical counterparts of natural numbers as defined in the example 1. Let $\mathcal{C}$ be any branch in $\mathbb{T}$. Since $\mathbb{1} \in \mathcal{C}$, then*

$$\begin{aligned} \check{0}_{\mathcal{C}} &= \emptyset \ , \\ \check{1}_{\mathcal{C}} &= \big\{ \langle \check{0}, \mathbb{1} \rangle \big\}_{\mathcal{C}} = \{ \emptyset \} \ , \\ \check{2}_{\mathcal{C}} &= \big\{ \langle \check{0}, \mathbb{1} \rangle , \langle \check{1}, \mathbb{1} \rangle \big\}_{\mathcal{C}} = \{ 0, 1 \} = \{ \emptyset, \{ \emptyset \} \} \ . \end{aligned}$$

We see that $\check{0}_{\mathcal{C}} = 0$, $\check{1}_{\mathcal{C}} = 1$, $\check{2}_{\mathcal{C}} = 2$ and so on.

An interpretation of a metaset is influenced not only by the elements from its range, but also elements of ranges of its potential elements (i.e., $\bigcup_{\mu\in\operatorname{dom}(\tau)} \operatorname{ran}(\mu)$), as well as the ranges of potential elements of these elements ($\bigcup_{\mu\in\operatorname{dom}(\tau)} \bigcup_{\nu\in\operatorname{dom}(\mu)} \operatorname{ran}(\nu)$), and so on. The deep range of the metaset contains all the conditions which determine the interpretations (see lemma 1). When it is finite, then the metaset has particularly regular properties which we discuss in the sequel.

Let τ be a metaset with a finite range and let $p \in \mathbb{T}$ be a condition such, that there exists no $q < p$ in $\operatorname{ran}(\tau)$. Such p exists, since the range is finite. For any branch $\mathcal{C}^p$ containing p, the interpretation $\tau_{\mathcal{C}^p}$ is comprised of the interpretations of potential elements from the same subset $D_p \subset \operatorname{dom}(\tau)$, namely

$$\begin{aligned} D_p &= \{\, \sigma \in \operatorname{dom}(\tau) \colon \exists_{r\in\mathbb{T}}\ \langle \sigma, r\rangle \in \tau \wedge r \in \mathcal{C}^p \,\} && (9)\\ &= \{\, \sigma \in \operatorname{dom}(\tau) \colon \exists_{r\in\mathbb{T}}\ \langle \sigma, r\rangle \in \tau \wedge r \geq p \,\}\ . && (10)\end{aligned}$$

Thus, for any $\mathcal{C}^p$ containing p we have $\tau_{\mathcal{C}^p} = \{\, \sigma_{\mathcal{C}^p} : \sigma \in D_p \,\}$. The set D_p of potential elements determining the interpretations of τ is independent of particular branches containing p, since there are no conditions in $\operatorname{ran}(\tau)$ below p, which could affect the contents of $\tau_{\mathcal{C}^p}$. Unfortunately, for any $\sigma \in D_p$ and any branch $\mathcal{C}^p \ni p$, the interpretations $\sigma_{\mathcal{C}^p}$ themselves may vary, since each $\operatorname{ran}(\sigma)$ may contain different conditions below p, which affect the interpretations. There may exist different $p_1, p_2 < p$ such, that for branches $\mathcal{C}^1 \ni p_1$ and $\mathcal{C}^2 \ni p_2$ we have $\sigma_{\mathcal{C}^1} \neq \sigma_{\mathcal{C}^2}$. Usually this implies $\tau_{\mathcal{C}^1} \neq \tau_{\mathcal{C}^2}$ – like in the following example – although this is not a rule.

Example 4. *For a natural number n, let $\bar{\omega}^n = \left\{\, \langle \check{p}, p\rangle : \exists_{k\leq n}\, p \in 2^k \,\right\}$, where p is a node of the k-th level of $\mathbb{T}$, for any $k \leq n$, and $\check{p}$ denotes the metaset which is the canonical counterpart of the natural number, whose binary representation is p, for $p \neq \mathbb{1}$, and $\check{p} = \check{0}$, when $p = \mathbb{1}$. For instance,*

$$\begin{aligned} \bar{\omega}^0 &= \left\{\, \left\langle \check{0}, \mathbb{1} \right\rangle \,\right\}\ ,\\ \bar{\omega}^1 &= \bar{\omega}^0 \cup \left\{\, \left\langle \check{0}, [0] \right\rangle, \left\langle \check{1}, [1] \right\rangle \,\right\}\ ,\\ \bar{\omega}^2 &= \bar{\omega}^1 \cup \left\{\, \left\langle \check{0}, [00] \right\rangle, \left\langle \check{1}, [01] \right\rangle, \left\langle \check{2}, [10] \right\rangle, \left\langle \check{3}, [11] \right\rangle \,\right\}\ .\end{aligned}$$

Additionally, let $\bar{\omega}^\infty = \{\, \langle \check{p}, p\rangle : p \in \mathbb{T} \,\}$. Clearly, for any $n \in \mathbb{N}$, the deep range $\operatorname{drn}(\bar{\omega}^n)$ is finite and contains all the nodes from the levels up to n. On the other hand, $\operatorname{drn}(\bar{\omega}^\infty)$ is infinite and is equal to the whole set $\mathbb{T}$.

Let $\tau = \{\, \langle \bar{\omega}^\infty, [1]\rangle \,\}$ and $\sigma = \left\{\, \left\langle \bar{\omega}^1, [1] \right\rangle \,\right\}$. We clearly see, that

$$\operatorname{ran}(\tau) = \operatorname{ran}(\sigma) = \{\, [1] \,\}\ . \tag{11}$$

However, $\operatorname{drn}(\tau) = \mathbb{T}$*, whereas* $\operatorname{drn}(\sigma) = \{\mathbb{1}, [0], [1]\}$*. So, although their ranges are equal, their deep ranges are not.*

If $\mathcal{C}$ *is any branch containing* $[1]$*, then* $\sigma_{\mathcal{C}} = \{\bar{\omega}^1_{\mathcal{C}}\} = \{\{0,1\}\}$*. Thus, each branch containing* $[1]$ *produces identical interpretations of* σ*. For any branch* $\mathcal{C}$*, the interpretation* $\bar{\omega}^\infty_{\mathcal{C}}$ *is a set of natural numbers of form* $\{\check{p}_{\mathcal{C}} : p \in \mathcal{C}\} = \{p : p \in \mathcal{C}\}$*, where* $p \in \mathbb{T}$ *is treated as natural number in binary notation, and therefore* $0 \leq p < 2^{|p|}$*, for each level number* $|p|$*.*[4] *Now, let* $\mathcal{C}'$ *be the leftmost branch containing the condition* $[1]$*, i.e.,* $\mathcal{C}' = \{\mathbb{1}, [1], [10], [100], \ldots\}$*, and let* $\mathcal{C}''$ *be the rightmost branch in the tree* $\mathbb{T}$*:* $\mathcal{C}'' = \{\mathbb{1}, [1], [11], [111], \ldots\}$*. We check that* $\tau_{\mathcal{C}'} = \{\{0,1,2,4,8,\ldots\}\}$*, and* $\tau_{\mathcal{C}''} = \{\{0,1,3,7,\ldots\}\}$*, where* n *is the level of* $\mathbb{T}$*. Thus, different branches containing* $[1]$ *may produce different interpretations of* τ*, even though there are no conditions stronger than* $[1]$ *in* $\operatorname{ran}(\tau)$*. However, there are many such conditions in* $\operatorname{drn}(\tau)$*, which make the interpretations of* $\bar{\omega}^\infty$ *variable, affecting thus the interpretations of* τ *itself.*

Based on the above example we see that interpretations of the metaset τ are influenced only by conditions from $\operatorname{drn}(\tau)$. The set $\operatorname{drn}(\tau)$ entirely determines all the interpretations of τ. If $\mathcal{C}'$ and $\mathcal{C}''$ are branches which differ only outside of $\operatorname{drn}(\tau)$, then they give equal interpretations.

Lemma 1. *Let* τ *be a metaset and let* $\mathcal{C}'$ *and* $\mathcal{C}''$ *be branches.*

$$\mathcal{C}' \cap \operatorname{drn}(\tau) = \mathcal{C}'' \cap \operatorname{drn}(\tau) \quad \to \quad \tau_{\mathcal{C}'} = \tau_{\mathcal{C}''} .$$

Proof. By induction on the relation of being a potential element (which is well founded). First, note that if we assume the left hand side of the implication, then $\operatorname{ran}(\tau) \cap \mathcal{C}' = \operatorname{ran}(\tau) \cap \mathcal{C}''$. Indeed, if $p \in \operatorname{ran}(\tau) \cap \mathcal{C}'$, then since $p \in \mathcal{C}' \cap \operatorname{drn}(\tau) = \mathcal{C}'' \cap \operatorname{drn}(\tau)$ we also have $p \in \mathcal{C}''$ what implies $p \in \operatorname{ran}(\tau) \cap \mathcal{C}''$.

Directly from the definition we have $\tau_{\mathcal{C}'} = \{\sigma_{\mathcal{C}'} : \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C}'\}$. Assuming that the thesis holds for the potential elements $\sigma \in \operatorname{dom}(\tau)$, and taking into account that $\operatorname{ran}(\tau) \cap \mathcal{C}' = \operatorname{ran}(\tau) \cap \mathcal{C}''$ we conclude the thesis for τ. □

Whenever we assume that the given metaset τ belongs to the class $\mathfrak{MR}$, we want to assure, that interpretations of potential elements of τ given by different branches containing some strong enough p, are all pairwise equal and therefore do not affect the interpretation of τ itself.

[4] Recall, that $|p|$ is the number of ordered pairs in the function p, i.e., the length of the sequence p, which is equal to the level number of the level containing the node p.

5 Forcing

In this section we define and investigate a relation between a condition and a sentence. This relation, called *forcing* relation [2], is designed to describe the level of confidence or certainty assigned to the sentence. The level is evaluated by means of nodes of $\mathbb{T}$. The root condition $\mathbb{1}$ specifies the absolute certainty, whereas its descendants represent less certain degrees. The sentences are classical set theory formulas, where free variables are substituted by metasets and bound variables range over the class of metasets.

Given a branch $\mathcal{C}$, we may substitute particular metasets in the sentence $\sigma \in \tau$ with their interpretations which are ordinary crisp sets, e.g.: $\sigma_\mathcal{C} \in \tau_\mathcal{C}$. The resulting sentence is a ZFC sentence expressing some property of the crisp sets $\tau_\mathcal{C}$ and $\sigma_\mathcal{C}$, the membership relation in this case. Such sentence may be either true or false, depending on $\tau_\mathcal{C}$ and $\sigma_\mathcal{C}$.

For the given metaset τ each condition $p \in \mathbb{T}$ specifies a family of interpretations of τ: they are determined by all the branches $\mathcal{C}$ containing this particular condition p. If for each such branch the resulting sentence – after substituting metasets with their interpretations – has the same logical value, then we may think of a conditional truth or falsity of the given sentence, which is qualified by the condition p. Therefore, we may consider p as the certainty degree for the sentence.

Let Φ be a formula built using some of the following symbols: variables $(x^1, x^2, \ldots)$, the constant symbol $(\emptyset)$, the relational symbols $(\in, =, \subset)$, logical connectives $(\wedge, \vee, \neg, \rightarrow)$, quantifiers $(\forall, \exists)$ and parentheses. If we substitute each free variable x^i $(i = 1 \ldots n)$ with some metaset ν^i, and restrict the range of each quantifier to the class of metasets $\mathfrak{M}$, then we get as the result the sentence $\Phi(\nu^1, \ldots, \nu^n)$ of the metaset language, which states some property of the metasets $\nu^1, \ldots, \nu^n$. By the *interpretation* of this sentence, determined by the branch $\mathcal{C}$, we understand the sentence $\Phi(\nu^1_\mathcal{C}, \ldots, \nu^n_\mathcal{C})$ denoted shortly with $\Phi_\mathcal{C}$. The sentence $\Phi_\mathcal{C}$ is the result of substituting free variables of the formula Φ with the interpretations $\nu^i_\mathcal{C}$ of the metasets ν^i, and restricting the range of bound variables to the class of all sets $\mathbf{V}$. In other words, we replace the metasets in the sentence Φ with their interpretations. The only constant $\emptyset$ in Φ as well as in $\Phi_\mathcal{C}$ denotes the empty set which is the same set in both cases: as a crisp set and as a metaset.

Definition 7. *Let $x^1, x^2, \ldots x^n$ be all free variables of the formula Φ and let $\nu^1, \nu^2, \ldots \nu^n$ be metasets. We say that the condition $p \in \mathbb{T}$ forces the sentence $\Phi(\nu^1, \nu^2, \ldots \nu^n)$, whenever for each branch $\mathcal{C} \subset \mathbb{T}$ containing the* *condition p, the sentence* $\Phi(\nu^1_{\mathcal{C}}, \nu^2_{\mathcal{C}}, \dots \nu^n_{\mathcal{C}})$ *is true. We denote the forcing relation with the symbol* $\Vdash$*. Thus,*

$$p \Vdash \Phi(\nu^1, \dots \nu^n) \quad \textit{iff} \quad \textit{for each branch } \mathcal{C} \ni p \textit{ holds } \Phi(\nu^1_{\mathcal{C}}, \dots \nu^n_{\mathcal{C}}) \ .$$

We use the abbreviation $p \nVdash \Phi$ for expressing the negation $\neg(p \Vdash \Phi)$. It this case, not for each branch $\mathcal{C}$ containing p the sentence $\Phi_{\mathcal{C}}$ holds, however, such branches may exist. Furthermore, the symbol $\notin$ in the formula $\mu \notin \tau$ will stand for $\neg(\mu \in \tau)$, and similarly, $\mu \neq \tau$ will stand for $\neg(\mu = \tau)$.

The key idea of the forcing relation lies in transferring properties from crisp sets onto metasets. Let a property described by a formula $\Phi(x)$ be satisfied by all crisp sets of form $\nu_{\mathcal{C}}$, where ν is a metaset and $\mathcal{C}$ is a branch in $\mathbb{T}$. In other words, $\Phi(\nu_{\mathcal{C}})$ holds for all the sets which are interpretations of the metaset ν given by all branches $\mathcal{C}$ in $\mathbb{T}$. Then we might think that this property also "holds" for the metaset ν, and we formulate this fact by saying that $\mathbb{1}$ forces $\Phi(\nu)$. If $\Phi(\nu_{\mathcal{C}})$ holds only for branches $\mathcal{C}$ containing some condition p, then we might think that it "holds to the degree p" for the metaset ν; we say that p forces $\Phi(\nu)$ in such case. Since we try to transfer – or force – satisfiability of some property from crisp sets onto metasets, we call this mechanism *forcing*.[5] The next example shows how to transfer the property of being equal onto two specific metasets.

Example 5. *Let* $\tau = \{ \langle \check{0}, p \rangle \}$*,* $\sigma = \{ \langle \check{0}, p \cdot 0 \rangle , \langle \check{0}, p \cdot 1 \rangle \}$*, where* $p \in \mathbb{T}$*. Let* $\mathcal{C}$ *be a branch.*

$$\begin{array}{rcccc} p \cdot 0 \in \mathcal{C} & \to & \tau_{\mathcal{C}} = \{ \emptyset \} \wedge \sigma_{\mathcal{C}} = \{ \emptyset \} & \to & \tau_{\mathcal{C}} = \sigma_{\mathcal{C}} \ , \\ p \cdot 1 \in \mathcal{C} & \to & \tau_{\mathcal{C}} = \{ \emptyset \} \wedge \sigma_{\mathcal{C}} = \{ \emptyset \} & \to & \tau_{\mathcal{C}} = \sigma_{\mathcal{C}} \ , \\ p \notin \mathcal{C} & \to & \tau_{\mathcal{C}} = \emptyset \wedge \sigma_{\mathcal{C}} = \emptyset & \to & \tau_{\mathcal{C}} = \sigma_{\mathcal{C}} \ . \end{array}$$

Of course, the last case is possible only when $p \neq \mathbb{1}$*, since the root of* $\mathbb{T}$ *is contained in each branch. We see, that the interpretations of* τ *and* σ *are always pairwise equal, although they are different sets depending on the chosen branch* $\mathcal{C}$*. Analyzing only the structure of* τ *and* σ *we may easily conclude that* $p \Vdash \tau = \sigma$*. However, since for any branch* $\mathcal{C}$ *which does not contain* p *the interpretations of* τ *and* σ *are both empty, then also* $\mathbb{1} \Vdash \tau = \sigma$*.*

The following two lemmas expose the most fundamental and significant features of the forcing relation. The first says that forcing is propagated

[5]This mechanism is similar to, and in fact was inspired by the method of forcing in the classical set theory [2]. It has not much in common with the original.

down the branch, i.e., if a condition p forces Φ, then stronger conditions force Φ too. However, weaker conditions do not have to force it. It should be understood that the stronger conditions carry more detailed information above the weaker ones.

Lemma 2. *Let $p, q \in \mathbb{T}$ and let Φ be a sentence. If p forces Φ and q is stronger than p, then q forces Φ too:*

$$p \Vdash \Phi \wedge q \leq p \quad \rightarrow \quad q \Vdash \Phi \ .$$

Proof. If $q \leq p$, then each branch containing q also contains p. If $\mathcal{C}$ is any such branch and $p \Vdash \Phi$, then $\Phi_{\mathcal{C}}$ holds. Because it is true for all $\mathcal{C} \ni q$, then we have $q \Vdash \Phi$. □

On the other hand, a finite maximal antichain of conditions stronger than $p \in \mathbb{T}$ propagates forcing upwards to the condition p. Recall, that a set $R \subset \mathbb{T}$ is called an antichain when all its members are pairwise incomparable. It is a maximal antichain in $\mathbb{T}$ when each $q \in \mathbb{T}$ is comparable to some element of R. It is a maximal antichain below p when each $q \leq p$ is comparable to some element of R and all the members of R are stronger than p.

Lemma 3. *Let $p \in \mathbb{T}$, $R \subset \mathbb{T}$ and let Φ be a sentence. If R is a finite maximal antichain below p and each $q \in R$ forces Φ, then p also forces Φ.*

Proof. $p \Vdash \Phi$ whenever for each branch $\mathcal{C} \ni p$ holds $\Phi_{\mathcal{C}}$. Since R is a finite maximal antichain whose elements are stronger than p, then each branch containing p must also contain some element $q \in R$. Each such q forces Φ, so for any branch $\mathcal{C} \ni p$ we have $\Phi_{\mathcal{C}}$. □

The example 5 shows p, τ, σ such, that $p \Vdash \tau = \sigma$. Inspecting the structure of τ and σ we conclude that also $p \cdot 0 \Vdash \tau = \sigma$ and $p \cdot 1 \Vdash \tau = \sigma$, what is confirmed by the lemma 2. On the other hand, the conditions $p \cdot 0$, $p \cdot 1$ form the final maximal antichain below p. Since they both force $\tau = \sigma$, then – by the lemma 3 – their parent p must force the sentence too.

5.1 Forcing and Certainty Degrees

If we treat conditions as certainty degrees for sentences, then the stronger condition specifies the degree which is less than the degree specified by the weaker one (assuming the conditions are different). Indeed, by the above lemmas $r \Vdash \Psi$ is equivalent to the conjunction $r \cdot 0 \Vdash \Psi \wedge r \cdot 1 \Vdash \Psi$

meaning that the certainty degree specified by r is equal to the "sum" of certainty degrees specified by both $r \cdot 0$ and $r \cdot 1$ taken together. But if it happens that $r \cdot 0 \Vdash \Psi$ and $r \cdot 1 \nVdash \Psi$, then also $r \nVdash \Psi$. In such case the $r \cdot 0$ contributes only a half of the certainty degree specified by r – another half of it could be contributed by $r \cdot 1$, but is not in this case. The root $\mathbb{1}$, being the largest element in $\mathbb{T}$, specifies the highest certainty degree. The ordering of certainty degrees is consistent with the ordering of conditions in $\mathbb{T}$. We stress that the term certainty degree is used informally in this paper. We define now other precise terms for measuring the certainty of sentences.

For the given sentence Φ, we call the set $\bar{\mathcal{T}}_\Phi = \{ p \in \mathbb{T} \colon p \Vdash \Phi \}$ the *certainty set* for Φ. It contains all the conditions which force the given sentence and it gives a measure of certainty that the sentence is true. Members of this set are called *certainty factors* for Φ. Each certainty factor contributes to the overall degree of certainty that the sentence is true, which is represented by the certainty set.

By the lemma 2, if there exist a $p \in \mathbb{T}$ which forces Φ, then there exist infinitely many other conditions which force Φ too. Among them are all those stronger than p. Therefore, the whole certainty set is equivalent to the set of its maximal elements. Since,

$$p \Vdash \Phi \quad \rightarrow \quad \exists_q \, p \le q \wedge q \in \max\{\bar{\mathcal{T}}_\Phi\} \wedge q \Vdash \Phi \; , \tag{12}$$

then each $p \in \bar{\mathcal{T}}_\Phi \setminus \max\{\bar{\mathcal{T}}_\Phi\}$ is redundant. The substantial information concerning the conditions which force Φ is contained in $\max\{\bar{\mathcal{T}}_\Phi\}$ exclusively. Forcing of Φ by any stronger conditions may be concluded by applying the lemma 2. Thus we come to the following concept of certainty degree for sentences.

Definition 8. *Let Φ be a sentence. The set of maximal elements of the certainty set for Φ:*

$$\mathcal{T}_\Phi = \max \{ p \in \mathbb{T} \colon p \Vdash \Phi \}$$

is called the certainty grade for Φ. If the certainty set is empty, then the certainty grade is empty too.

One may easily see that $\mathcal{T}_\Phi$ forms an antichain. When the certainty set is equal to the whole tree $\mathbb{T}$, then the certainty grade is the singleton containing only the root: $\mathcal{T}_\Phi = \{ \mathbb{1} \}$. We may assign numerical values to certainty grades with the following formula.

$$\mathcal{V}_\Phi = \sum_{p \in \mathcal{T}_\Phi} \frac{1}{2^{|p|}} \; , \tag{13}$$

where $|p| = n$ is the number of pairs in the function $p\colon n \mapsto 2$, i.e., the length of the binary sequence p, or simply the level of the tree $\mathbb{T}$ where p belongs. The value $\mathcal{V}_\Phi$ is called the *certainty value* for Φ. One may easily see that whenever no p forces Φ, then $\mathcal{V}_\Phi = 0$ and if each $p \in \mathbb{T}$ forces Φ, then $\mathcal{V}_\Phi = 1$. Therefore, $\mathcal{V}_\Phi \in [0, 1]$.

5.2 Membership and Non-membership

We do not give thorough presentation of relations for metasets in this paper. For completeness, we supply only the definitions of conditional membership and non-membership. Other relations, like conditional equality and non-equality, are defined similarly – by means of the forcing mechanism.

In fact, we define an infinite number of membership relations. Each of them designates the membership satisfied to some degree specified by a node of the binary tree. Moreover, any two metasets may be simultaneously in multiple membership relations qualified by different conditions.

Definition 9. *We say that the metaset μ belongs to the metaset τ under the condition $p \in \mathbb{T}$, whenever $p \Vdash \mu \in \tau$. We use the notation $\mu \epsilon_p \tau$.*

In other words, $\mu \epsilon_p \tau$ whenever for each branch $\mathcal{C} \subset \mathbb{T}$ containing p holds $\mu_\mathcal{C} \in \tau_\mathcal{C}$. The conditional membership reflects the idea that a metaset μ belongs to a metaset τ whenever some conditions are fulfilled. The conditions are represented by nodes of $\mathbb{T}$.

Each $p \in \mathbb{T}$ specifies another relation ϵ_p. Different conditions specify membership relations which are satisfied with different certainty factors. The lemmas 2 and 3 prove that the relations are not independent. For instance, $\mu \epsilon_p \tau$ is equivalent to $\mu \epsilon_{p\cdot 0} \tau \wedge \mu \epsilon_{p\cdot 1} \tau$, i.e., being a member under the condition p is equivalent to being a member under conditions $p \cdot 0$ and $p \cdot 1$ simultaneously.

We introduce another set of relations for expressing non-membership. The reason for this is due to the fact that $p \nVdash \mu \in \tau$ is not equivalent to $p \Vdash \mu \notin \tau$. Indeed, $p \nVdash \mu \in \tau$ means, that it is not true that for each branch $\mathcal{C}$ containing p holds $\mu_\mathcal{C} \in \tau_\mathcal{C}$, however such branches may exist. On the other hand, $p \Vdash \mu \notin \tau$ means that for each $\mathcal{C} \ni p$ holds $\mu_\mathcal{C} \notin \tau_\mathcal{C}$. That is why we need another relation "is not a member under the condition p".

Definition 10. *We say that the metaset μ does not belong to the metaset τ under the condition $p \in \mathbb{T}$, whenever $p \Vdash \mu \notin \tau$. We use the notation $\mu \not\epsilon_p \tau$.*

Thus, $\mu \notin_p \tau$, whenever for each branch $\mathcal{C}$ containing p the set $\mu_{\mathcal{C}}$ is not a member of the set $\tau_{\mathcal{C}}$. Contrary to the classical case, where a set is either a member of another or it is not at all, for two metasets it is possible that they are simultaneously in different membership and non-membership relations. This resembles intuitionistic fuzzy sets [1], where membership of an element in such set is characterized by two values given by the membership function and the non-membership function. The following example illustrates this phenomenon and we elaborate more on this in the corollary 1.

Example 6. *Let $\tau = \{ \langle \check{0}, p \rangle \}$, where $p \neq \mathbb{1}$. If $\mathcal{C}$ is a branch, then*

$$\begin{aligned} p \in \mathcal{C} &\quad\to\quad \tau_{\mathcal{C}} = \{ \emptyset \} \ , \\ p \notin \mathcal{C} &\quad\to\quad \tau_{\mathcal{C}} = \emptyset \ . \end{aligned}$$

Thus, $p \Vdash \check{0} \in \tau$, so $\check{0} \epsilon_p \tau$. If $q \in \mathbb{T}$ is incomparable to p – for instance if $p = [0]$ and $q = [1]$ – then $q \Vdash \check{0} \notin \tau$, since for any branch $\mathcal{C}$ containing q we have $\tau_{\mathcal{C}} = \emptyset$. Therefore, $\check{0} \notin_q \tau$. We see, that $\check{0}$ is a member of τ to the degree p and simultaneously it is not a member to the degree q, i.e., $\check{0} \epsilon_p \tau \wedge \check{0} \notin_q \tau$ is true. It is worth noting, that in this case also $\check{0} \epsilon_{p \cdot 0} \tau \wedge \check{0} \epsilon_{p \cdot 1} \tau$, as well as $\check{0} \notin_{q \cdot 0} \tau \wedge \check{0} \notin_{q \cdot 1} \tau$.

Of course, it is not possible that $\sigma \epsilon_p \tau$ and at the same time $\sigma \notin_p \tau$, since it is false that $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$ and $\sigma_{\mathcal{C}} \notin \tau_{\mathcal{C}}$, for any branch $\mathcal{C}$.

6 Finite Decidability

The forcing relation assigns certainty grades to sentences. There exist sentences, like $\check{0} \neq \check{0}$, which cannot be forced by any condition, since their interpretations are always false. Their certainty grades exist but they are empty sets. On the other hand, there are sentences whose interpretations are always true, like $\forall_\tau \, \tau = \tau$. Such sentences are forced by each condition, so their certainty grades are equal to the singleton $\{ \mathbb{1} \}$. Finally, there exist sentences, whose interpretations are either true or false, depending on the branch (cf. example 6). There arises a natural question: is it possible to assign a non-empty certainty grade to each sentence which is true in some interpretation? Formally, for the given sentence Φ, if there exists a branch $\mathcal{C}$ such, that $\Phi_{\mathcal{C}}$ is true, then does there exist a $p \in \mathbb{T}$ such, that $p \Vdash \Phi$? And more generally: since for each branch $\mathcal{C}$ the interpretation $\Phi_{\mathcal{C}}$ of the given sentence Φ is either true or false, then does there exist a $p \in \mathcal{C}$ such, that either $p \Vdash \Phi$ or $p \Vdash \neg\Phi$? Surprisingly, this is not true in general, however this is true for the sentences involving finite deep range metasets only, what we prove in this section.

Let $\Phi(x_1,\ldots,x_n)$ be a formula with all free variables shown, let $\mathfrak{C}$ be some class of metasets and let $\mu_1,\ldots,\mu_n \in \mathfrak{C}$ be metasets. If we substitute each free variable x_i in the formula Φ with the corresponding metaset $\mu_i \in \mathfrak{C}$ and restrict the range of each quantifier to the class $\mathfrak{M}$ of metasets, then we call the resulting sentence $\Phi(\mu_1,\ldots,\mu_n)$ a $\mathfrak{C}$-*sentence*. We focus here mainly on $\mathfrak{MR}$-sentences which involve exclusively metasets with finite deep ranges. A $\mathfrak{MR}$-sentence $\Phi(\mu_1,\ldots,\mu_n)$ is a sentence of the metaset language expressing some property of the metasets $\mu_1,\ldots,\mu_n$.

The first important fact that we prove says, that if a $\mathfrak{MR}$-sentence is true in some interpretation, then it must be forced by some condition. As a consequence, it is true in an infinite number of interpretations determined by other branches containing the condition. The condition itself becomes a certainty factor for the sentence and the certainty value for the sentence is greater than 0. This property is not satisfied for metasets in general: there are sentences involving metasets with infinite deep ranges, which are not forced by any condition, although they are true in particular interpretations (see example 7).

We split the theorem into two parts: the first focuses on atomic sentences and the second generalizes the result to arbitrary sentences. By atomic formula we understand a formula which contains neither logical connectives nor quantifiers: it consists of two terms and one relational symbol, so it is built of at most 2 variables, the constant $\emptyset$, the relations $\in$, $=$, $\subset$ and their negations $\notin, \neq, \not\subset$. If $\Phi(x,y)$ is an atomic formula and τ, η are metasets, then $\Phi(\tau,\eta)$ is an atomic sentence.

Lemma 4. *Let $\Phi(\tau,\eta)$ be an atomic sentence, where $\tau,\sigma \in \mathfrak{MR}$. If there exist a branch $\mathcal{C}$ such, that $\Phi(\tau_{\mathcal{C}},\eta_{\mathcal{C}})$ is true, then there exists $q \in \mathcal{C}$ such, that $q \Vdash \Phi(\tau,\eta)$.*

Proof. Let τ, η be metasets in the atomic sentence $\Phi(\tau,\eta)$. Let $R \subset \mathbb{T}$ be the union of deep ranges of τ and η: $R = \operatorname{drn}(\tau) \cup \operatorname{drn}(\eta)$, and let $\bar{R}$ be the set of all the conditions weaker than those from R.

$$\bar{R} = \{\, s \in \mathbb{T} \colon \exists_{q \in R}\, q \leq s \,\} \ .$$

If $\tau = \emptyset = \eta$, then $\operatorname{drn}(\tau) = \operatorname{drn}(\eta) = \emptyset$ and $R = \bar{R} = \emptyset$. Clearly, $R \subset \bar{R}$.

Let $\mathcal{C}$ be a branch such, that $\Phi(\tau_{\mathcal{C}},\eta_{\mathcal{C}})$ is true. We find the $q \in \mathcal{C}$ such, that $q \Vdash \Phi(\tau,\eta)$. Initially, if $R \neq \emptyset$, then let q' be the least element (the strongest condition) of the set $\bar{R}$, which lies on the branch $\mathcal{C}$. If $R = \emptyset$, then let $q' = \mathbb{1}$. Note, that $\mathcal{C} \cap \bar{R} \neq \emptyset$ for $R \neq \emptyset$, since the intersection contains at least $\mathbb{1}$.

$$q' = \begin{cases} \min\left(\mathcal{C} \cap \bar{R}\right) & \text{iff} \quad R \neq \emptyset \ , \\ \mathbb{1} & \text{iff} \quad R = \emptyset \ . \end{cases}$$

For instance, if $R = \{\,[0], [011]\,\}$, then $\bar{R} = \{\,\mathbb{1}, [0], [01], [011]\,\}$, $\min(R) = \min(\bar{R}) = [011]$, and for the sample branches $\mathcal{C}_1$, $\mathcal{C}_2$ we have:

$$\mathcal{C}_1 = \{\,\mathbb{1}, [0], [01], [010], \dots\} \quad \to \quad q' = \min(\mathcal{C}_1 \cap \bar{R}) < \min(\mathcal{C}_1 \cap R) \ , \tag{14}$$
$$\mathcal{C}_2 = \{\,\mathbb{1}, [0], [01], [011], \dots\} \quad \to \quad q' = \min(\mathcal{C}_2 \cap \bar{R}) = \min(\mathcal{C}_2 \cap R) \ , \tag{15}$$

since $\min(\mathcal{C}_1 \cap \bar{R}) = [01]$ and $\min(\mathcal{C}_1 \cap R) = [0]$ and $\min(\mathcal{C}_2 \cap \bar{R}) = [011]$.

If q' is also a minimal element in R – i.e., no condition in R is strictly stronger than q', like in (15) – or if $q' = \mathbb{1}$, then $q = q'$ and we are done, since none of descendants of q' affect interpretations of τ and η. Consequently (cf. lemma 1), all the branches containing q' give the same interpretations for τ and η, and the same logical value for $\Phi(\tau_{\mathcal{C}}, \eta_{\mathcal{C}})$, which is true. Therefore, $q' \Vdash \Phi(\tau, \eta)$.

However, it is possible that there exists $s < q' = \min(\mathcal{C} \cap \bar{R})$ such, that $s \in R$ and $s \notin \mathcal{C}$, like in (14). In such case we take as q one of the direct descendants of q' – the one which lies on $\mathcal{C}$. Therefore, the q is defined as follows:

$$q = \begin{cases} q' & \text{iff} \quad \forall_{s<q'}\ s \notin R \ , \\ q' \cdot 0 & \text{iff} \quad \exists_{s<q'}\ s \in R \wedge q' \cdot 0 \in \mathcal{C} \ , \\ q' \cdot 1 & \text{iff} \quad \exists_{s<q'}\ s \in R \wedge q' \cdot 1 \in \mathcal{C} \ . \end{cases}$$

Of course, it is not possible that $q' \cdot 0 \in \mathcal{C}$ and $q' \cdot 1 \in \mathcal{C}$ at the same time. Therefore, if the first case does not hold, then the remaining two are mutually exclusive.

Why does $q \Vdash \Phi(\tau, \eta)$? Take arbitrary branches $\mathcal{C}'$ and $\mathcal{C}''$ containing q. Clearly $R \cap \mathcal{C}' = R \cap \mathcal{C}''$, so $\mathcal{C}' \cap \mathrm{drn}(\tau) = \mathcal{C}'' \cap \mathrm{drn}(\tau)$ and by the lemma 1 we have $\tau_{\mathcal{C}'} = \tau_{\mathcal{C}''}$. Similarly for η. As we see, the branches containing q give identical interpretations of the metasets that are subject to the relation stated by Φ. This implies that this relation is preserved in all such interpretations (by the assumption it holds for $\mathcal{C}$). Therefore, $q \Vdash \Phi(\tau, \eta)$. □

Thus, if $\tau, \eta \in \mathfrak{MR}$ and for some branch $\mathcal{C}$ the atomic sentence $\Phi(\tau_{\mathcal{C}}, \eta_{\mathcal{C}})$ is true, then for some $p \in \mathcal{C}$ holds $p \Vdash \Phi(\tau, \eta)$. If $\mathcal{C}'$ is any other branch containing p, then $\Phi(\tau_{\mathcal{C}'}, \eta_{\mathcal{C}'})$ is also true. The condition p is a certainty factor for $\Phi(\tau,\eta)$ and the certainty value for this sentence is equal at least $\frac{1}{2^{|p|}}$. This property may be generalized to arbitrary, non-atomic $\mathfrak{MR}$-sentences.

Theorem 1. *Let $\Phi(x^1,\ldots,x^n)$ be a formula with all free variables shown and let $\tau^1,\ldots,\tau^n \in \mathfrak{MR}$. If for some branch $\mathcal{C}$ the sentence $\Phi(\tau^1_{\mathcal{C}},\ldots,\tau^n_{\mathcal{C}})$ is true, then there exists a condition $p \in \mathcal{C}$ such, that $p \Vdash \Phi(\tau^1,\ldots,\tau^n)$.*

Proof. Similarly to the proof of the lemma 4, let R be the union of deep ranges of the metasets $\tau^1,\ldots,\tau^n$: $R = \bigcup_{i=1}^{n} \operatorname{drn}(\tau^i)$. Let $\mathcal{C}$ be the branch such, that $\Phi(\tau^1_{\mathcal{C}},\ldots,\tau^n_{\mathcal{C}})$ is true, and let $p \in \mathcal{C}$ be any condition which has no descendants belonging to R: $\neg\exists_{q\in R}\, q \leq p$ (when $R = \emptyset$, then take $p = \mathbb{1}$). Such p exists, since R is finite and it may be constructed similarly as in the proof of the lemma 4.

Why does $p \Vdash \Phi(\tau^1,\ldots,\tau^n)$ hold? It does, since all the branches containing p give equal interpretations of the metasets $\tau^1,\ldots,\tau^n$. Indeed, let $p \in \mathcal{C}',\mathcal{C}''$. We have:

$$\tau^1_{\mathcal{C}'} = \tau^1_{\mathcal{C}''} \wedge \ldots \wedge \tau^n_{\mathcal{C}'} = \tau^n_{\mathcal{C}''}\ ,$$

and therefore also

$$\Phi(\tau^1_{\mathcal{C}'},\ldots,\tau^n_{\mathcal{C}'}) \quad \leftrightarrow \quad \Phi(\tau^1_{\mathcal{C}''},\ldots,\tau^n_{\mathcal{C}''})\ .$$

□

The theorem 1 says, that any $\mathfrak{MR}$-sentence true in some interpretation is forced by some condition. The following example justifies that the assumption on finiteness of deep ranges in the above theorems is necessary. It also shows that this property is not valid in general. We construct metasets $\sigma \notin \mathfrak{MR}$ and $\tau \in \mathfrak{MR}$ such, that $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$ for some branch $\mathcal{C}$, but at the same time, for any $p \in \mathcal{C}$ holds $p \nVdash \sigma \in \tau$.

Example 7. *Recall that $\omega = \{0,1,\ldots\}$ is the set of finite ordinals and $\check{\omega} = \{\langle\check{0},\mathbb{1}\rangle,\langle\check{1},\mathbb{1}\rangle,\ldots\}$ is its canonical counterpart. Let $\tau = \{\langle\check{\omega},\mathbb{1}\rangle\}$ and let $\sigma = \{\langle\check{n},p_n\rangle : n \in \omega\}$, where $\check{n}$ is the canonical counterpart of n, and $p_0 = \mathbb{1}$, $p_1 = [1]$, $\ldots$, i.e., p_n contains exactly n occurrences of 1; it is a constant function $p_n\colon n \mapsto \{1\}$. Let $\mathcal{C}_1 = \{p_n\colon n \in \omega\}$ be the rightmost branch in $\mathbb{T}$, comprised of 1s exclusively: $\mathcal{C}_1 = [111\ldots]$. It is clear that $\sigma_{\mathcal{C}_1} = \omega$. Of course, $\tau_{\mathcal{C}_1} = \{\omega\}$, so $\sigma_{\mathcal{C}_1} \in \tau_{\mathcal{C}_1}$. However, for no $p_n \in \mathcal{C}_1$ it is true that $p_n \Vdash \sigma \in \tau$. Indeed, if $\mathcal{C}$ is any branch containing $p_n \cdot 0$, then $\sigma_{\mathcal{C}} = \{0,\ldots,n\}$ but $\tau_{\mathcal{C}} = \{\omega\}$ still, so $\sigma_{\mathcal{C}} \notin \tau_{\mathcal{C}}$. Therefore, $p_n \nVdash \sigma \in \tau$.*

If $p \in \mathbb{T}$ is arbitrary condition and $\mathcal{C}^p$ is any branch containing p, different than the rightmost branch $\mathcal{C}_1$, then $\mathcal{C}^p$ must also contain some p_n. For instance, if $\mathcal{C}_0 = [0....]$ is any branch containing $[0]$, then $p_0 = \mathbb{1} \in \mathcal{C}_0$. Since $\tau_{\mathcal{C}^p} = \{\, \omega \,\}$ and $\sigma_{\mathcal{C}^p} = \{\, 0, \ldots, n \,\}$, for the largest n such, that $p_n \in \mathcal{C}^p$, then no $p \in \mathbb{T}$ forces $\sigma \in \tau$. Consequently, for all $p \in \mathbb{T}$ we have $p \nVdash \sigma \in \tau$. Furthermore, for any $p \notin \mathcal{C}_1$, since for any branch $\mathcal{C}^p$ containing p holds $\sigma_{\mathcal{C}^p} \neq \omega$, then $p \Vdash \sigma \notin \tau$.

Thus, we have shown, that even though σ belongs to τ in some interpretation, then this fact is not forced by any condition. Moreover, there exists $q \in \mathbb{T}$ such, that $q \Vdash \sigma \notin \tau$. The reason for this strange behavior is that the deep range of σ is infinite.

For the given sentence Φ and for any branch $\mathcal{C}$, either $\Phi_\mathcal{C}$ or $\neg\Phi_\mathcal{C}$ is true. Thus, if Φ is a $\mathfrak{MR}$-sentence, then for any branch $\mathcal{C}$ we may find a condition $p \in \mathcal{C}$ which decides Φ: either $p \Vdash \Phi$ or $p \Vdash \neg\Phi$. Consequently, for any p in $\mathbb{T}$ there exists a $q \leq p$ which decides Φ, i.e., q forces either the sentence or its negation. Recall (example 6), that there may exist different branches $\mathcal{C}'$ and $\mathcal{C}''$ such, that $\Phi_{\mathcal{C}'}$ and $\neg\Phi_{\mathcal{C}''}$ hold simultaneously. Therefore, there may exist $p \neq q$ such, that $p \Vdash \Phi$ and $q \Vdash \neg\Phi$. The following corollary summarizes this property. It says that each $\mathfrak{MR}$-sentence is decided by some condition: either the sentence and/or its negation is forced. It may seem strange, that sentences and their negations may be forced simultaneously, by different conditions.

Corollary 1. *Let $\Phi(x^1, \ldots, x^n)$ be a formula with free variables $x^1, \ldots, x^n$ and let $\tau^1, \ldots, \tau^n \in \mathfrak{MR}$. Exactly one of the following holds*

$$\mathbb{1} \Vdash \Phi(\tau^1, \ldots, \tau^n) \,,$$
$$\mathbb{1} \Vdash \neg\Phi(\tau^1, \ldots, \tau^n) \,,$$
$$\exists\, p, q \in \mathbb{T}\colon\ \big(p \Vdash \Phi(\tau^1, \ldots, \tau^n) \wedge q \Vdash \neg\Phi(\tau^1, \ldots, \tau^n)\big) \ .$$

Proof. If $\mathbb{1} \nVdash \Phi$, then there exists a branch $\mathcal{C}'$ such, that $\neg\Phi_{\mathcal{C}'}$. By the theorem 1 there exists q such, that $q \Vdash \neg\Phi$. If $\mathbb{1} \Vdash \neg\Phi$, then by lemma 2 it implies $q \Vdash \neg\Phi$. Otherwise, if $\mathbb{1} \nVdash \neg\Phi$, then there exists a branch $\mathcal{C}''$ such, that $\Phi_{\mathcal{C}''}$ holds. Applying the theorem again we obtain $p \in \mathcal{C}''$ such, that $p \Vdash \Phi$. Note, that p is incomparable to q (by lemma 2). □

If a sentence involves metasets whose deep ranges are not finite, then it is possible, that neither the sentence nor its negation is forced by any condition. The following example demonstrates metasets σ, τ such, that both $p \not\Vdash \sigma \in \tau$ and $p \not\Vdash \sigma \notin \tau$, for all $p \in \mathbb{T}$. Of course, each interpretation of the sentence is either true or false.

Example 8. *Let* $\sigma = \{ \langle \check{n}, p \rangle : p \in \mathbb{T} \wedge n = \Sigma_{i \in \mathrm{dom}(p)} p(i) \}$, $\tau = \{ \langle \check{\omega}, \mathbb{1} \rangle \}$. *Recall, that conditions are functions* $p \colon m \mapsto 2$ *with domains in* ω. *Each ordered pair in* σ *is comprised of an arbitrary condition* $p \in \mathbb{T}$ *and the canonical counterpart* $\check{n}$ *of* $n \in \omega$, *which is the number of occurrences of* 1 *in the binary representation of* p: $n = \Sigma_{i \in \mathrm{dom}(p)} p(i)$. *In other words*

$$\sigma = \{ \langle \check{n}, p_n \rangle : n \in \omega \text{ and } p_n \text{ has exactly } n \text{ occurrences of } 1 \} \ .$$

For instance: p_0 *may be* [0], [00], *etc.,* p_1 *may be of form* [100], [01], [0010].

If $\mathcal{C}$ *is a branch containing a finite number of* 1 *and infinite number of* 0, *i.e.,* $\Sigma_{i \in \omega} \mathcal{C}(i) = n < \infty$, *then* $\sigma_{\mathcal{C}} = \{ 0, \ldots, n \}$, *so* $\sigma_{\mathcal{C}} \notin \tau_{\mathcal{C}} = \{ \omega \}$. *If, on the other hand,* $\mathcal{C}$ *contains infinite number of* 1, *then* $\sigma_{\mathcal{C}} = \omega$, *since for any* $n \in \omega$ *there exists at least one condition* $p_n \in \mathcal{C}$ *such, that* $n = \Sigma_{i \in \mathrm{dom}(p_n)} p_n(i)$ *and* $\langle \check{n}, p_n \rangle \in \sigma$. *In such case we have* $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$. *Thus, for an arbitrary* $p \in \mathbb{T}$ *holds* $p \not\Vdash \sigma \in \tau$ *as well as* $p \not\Vdash \sigma \notin \tau$, *since for* $\mathcal{C}$ *containing infinitely may* 1 *the membership holds in interpretations, whereas for the remaining ones – it does not hold.*

Let Φ denote the sentence $\sigma \in \tau$. The example shows that although for each branch $\mathcal{C}$ either $\Phi_{\mathcal{C}}$ or $\neg\Phi_{\mathcal{C}}$ holds, the certainty sets for both Φ and $\neg\Phi$ are empty. Therefore also certainty values $\mathcal{V}_{\Phi}$ and $\mathcal{V}_{\neg\Phi}$ are equal 0. The difference $1 - (\mathcal{V}_{\Phi} + \mathcal{V}_{\neg\Phi})$ is the measure of uncertainty of the sentence Φ. Since it is equal to 1 in this case, then we say that Φ is totally uncertain – we cannot say anything about truth or falsity of Φ. The example 8 may be modified so, that both certainty values $\mathcal{V}_{\Phi}$, $\mathcal{V}_{\neg\Phi}$, as well as the uncertainty value $1 - (\mathcal{V}_{\Phi} + \mathcal{V}_{\neg\Phi})$ are positive [7]. This idea is the basis for representing intuitionistic fuzzy sets [1] by metasets.

We now show that for any $\mathfrak{MR}$-sentence Φ the certainty grade for Φ complements the certainty grade for $\neg\Phi$, i.e., their union forms a maximal antichain in $\mathbb{T}$. Consequently, the sum of certainty values for Φ and $\neg\Phi$ is equal to 1. It means that $\mathfrak{MR}$-sentences admit no hesitancy degree.

Let $\Phi(x^1, \ldots, x^n)$ be a formula with all free variables shown and let $\tau^1, \ldots, \tau^n \in \mathfrak{MR}$. Let $\mathrm{drn}(\Phi)$ denote the union of deep ranges of these metasets:

$$\mathrm{drn}(\Phi) = \mathrm{drn}(\tau^1) \cup \ldots \cup \mathrm{drn}(\tau^n) \ . \tag{16}$$

Let l_{Φ} be the greatest level number of conditions in $\mathrm{drn}(\Phi)$ (it is well defined since drn(Φ) is finite):[6]

$$l_\Phi = \max\{\, |p| \colon p \in \mathrm{drn}(\Phi) \,\} \ . \tag{17}$$

We call l_Φ the *deciding level* for the $\mathfrak{MR}$-sentence Φ. It has the following property.

Lemma 5. *If Φ is a $\mathfrak{MR}$-sentence and l_Φ is the deciding level for Φ, then the following holds*

$$p \in 2^{l_\Phi} \quad \to \quad p \Vdash \Phi \vee p \Vdash \neg\Phi \ .$$

Proof. Let $\tau^1, \ldots, \tau^n \in \mathfrak{MR}$ be all metasets occurring in Φ (not bound by quantifiers). Take arbitrary $p \in 2^{l_\Phi}$ and let us assume that $p \nVdash \Phi$. By the definition there exists a branch $\mathcal{C} \ni p$ such, that $\neg\Phi_{\mathcal{C}}$ is true. Let $\mathcal{C}'$ be another branch containing p. There are no elements of the set drn(Φ), which are less than p. Therefore, $\mathcal{C} \cap \mathrm{drn}(\Phi) = \mathcal{C}' \cap \mathrm{drn}(\Phi)$ and for each $i = 1, \ldots, n$ also $\mathcal{C} \cap \mathrm{drn}(\tau^i) = \mathcal{C}' \cap \mathrm{drn}(\tau^i)$. By the lemma 1 we conclude $\tau^i_{\mathcal{C}} = \tau^i_{\mathcal{C}'}$ for each τ^i. Obviously,

$$\neg\Phi(\tau^1_{\mathcal{C}}, \ldots, \tau^n_{\mathcal{C}}) \wedge \bigwedge_{i=1}^{i=n} \tau^i_{\mathcal{C}} = \tau^i_{\mathcal{C}'} \ . \tag{18}$$

implies $\neg\Phi(\tau^1_{\mathcal{C}'}, \ldots, \tau^n_{\mathcal{C}'})$. Since for each branch $\mathcal{C}' \ni p$ holds $\neg\Phi(\tau^1_{\mathcal{C}'}, \ldots, \tau^n_{\mathcal{C}'})$, then $p \Vdash \neg\Phi$. □

Note, that by lemma 2, levels below the deciding level have the same property too. For each $p \in \mathbb{T}$ such, that $|p| \geq l_\Phi$, either $p \Vdash \Phi$ or $p \Vdash \neg\Phi$. Nonetheless, there still may exist conditions with $|p| < l_\Phi$, which do not force anything. For instance, if $p \cdot 0 \Vdash \Phi$ and $p \cdot 1 \Vdash \neg\Phi$, then $p \nVdash \Phi$ and $p \nVdash \neg\Phi$. Such conditions occur in upper levels of $\mathbb{T}$, near the root.

Each level is a maximal antichain in $\mathbb{T}$. Levels below the deciding level contain the whole information concerning truth or falsity of interpretations of the given sentence; each condition on such level either forces the sentence or its negation. The notion of maximal antichain generalizes level with respect to this property. A maximal antichain of conditions below the deciding level also carries all the information about the sentence and

[6] Recall that each $p \in \mathbb{T}$ is a function valued in $2 \in \omega$ and $|p|$ is its cardinality – the number of ordered pairs. In other words it is the length of the sequence p. The set of all p of equal length l is the l-th level in $\mathbb{T}$ denoted with 2^l.

it consists of pairwise incomparable elements, however it may contain conditions from different levels. Maximal antichains which decide a sentence may contain elements from levels above the deciding level too. The least such antichain is given by the following theorem.

Theorem 2. *If Φ is a $\mathfrak{MR}$-sentence, then the union $\mathcal{T}_\Phi \cup \mathcal{T}_{\neg\Phi}$ of certainty grades for Φ and $\neg\Phi$ is a finite maximal antichain in $\mathbb{T}$.*

Proof. The union forms an antichain, since both $\mathcal{T}_\Phi$ and $\mathcal{T}_{\neg\Phi}$ are antichains and they contain pairwise incomparable elements. Indeed, if $p' \in \mathcal{T}_\Phi$ and $p'' \in \mathcal{T}_{\neg\Phi}$, then $p' \Vdash \Phi$ and $p'' \Vdash \neg\Phi$, so if $p' \leq p''$, then by the lemma 2, $p' \Vdash \neg\Phi$, what implies $p' \Vdash \Phi \wedge \neg\Phi$, a contradiction. Similarly, when $p' \geq p''$.

The union is finite, since both $\mathcal{T}_\Phi$ and $\mathcal{T}_{\neg\Phi}$ are finite. By the lemma 5, for any $\mathfrak{MR}$-sentence, all elements of its certainty grade lie on levels of $\mathbb{T}$, which are not greater than the deciding level, so there is a finite number of them.

We show that it is maximal in $\mathbb{T}$, what means that each element in $\mathbb{T}$ is comparable to some element of the antichain. Assume contrary and let p be any condition which is incomparable to all elements of $\mathcal{T}_\Phi \cup \mathcal{T}_{\neg\Phi}$. If $|p| \geq l_\Phi$, i.e., p belongs to a level which is greater than or equal than the deciding level for Φ, then either $p \Vdash \Phi$ or $p \Vdash \neg\Phi$. By the definition 8 it must be stronger than some element of $\mathcal{T}_\Phi \cup \mathcal{T}_{\neg\Phi}$ – a contradiction. If $|p| < l_\Phi$, then let $q \in 2^{l_\Phi}$ be any descendant of p. Clearly, q is incomparable to each element of the union too. However, by the lemma 5, q forces either Φ or $\neg\Phi$, so as previously, it must be stronger than some element of $\mathcal{T}_\Phi \cup \mathcal{T}_{\neg\Phi}$. Again, a it contradicts the assumption that p is incomparable to elements of the union. □

We use sets of nodes of the binary tree instead of numbers to express degrees of certainty, since they are more general – sets carry more information than just numbers. In the language of sets of nodes, the maximal antichains play role similar to the certainty value of 1: they represent the highest degree of information available. At the same time they contain no redundant elements. We now give precise formulation of this observation. Recall, that by the equation (13), the certainty value for the sentence Φ is equal to the sum $\sum_{p\in\mathcal{T}_\Phi} \frac{1}{2^{|p|}}$.

Proposition 3. *If $A \subset \mathbb{T}$ is a maximal antichain in $\mathbb{T}$, then $\Sigma_{p\in A} \frac{1}{2^{|p|}} = 1$.*

Proof. Each $p \neq \mathbb{1}$ is a binary sequence which represents a natural number $\#p = \Sigma_{i\in \mathrm{dom}(p)}\, p(i) \cdot 2^i$. Therefore, each $p \neq \mathbb{1}$ corresponds to an interval

$\bar{p} = [\frac{\#p}{2^{|p|}}, \frac{\#p+1}{2^{|p|}}) \subset [0,1]$ and $\mathbb{1}$ corresponds to $I = [0,1)$. The length of each interval is $\frac{1}{2^{|p|}}$. For incomparable p and q, the corresponding intervals are disjoint: $\bar{p} \cap \bar{q} = \emptyset$. Indeed, if $\bar{p} \cap \bar{q} \neq \emptyset$, then there must exist some $r \in \mathbb{T}$ such, that $\bar{r} \subset \bar{p} \cap \bar{q}$. Since $\bar{r} \subset \bar{p}$, then $r \leq p$, and similarly $r \leq q$. This implies $p \leq q$ or $q \leq p$, so they are comparable.

We now show, that the measure of $\bigcup_{p\in A} \bar{p}$ is equal 1. Clearly, it cannot be grater than 1, so if it is less, then let $u \subset I \setminus \bigcup_{p\in A} \bar{p}$ be an open interval. There must exist $s \in \mathbb{T}$ such, that $\bar{s} \subset u$. If s is comparable to some $p \in A$, then $\bar{s} \cap \bar{p} \neq \emptyset$, so $\bar{s} \cap \bigcup_{p\in A} \bar{p}$ is non-empty, what contradicts $\bar{s} \subset u$. Thus, assuming that the measure of $\bigcup_{p\in A} \bar{p}$ is less than 1 we found s incomparable to all elements of A, what contradicts its maximality.

To finish the proof, note that the measure of each $\bar{p}$ is $\frac{1}{2^{|p|}}$, the measure of $\bigcup_{p\in A} \bar{p}$ is 1 and they are all pairwise disjoint. □

Using the proposition 3 we may reformulate the theorem 2 in terms of certainty values.

Corollary 2. *If Φ is a $\mathfrak{MR}$-sentence, then $\mathcal{V}_\Phi + \mathcal{V}_{\neg\Phi} = 1$.*

We may easily calculate certainty values for $\mathfrak{MR}$-sentences applying the the lemma 5. Let

$$T_\Phi = \left\{ p \in 2^{l_\Phi} : p \Vdash \Phi \right\} \quad \text{and} \quad F_\Phi = \left\{ p \in 2^{l_\Phi} : p \Vdash \neg\Phi \right\} . \tag{19}$$

By the lemma we have $T_\Phi \cup F_\Phi = 2^{l_\Phi}$ – these sets fill the whole deciding level. Since there are 2^{l_Φ} elements on the l_Φ-th level, then

$$\mathcal{V}_\Phi = \frac{|T_\Phi|}{2^{l_\Phi}} \quad \text{and} \quad \mathcal{V}_{\neg\Phi} = \frac{|F_\Phi|}{2^{l_\Phi}} . \tag{20}$$

We apply here lemmas 2, 3 and take into account that for any $p \in \mathbb{T}$ holds $\frac{1}{2^{|p|}} = \frac{1}{2^{|p\cdot 0|}} + \frac{1}{2^{|p\cdot 1|}}$.

Corollary 3. *If Φ is a $\mathfrak{MR}$-sentence, then $\mathcal{T}_\Phi \cup \mathcal{T}_{\neg\Phi}$ intersects all branches in $\mathbb{T}$.*

Each maximal finite antichain in $\mathbb{T}$ intersects all the branches. This is not true in general, for infinite maximal antichains. For instance $A = \{ [0], [10], [110], \dots \}$ is an infinite maximal antichain, since each node in $\mathbb{T}$ is comparable to some element of A, and it is comprised of pairwise incomparable elements. However, it does not intersect the rightmost branch $\mathcal{C}_1 = \{ \mathbb{1}, [1], [11], [111], \dots \}$. The example 7 demonstrates Φ – which is not a $\mathfrak{MR}$-sentence – such, that $\Phi_{\mathcal{C}_1}$ is true, although no condition in $\mathbb{T}$ forces Φ. Moreover, for each $p \in A$ holds $p \Vdash \neg\Phi$. Therefore, $\mathcal{V}_\Phi = 0$ and $\mathcal{V}_{\neg\Phi} = 1$!

7 Summary

We have introduced the concept of metaset – set with partial membership relation. We have defined fundamental techniques of interpretation and forcing and we have showh how to evaluate certainty degrees for sentences of the metaset language. In this paper we have focused on specific properties of metasets with finite deep ranges.

It turns out that several important results may be obtained for sentences involving only metasets from the class $\mathfrak{MR}$. One of the most significant is that for such sentences the certainty values of the sentence and its negation sum up to unity, what is not true in general. Therefore, there is no hesitancy of membership for such metasets. The membership and non-membership degrees, when expressed as numbers (certainty values) sum up to 1, and when expressed as sets of conditions (certainty grades) sum up to a finite maximal antichain in $\mathbb{T}$.

The class of metasets with finite deep ranges is especially important due to the fact, that metasets implementable in computers are hereditarily finite and thus they have finite deep ranges too. Therefore, the presented results are significant for computer applications of metasets [6]. Usually, when trying to implement some mathematical theory in computers we encounter a variety of limitations caused by the finiteness of machine world. This is not the case for the computer-oriented theory of metasets. When restricting the domain of discourse to the class of computer representable metasets we obtain additional, important results.

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