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Meta Sets – Another Approach to Fuzziness

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1.1 Preface

In this chapter we present the concept of a meta set, which is an alternative to a fuzzy set [6]. Similarly to fuzzy sets, the meta sets are meant to describe and represent imprecise data or collections. However, meta sets are better fitted within the classical set theory. In particular, "elements" of meta sets are also meta sets. The language of meta sets resembles the language of the Zermelo–Fraenkel set theory [2] (ZFC) and many properties of crisp sets are reflected in the meta sets theory.

As oppose to fuzzy sets, which involve quite complex ideas like real function, meta sets are defined using simple – from the set-theoretic point of view – and well known notions. This enables easier and more efficient algorithmisation and computer implementations of relations and operations for meta sets.

The definition of a meta set, although similar to the definition of a fuzzy set, is much more general. In fact, meta sets generalise fuzzy sets, or even intuitionistic fuzzy sets [1], as they allow for expressing a hesitancy degree.

In practical applications we mostly deal with finite sets. Therefore we have distinguished a subclass of meta sets which correspond to finite sets. We have managed to define basic algebraic operations for such sets, and have proved that they satisfy the axioms of Boolean algebra.

Although the algebraic operations are the main topic of this chapter, we start with the general introduction to the concept of a meta set. The section 1.2 establishes some well known definitions and notations. The section 1.3 presents fundamentals of meta sets. In the section 1.4 we introduce some important class of meta sets and define basic relations and operations for them. Finally, the section 1.5 contains the proof that these operations satisfy the Boolean algebra axioms.

1.2 Preliminary Definitions and Terminology

We will denote the binary tree (the full and infinite one) with the symbol $\mathbb{T}$. The root of the binary tree, denoted with $\mathbb{1}$, is its largest element. Nodes of the tree $\mathbb{T}$ will be called *conditions*. Thus, for all $p \in \mathbb{T}$, we have $p \leq \mathbb{1}$. Comparable conditions (either $p \leq q$ or $p \geq q$), are denoted with the symbol $p \top q$. Incomparable ones $(\neg (p \leq q) \wedge \neg (p \geq q))$ are denoted with $p \perp q$. If $p, q \in \mathbb{T}$ are arbitrary conditions, then we say that the condition p is *stronger* than the condition q, whenever $p \leq q$. If $p \geq q$, then we say that the condition p is *weaker* than the condition q. A stronger condition is meant to designate a stipulation which is harder to satisfy than the one described by some weaker condition.

A condition in the binary tree $\mathbb{T}$ may be viewed as a finite binary sequence. We will specify a condition using square brackets surrounding consecutive elements of the appropriate sequence, as depicted on the Fig. 1.1: [0] and [1] are direct descendants of the root $\mathbb{1}$. [00], [01], [10], [11] is the second generation, and so on.

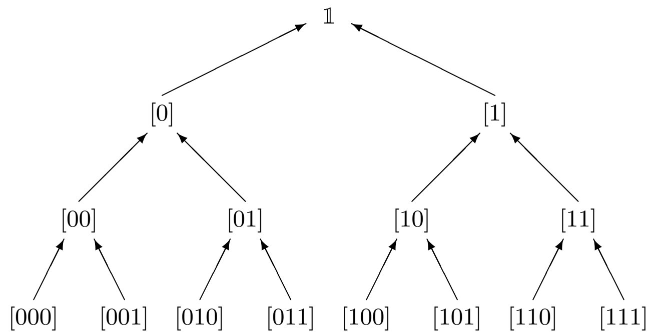

Fig. 1.1. Conditions and the order in the binary tree $\mathbb{T}$. Arrows point at the larger element, i.e., the weaker condition

A set $C \subset \mathbb{T}$ is called a *chain* in $\mathbb{T}$, if $\forall_{p,q \in C}\, (p \leq q \vee q \leq p)$. A set $A \subset \mathbb{T}$ is called *antichain* in $\mathbb{T}$, if $\forall_{p,q \in A}\, (p \neq q \rightarrow p \perp q)$. Thus, a chain consists of pairwise comparable conditions, whereas an antichain consists of mutually incomparable conditions. The empty set $\emptyset$ is a chain, as well as an antichain. On the Fig. 1.1, the elements $\{\, [00], [01], [100] \,\}$ form a sample antichain. A *maximal antichain* is an antichain which cannot be extended by adding new elements – it is a maximal element with respect to inclusion of antichains. Examples of maximal antichains on the Fig. 1.1 are $\{\, [0], [1] \,\}$ or $\{\, [00], [01], [1] \,\}$ or even $\{\, \mathbb{1} \,\}$. A *branch* is a maximal chain in the tree $\mathbb{T}$. Note that $p \top q$ only, if there exists a branch containing p and q simultaneously. Similarly, $p \perp q$ whenever no branch contains both p and q. Let $R \subset \mathbb{T}$ and $p \in \mathbb{T}$. If R includes as a subset an antichain A such that $\forall_{q \in A}\, (q \leq p)$, then we say, that

R includes an antichain *below* p. R includes a maximal antichain below p if the antichain A cannot be extended to another antichain below p by adding elements stronger than p.

A *level* in the tree $\mathbb{T}$ is the set of all conditions of the same length seen as binary sequences. The level *number* is the length of the condition. Thus, the level number 0 contains only the root $\mathbb{1}$, and the level number 1 contains the elements [0] and [1]. The Fig. 1.1 displays the levels $0 \ldots 3$ of the binary tree. A *subtree rooted at* a condition p is the full subtree of the tree $\mathbb{T}$, whose root is the element p. It consists of all the conditions stronger than p (including p). On the Fig. 1.1 the subtree rooted at [01] consists of the conditions $\{ [01], [010], [011] \}$.

1.3 Meta Sets

A meta set is a set, which is not fully precised, but – potentially – it might be precised in various ways. It might acquire various particular representations, which are ordinary crisp sets, depending on some external circumstances. These external circumstances will be formalised as interpretations of the meta set determined by branches in the binary tree $\mathbb{T}$. The properties of the crisp sets which are interpretations of a meta set determine the properties of the meta set itself.

1.3.1 Fundamental Definitions

Elements of crisp sets are other crisp sets. Similarly, elements of meta sets should be other meta sets. However, being an element of a meta set means much more than in the case of a crisp set, as it must consider the degree of partial membership of the element to the meta set. Because of this reason, the actual elements of a meta set (viewed as a crisp set) are ordered pairs. The first element of such a pair is a meta set – the potential element. The second element of the pair is a condition in the binary tree $\mathbb{T}$, which determines the degree of membership.

Definition 1. *A meta set is a crisp set which is either the empty set $\emptyset$, or which has the form:*

$$\tau = \{ \langle \sigma, p \rangle : \sigma \textit{ is a meta set, } p \in \mathbb{T} \} \ .$$

Here $\mathbb{T}$ is the binary tree and $\langle \cdot, \cdot \rangle$ denotes an ordered pair.

Note, that the above definition is recursive, however, founded by the empty set $\emptyset$ which itself is a meta set too. We denote meta sets with small Greek letters: τ, η, σ. The class of all meta sets is denoted with the letter $\mathfrak{M}$.

Formally, this is a definition by induction on the well founded relation $\in$. The well foundedness of $\in$ is directly implied by the Axiom of Foundation in the Zermelo–Fraenkel set theory[1]. A justification for such type of definition is presented in the discussion following the definition of a $\mathbb{P}$-name[2].

The first element of an ordered pair contained in a meta set τ, which is another meta set, is called a *potential element* of τ. Thus meta sets are potential elements of other meta sets, whereas their real elements (from the crisp sets point of view) are ordered pairs.

We may perceive a meta set as a crisp set, whose elements (as well as elements of elements, and so on) are labelled with nodes of the tree $\mathbb{T}$. Each potential element may be labelled with multiple different labels constituting this way multiple pairs which are elements of the meta set.

From the point of view of the crisp set theory a meta set is a relation (i.e. a subset of a Cartesian product) between the set of its potential elements and the binary tree $\mathbb{T}$. Mostly, this relation is not a function, as it is in the case of fuzzy sets, as each potential element may be labelled with different conditions.

Definition 2. *The domain of a meta set τ, denoted with $\mathrm{dom}(\tau)$, is the set of its potential elements:*

$$\mathrm{dom}(\tau) = \{\, \sigma \colon\ \langle \sigma, p \rangle \in \tau \,\} \ .$$

Definition 3. *The range of the meta set τ is the set:*

$$\mathrm{ran}(\tau) = \{\, p \colon\ \langle \sigma, p \rangle \in \tau \,\} \ .$$

Thus, the domain of a meta set is the domain of the relation which the meta set is. According to this we easily see that:

$$\tau \subset \mathrm{dom}(\tau) \times \mathrm{ran}(\tau) \subset \mathrm{dom}(\tau) \times \mathbb{T} \ . \tag{1.1}$$

Definition 4. *Let τ and σ be arbitrary meta sets. The set*

$$\tau[\sigma] = \{\, p \in \mathbb{T} \colon\ \langle \sigma, p \rangle \in \tau \,\}$$

is called the image of the meta set τ at the meta set σ in the tree $\mathbb{T}$.

The image $\tau[\sigma]$ might be the empty set $\emptyset$, if σ is not a potential element of τ. Generally, the image $\tau[\sigma]$ is a set of conditions describing the degree of membership of σ in τ. We can easily see that:

$$\mathrm{ran}(\tau) = \bigcup_{\sigma \in \mathrm{dom}(\tau)} \tau[\sigma] \ , \tag{1.2}$$

$$\tau = \bigcup_{\sigma \in \mathrm{dom}(\tau)} \{\, \sigma \,\} \times \tau[\sigma] \ . \tag{1.3}$$

Let us consider some examples. The simplest meta set is the empty set $\emptyset$. It may be used as a potential element of other meta sets:

[1] theorem 4.1 in [2, Ch. III, §4]
[2] definition 2.5 in [2, Ch. VII, §2]

$$\tau = \{\, \langle \emptyset, p \rangle \,\} \ , \qquad \tau[\emptyset] = \{\, p \,\} \ , \qquad \mathrm{dom}(\tau) = \{\, \emptyset \,\} \ , \quad \mathrm{ran}(\tau) = \{\, p \,\} \ ,$$
$$\sigma = \{\, \langle \emptyset, p \rangle , \langle \emptyset, q \rangle \,\} \ , \quad \sigma[\emptyset] = \{\, p, q \,\} \ , \quad \mathrm{dom}(\sigma) = \{\, \emptyset \,\} \ , \quad \mathrm{ran}(\sigma) = \{\, p, q \,\} \ .$$

In the first case the degree of membership of $\emptyset$ in τ is represented by the one-element subset of $\mathbb{T}$ which is $\{\, p \,\}$. In the second example the degree of membership is represented by two-element subset (assuming $p \neq q$): $\{\, p, q \,\}$.

As we will see further, if $p \top q$, then the stronger condition will not contribute any additional membership information above the weaker one, the stronger condition is in such case redundant. On the other hand, if $p \perp q$, then both conditions contribute independent membership information and together, as $\{\, p, q \,\}$, describe the degree of the membership of $\emptyset$ in τ.

It is easy to reflect ordinary crisp sets within the class of meta sets. Similarly to the definition 1 of a meta set, we define by induction on the $\in$ relation the class of canonical meta sets, which correspond to crisp sets.

Definition 5. *A meta set $\check{\tau}$ is called a canonical meta set, if it is the empty set, or if it has the form:*

$$\check{\tau} = \{\, \langle \check{\sigma}, \mathbb{1} \rangle : \check{\sigma} \text{ is a canonical meta set} \,\} \ .$$

We denote the class of canonical meta sets with the symbol $\mathfrak{M}^{\mathfrak{c}}$. Thus, a canonical meta set is a meta set whose domain includes only canonical meta sets or is empty, and whose range $\mathrm{ran}(\check{\tau}) \subset \{\, \mathbb{1} \,\}$ contains at most one element $\mathbb{1} \in \mathbb{T}$ which is the root of the tree $\mathbb{T}$. We decorate variables corresponding to canonical meta sets with the ˇ (`\check`) accent.

Another very important class of meta sets constitute meta sets which are hereditarily finite sets.

Definition 6. *A meta set τ is a hereditarily finite meta set, if its domain and range are finite sets, and each potential element is also a hereditarily finite meta set.*

We denote the class of hereditarily finite meta sets with the symbol $\mathfrak{MF}$. In other words:

$$\tau \in \mathfrak{MF} \quad \text{if} \quad |\mathrm{dom}(\tau)| < \aleph_0 \wedge |\mathrm{ran}(\tau)| < \aleph_0 \wedge \forall_{\sigma \in \mathrm{dom}(\tau)}\, \sigma \in \mathfrak{MF} \ . \qquad (1.4)$$

1.3.2 Interpretations of Meta Sets

An interpretation of a meta set is a crisp set. It represents some point of view on the meta set. Each meta set may have many different interpretations. In general there may be continuum ($2^{\aleph_0}$) of them. The properties of interpretations imply the properties of the meta set.

An interpretation of a meta set is determined by a branch in the tree $\mathbb{T}$.

Definition 7. *Let τ be a meta set and let $\mathcal{C} \subset \mathbb{T}$ be a branch. The interpretation of the meta set τ, given by the branch $\mathcal{C}$, is the crisp set:*

$$\tau_{\mathcal{C}} = \{\, \sigma_{\mathcal{C}} \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \,\} \ .$$

The process of generating the interpretation of the meta set consists in two stages. In the first stage we remove all the ordered pairs, whose second elements are conditions which do not belong to the branch $\mathcal{C}$. The second stage replaces the remaining pairs with their first elements which are other meta sets. This two-stage process is repeated at all levels of membership hierarchy. As the result we obtain a crisp set.

Let us have a look at some examples. $0 = \emptyset$, $1 = \{ 0 \}$, and $2 = \{ 0, 1 \}$ are initial ordinal numbers. $\check{0} = 0$, $\check{1} = \{ \langle \check{0}, \mathbb{1} \rangle \}$ and $\check{2} = \{ \langle \check{0}, \mathbb{1} \rangle, \langle \check{1}, \mathbb{1} \rangle \}$ are canonical meta sets corresponding to these ordinals. For an arbitrary branch $\mathcal{C} \subset \mathbb{T}$:

$$\begin{aligned}
&\emptyset_{\mathcal{C}} = \emptyset = 0 \ , \\
&\check{1}_{\mathcal{C}} = \{ \langle \emptyset, \mathbb{1} \rangle \}_{\mathcal{C}} = \{ \emptyset \} = 1 \ , \\
&\check{2}_{\mathcal{C}} = \big\{ \langle \emptyset, \mathbb{1} \rangle, \big\langle \{ \langle \emptyset, \mathbb{1} \rangle \}, \mathbb{1} \big\rangle \big\}_{\mathcal{C}} = \{ \emptyset, \{ \emptyset \} \} = \{ 0, 1 \} = 2 \ .
\end{aligned}$$

Indeed, $\mathbb{1} \in \mathcal{C}$ for all $\mathcal{C}$, so interpretations of the given canonical meta set are independent of the chosen branch $\mathcal{C}$. For all branches they are equal crisp sets. Therefore, we may treat them as crisp sets.

Proposition 1. *If $\mathcal{C}'$ and $\mathcal{C}''$ are different branches and $\check{\tau}$ is a canonical meta set, then:*

$$\check{\tau}_{\mathcal{C}'} = \check{\tau}_{\mathcal{C}''} \, .$$

Now, let $p, q \in \mathbb{T}$ and $p \perp q$, for instance: $p = [01]$, $q = [00]$. Further, let

$$\sigma = \{ \langle \check{1}, p \rangle, \langle \check{2}, q \rangle \} \ .$$

If $\mathcal{C}$ is a branch, then we may easily see that:

$$\begin{aligned}
p \in \mathcal{C} \quad &\to \quad \sigma_{\mathcal{C}} = \{ 1 \} \ , && \text{(since } q \notin \mathcal{C}) \\
q \in \mathcal{C} \quad &\to \quad \sigma_{\mathcal{C}} = \{ 2 \} \ , && \text{(since } p \notin \mathcal{C}) \\
p \notin \mathcal{C} \wedge q \notin \mathcal{C} \quad &\to \quad \sigma_{\mathcal{C}} = 0 = \emptyset \ . && \text{(in this case } [1] \in \mathcal{C})
\end{aligned}$$

The above three cases are mutually exclusive, because $p \perp q \perp [1]$, so these conditions cannot lie on the same branch. It turns out that depending on the selected branch $\mathcal{C}$ we obtain different crisp sets as interpretations of the given meta set σ.

1.4 First Order Meta Sets

The first order meta sets constitute a very important subclass of meta sets, especially from the point of view of computer applications. They may be viewed as meta sets whose potential elements are crisp sets. The first order meta sets resemble fuzzy sets, as they represent "fuzzy" collections of "crisp" entities. In this case the membership relation becomes "fuzzy" only on the first level of the membership hierarchy.

1.4.1 Introduction

In general, interpretations of potential elements of meta sets may vary depending on the branch determining the interpretation. Consider for instance $\tau = \{ \langle \emptyset, p \rangle \}$ and $\sigma = \{ \langle \tau, \mathbb{1} \rangle \}$, where $p \neq \mathbb{1}$ is an arbitrary condition. Depending on the branch $\mathcal{C}$, the set $\sigma_{\mathcal{C}}$ may have variable contents. It will always contain a single element, however this element may be different for different branches.

$$\sigma_{\mathcal{C}} = \{ \tau_{\mathcal{C}} \} = \begin{cases} \{ \{ \emptyset \} \} & \text{if } p \in \mathcal{C}, \quad \text{since} \quad \tau_{\mathcal{C}} = \{ \emptyset \} \ , \\ \{ \emptyset \} & \text{if } p \notin \mathcal{C}, \quad \text{since} \quad \tau_{\mathcal{C}} = \emptyset \ . \end{cases}$$

This variability of elements makes analysis of meta sets difficult. Besides, in many circumstances – especially in applications – we would like to have meta sets, whose elements are identical in all interpretations. The first order meta sets satisfy this requirement.

From the above example it is also evident, that our construction does not follow the path of generalising the classical type 1 fuzzy sets to the type 2 fuzzy sets [7]. The meta sets of higher orders are ordinary meta sets, but their "elements" are variable in the manner presented above, i.e. they vary depending on interpretations.

Elements of a first order meta set are ordered pairs of form $\langle \check{\sigma}, p \rangle$. Its first element is a canonical meta set, which assures that elements of interpretations are always the same, independently of the branch determining the interpretation (see proposition 1).

Definition 8. *A meta set is called the first order meta set, when it is empty or it has the form:*

$$\tau^1 = \{ \langle \check{\sigma}, p \rangle : p \in \mathbb{T}, \textit{ and } \check{\sigma} \textit{ is a canonical meta set} \}$$

We denote the class of the first order meta sets with the symbol $\mathfrak{M}^1$. More important is its subclass of hereditarily finite meta sets (which are first order meta sets as well). We denote this class with the symbol $\mathfrak{MF}^1$. Thus:

$$\mathfrak{MF}^1 = \mathfrak{MF} \cap \mathfrak{M}^1 \ . \tag{1.5}$$

The potential elements of the considered here meta sets of the class $\mathfrak{MF}^1$ are canonical meta sets, which are hereditarily finite. We denote the class of such sets with the symbol $\mathfrak{MF}^c$. Thus:

$$\mathfrak{MF}^c = \mathfrak{MF} \cap \mathfrak{M}^c \ . \tag{1.6}$$

We will need some technical definitions to express relations between the meta sets in terms of subsets of the binary tree.

Definition 9. *We say that the set $R \subset \mathbb{T}$ covers $p \in \mathbb{T}$, whenever R contains a finite maximal antichain below p, or it contains a condition weaker than p.*

We use the symbol $R \mid p$ to denote that R covers p. If $R = \emptyset$, then the sentence $R \mid p$ (i.e. $\emptyset \mid p$) is false for each $p \in \mathbb{T}$. Note also that $\{ p \}$ covers p.

Definition 10. *Let Q, R are arbitrary subsets of $\mathbb{T}$. We say that Q and R are equivalent if:*

$$\forall_{q \in Q}\, R \mid q \ \wedge\ \forall_{r \in R}\, Q \mid r\ .$$

We denote the equivalence of the sets Q and R with the symbol $Q \parallel R$. Note that the sentences $Q \parallel \emptyset$ and $\emptyset \parallel R$ are always false for non-empty Q, R (as $\emptyset \mid p$ is false). On the other hand the sentence $\emptyset \parallel \emptyset$ is true.

The equivalence of the sets Q and R means, that if a branch $\mathcal{C}$ in $\mathbb{T}$ contains some condition from Q, then it must also contain a condition from R, and vice versa.

1.4.2 Relations

In this paper we define the membership relation of a hereditarily finite canonical meta set in a first order meta set, and we further focus on the relations and operations for such meta sets. The general definitions and discussion of conditional relations for meta sets, which are based entirely on the interpretation technique, are presented in [3].

Definition 11. *Let $\check{\sigma} \in \mathfrak{MF}^{\mathbf{c}}$, and $\tau \in \mathfrak{MF}^{\mathbf{1}}$. We say that $\check{\sigma}$ is a meta member of τ, if $\tau[\check{\sigma}]$ contains a finite maximal antichain in $\mathbb{T}$.*

We denote the meta membership of $\check{\sigma}$ in τ using the symbol $\check{\sigma} \mathrel{\epsilon} \tau$. In other words $\check{\sigma} \mathrel{\epsilon} \tau$, if each branch $\mathcal{C}$ contains some condition from the image $\tau[\check{\sigma}]$. This guarantees that $\check{\sigma}_{\mathcal{C}}$ is a member of $\tau_{\mathcal{C}}$ for any $\mathcal{C}$.

Definition 12. *Let $\check{\sigma} \in \mathfrak{MF}^{\mathbf{c}}$, $\tau \in \mathfrak{MF}^{\mathbf{1}}$, and $p \in \mathbb{T}$. We say, that $\check{\sigma}$ is a meta member of τ under the condition p ($\check{\sigma} \mathrel{\epsilon_p} \tau$), if $\tau[\check{\sigma}]$ covers p.*

Thus $\check{\sigma} \mathrel{\epsilon_p} \tau \ \leftrightarrow\ \tau[\check{\sigma}] \mid p$. The conditional membership is meant to describe the partial membership of an element to a set. The condition p measures the degree of the membership. The stronger condition, the weaker membership. On the other hand, the weakest condition $\mathbb{1}$ describes the full (unconditional) membership, i.e. $\check{\sigma} \mathrel{\epsilon_{\mathbb{1}}} \tau$ is equivalent to $\check{\sigma} \mathrel{\epsilon} \tau$.

Definition 13. *Let $\tau, \sigma \in \mathfrak{MF}^{\mathbf{1}}$. We say that τ is a meta subset of σ ($\tau \Subset \sigma$), if:*

$$\forall_{\check{\eta} \in \mathrm{dom}(\tau)} \forall_{q \in \tau[\check{\eta}]}\ \sigma[\check{\eta}] \mid q\ .$$

In other words $\tau \Subset \sigma$, if $\forall_{\check{\eta} \in \mathrm{dom}(\tau)} \forall_{q \in \tau[\check{\eta}]}\ \check{\eta} \mathrel{\epsilon_q} \sigma$. The definition says, that τ is a meta subset of σ, whenever for each potential element $\check{\eta}$ of τ, and for each condition q from the image $\tau[\check{\eta}]$, the image of σ at $\check{\eta}$ covers the condition q. It means that, $\sigma[\check{\eta}]$ contains a finite maximal antichain below q, or it contains a condition weaker than q.

Proposition 2. *Let* $\tau, \sigma \in \mathfrak{MF}^1$. *If* $\tau \mathrel{\tilde{\in}} \sigma$, *then* $\mathrm{dom}(\tau) \subset \mathrm{dom}(\sigma)$.

Proof. Directly from the definition. If $\check{\eta} \in \mathrm{dom}(\tau)$ and $q \in \tau[\check{\eta}]$, then $\sigma[\check{\eta}] \neq \emptyset$ must be true for $\sigma[\check{\eta}] \mid q$ to be true. Therefore, $\check{\eta} \in \mathrm{dom}(\sigma)$.

Definition 14. *Let* $\tau, \sigma \in \mathfrak{MF}^1$. *We say that* τ *is meta equal to* σ *(*$\tau \approx \sigma$*), whenever:*

$$\forall_{\check{\mu} \in \mathrm{dom}(\tau) \cup \mathrm{dom}(\sigma)}\ \tau[\check{\mu}] \parallel \sigma[\check{\mu}]\ .$$

It is possible to similarly define conditional versions of other relations for the first order hereditarily finite meta sets too. They reflect relations that are satisfied to some degree, other than certainty.

The presented here definitions of relations for $\mathfrak{MF}^1$ meta sets, as well as their conditional versions, are equivalent [4] to definitions for the general case, developed using interpretations.

1.4.3 Algebraic Operations

In this section we define basic algebraic operations like the sum, the intersection and the difference for the first order hereditarily finite meta sets,

Definition 15. *Let* $\tau,\ \eta \in \mathfrak{MF}^1$. *The meta sum of* τ *and* η*, denoted with the symbol* $\tilde{\cup}$*, is their set-theoretic sum:*

$$\tau \mathbin{\tilde{\cup}} \eta \ = \ \tau \cup \eta\ .$$

The following important facts are obvious, so they do not require proofs.

Lemma 1. $\tau, \eta \in \mathfrak{MF}^1 \ \to \ \tau \mathbin{\tilde{\cup}} \eta \in \mathfrak{MF}^1$.

Proposition 3. *If* $\tau, \eta \in \mathfrak{MF}^1$, *then* $\mathrm{dom}(\tau \mathbin{\tilde{\cup}} \eta) \ = \ \mathrm{dom}(\tau) \cup \mathrm{dom}(\eta)$.

The intersection of two meta sets is not so easy to define as the meta sum was. We will need some additional notions.

Definition 16. *Let* $P, Q \subset \mathbb{T}$ *are arbitrary subsets of the tree* $\mathbb{T}$. *The half convolution of the set* P *below* Q *is the set:*

$$P \lhd Q \ = \ \{\, p \in P \colon \exists_{q \in Q}\, q \geq p \,\}\ .$$

The half convolution of the set P *over* Q *is the set:*

$$P \rhd Q \ = \ Q \lhd P \ = \ \{\, q \in Q \colon \exists_{p \in P}\, p \geq q \,\}\ .$$

It is easy to see, that $P \lhd Q \subset P$. If $P = \emptyset$ or $Q = \emptyset$, then $P \lhd Q = \emptyset$. If $r \in P \lhd Q$ and $\mathcal{C} \subset \mathbb{T}$ is a branch containing r, then $\mathcal{C} \cap Q \neq \emptyset$, i.e. the branch $\mathcal{C}$ contains some element of Q too. This explains the meaning of the half convolution. Thus, the following implication holds for any branch $\mathcal{C}$:

$$\mathcal{C} \cap (P \lhd Q) \neq \emptyset \quad \to \quad \mathcal{C} \cap Q \neq \emptyset \wedge \mathcal{C} \cap P \neq \emptyset\ . \tag{1.7}$$

Definition 17. *Let $P, Q \subset \mathbb{T}$ are arbitrary subsets of the tree $\mathbb{T}$. The convolution of the sets P and Q is the set:*

$$P \diamond Q \ = \ (P \lhd Q) \cup (P \rhd Q) \ .$$

Directly from the definition we obtain:

$$P \diamond Q \ = \ \{ \, p \in P \colon \exists_{q \in Q} \, q \geq p \, \} \cup \{ \, q \in Q \colon \exists_{p \in P} \, p \geq q \, \} \ . \qquad (1.8)$$

If any of the sets P, Q is empty, then their convolution is empty too.

Let $r \in P \diamond Q$ and at the same time $r \in \mathcal{C}$, for some branch $\mathcal{C}$. If $r \in P$, then $\mathcal{C} \cap Q \neq \emptyset$, and conversely: if $r \in Q$, then $\mathcal{C} \cap P \neq \emptyset$.

We assume that the convolution and the half convolution operators have the same priority: higher than the sum and lower than the intersection. This is illustrated by the following equality.

$$P \cup Q \diamond R \cap S \ = \ P \cup (Q \diamond (R \cap S)) \ . \qquad (1.9)$$

Anyway, we will avoid ambiguous notation.

We will need the convolution to define the intersection of the hereditarily finite first order meta sets.

Definition 18. *Let $\tau, \eta \in \mathfrak{MF}^1$. The meta intersection of τ and η is the meta set:*

$$\tau \mathbin{\tilde{\cap}} \eta \ = \ \{ \, \langle \xi, p \rangle : \xi \in \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta) \wedge p \in \tau[\xi] \diamond \eta[\xi] \, \} \ .$$

The potential elements of the intersection $\tau \mathbin{\tilde{\cap}} \eta$ might be – but do not necessarily have to be – only those meta sets, which are simultaneously the potential elements of τ and η. In particular there may exist $\xi \in \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta)$ such, that $\tau[\xi] \diamond \eta[\xi] = \emptyset$, and then ξ is not a potential element of the intersection, as $\xi \notin \mathrm{dom}(\tau \mathbin{\tilde{\cap}} \eta)$. The degree of membership of a potential element to the intersection is determined by the degree of its membership to both arguments. Thus, directly from the definition we obtain:

Proposition 4. *If $\tau, \eta \in \mathfrak{MF}^1$, then* $\mathrm{dom}(\tau \mathbin{\tilde{\cap}} \eta) \subset \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta)$.

For the image of the intersection we have $\mathrm{ran}(\xi \mathbin{\tilde{\cap}} \mu) \subset \mathrm{ran}(\xi) \cup \mathrm{ran}(\mu)$, because for $\eta \in \mathrm{dom}(\xi \mathbin{\tilde{\cap}} \mu)$ holds $(\xi \mathbin{\tilde{\cap}} \mu)[\eta] = \xi[\eta] \diamond \mu[\eta] \subset \xi[\eta] \cup \mu[\eta]$. This implies the following property.

Lemma 2. $\tau, \eta \in \mathfrak{MF}^1 \ \rightarrow \ \tau \mathbin{\tilde{\cap}} \eta \in \mathfrak{MF}^1$.

The definition of the difference of meta sets is much more complex, than the definitions of sum and intersection. Contrary to the definition of the difference of crisp sets, in the case of meta sets the difference of τ and η contains not only those “elements” from τ, which are not “members” of η, but also such “elements”, that somehow occur in τ as well as in η. In particular, if τ “contains more” σ than η does, then the difference of τ and η should contain some quantity of σ. To express these subtleties we will need additional notions.

We start with introducing some usefull notation. Let $P \subset \mathbb{T}$ be a set of conditions from the tree $\mathbb{T}$. By $P^\top$ we understand the set of conditions comparable to elements of P:

$$P^\top = \{ q \in \mathbb{T} \colon \exists_{p \in P}\, p \top q \} \ . \tag{1.10}$$

Similarly, by $P^\perp$ we understand the set of conditions incomparable to any element of P:

$$P^\perp = \{ q \in \mathbb{T} \colon \forall_{p \in P}\, p \perp q \} \ . \tag{1.11}$$

Elements of $P^\top$ lie on branches determined by the elements of P. No element of the set $P^\perp$ lies on the same branch with any element of P. If $P = \emptyset$, then $P^\top = \emptyset$ and $P^\perp = \mathbb{T}$. It should also be clear that $P \subset P^\top$. On the other hand, if $\mathbb{1} \in P$, then $P^\top = \mathbb{T}$ and $P^\perp = \emptyset$. However, if $p \neq \mathbb{1}$, then $\{ p \}^\top$ consists of the subtree with the root p and a branch containing p. Moreover:

Proposition 5. *Let $P \subset \mathbb{T}$.*

$$P^\top \cup P^\perp = \mathbb{T} \ ,$$
$$P^\top \cap P^\perp = \emptyset \ .$$

Let $P = \{ [11] \}$. $P^\top$ consists of the subtree with the root [11] plus the element [1] plus the root $\mathbb{1}$. $P^\perp$ contains two subtrees with the roots [0] and [10], i.e. it contains conditions stronger than [0] and [10]. Note, that the conditions [0], [10] and [11] constitute a final maximal antichain.

Let $P \subset \mathbb{T}$ be a set of conditions. By $\max(P)$ we denote the set of maximal elements in P. Thus

$$p \in \max(P) \quad \text{if, and only if} \quad p \in P \wedge \forall_{q \in P}\ (q \geq p \rightarrow q = p) \ . \tag{1.12}$$

We see that $\max(\emptyset) = \emptyset$ and $\max(\mathbb{T}) = \{ \mathbb{1} \}$. An important property of the set $\max(P)$ is, that each element of P is comparable to some element of $\max(P)$. Moreover, each such element is stronger than its counterpart from $\max(P)$. In the above example $\max(P^\perp) = \{ [0], [10] \}$.

Proposition 6. *Let $P \subset \mathbb{T}$. The set $\max(P)$ of maximal elements in P is a maximal antichain in P.*

Proof. Elements of the set $\max(P)$ are pairwise incomparable, so it is an antichain. Moreover, each element of P is comparable to some element of $\max(P)$, so it is the maximal antichain in P.

Lemma 3. *If P is a finite subset of $\mathbb{T}$, then the set $\max(P^\perp)$ is a maximal finite antichain in $P^\perp$.*

Proof. If $\mathbb{1} \in P$, then $P^\perp = \emptyset$ and $\max(P^\perp) = \emptyset$, so obviously $\max(P^\perp)$ is a maximal finite antichain in $P^\perp$. Further we assume, that $\mathbb{1} \notin P$.

The fact, that $\max(P^\perp)$ is a maximal antichain in $P^\perp$ follows from the proposition 6. We show, that if P is finite, then $\max(P^\perp)$ is finite too.

Denote the set of conditions stronger than the given $p \in \mathbb{T}$ with the symbol $p^{\leq}$. In other words it is the subtree rooted at p: $p^{\leq} = \{\, q \in \mathbb{T} \colon q \leq p \,\}$. Note, that $\max(p^{\leq}) = \{\, p \,\}$, as well as:

$$\max(q^{\leq} \cup r^{\leq}) = \begin{cases} \{\, \max(q, r) \,\} & \text{if } \; q \top r \; , \\ \{\, q, r \,\} & \text{if } \; q \perp r \; . \end{cases}$$

The above formula may be generalised to an arbitrary number of operands.

For a condition $s \neq \mathbb{1}$, the set of conditions incomparable to s, i.e. $\{\, s \,\}^\perp$, is a finite sum of subtrees: $\{\, s \,\}^\perp = s_1^{\leq} \cup \ldots \cup s_n^{\leq}$, where n is the number of the tree level containing s, and s_i is a condition from the level i. For instance, the s_n is the only sibling of s, and the s_{n-1} is the sibling of the parent of s and s_n (if it exists, i.e. when s is not a direct descendant of the root). Thus, applying the above formula we see that:

$$\max(\{\, s \,\}^\perp) = \max(s_1^{\leq} \cup \ldots \cup s_n^{\leq}) \subset \{\, s_1, \ldots, s_n \,\}$$

is a finite set. Further, note that for $Q, R \subset \mathbb{T}$ holds $(Q \cup R)^\perp = Q^\perp \cap R^\perp$. If so, then for $P = \{\, p_1, \ldots, p_m \,\}$ we obtain:

$$P^\perp = \{\, p_1, \ldots, p_m \,\}^\perp = \{\, p_1 \,\}^\perp \cap \ldots \cap \{\, p_m \,\}^\perp = \bigcap_{i=1}^{m} \{\, p_i \,\}^\perp \; .$$

By substituting consecutive $\{\, p_i \,\}^\perp$ with sums we obtain:

$$P^\perp = \bigcap_{i=1}^{m} \{\, p_i \,\}^\perp = \bigcap_{i=1}^{m} \bigcup_{j=1}^{n_i} p_{ij}^{\leq} \; .$$

By multiplying the appropriate sums we obtain the equality:

$$P^\perp = \bigcup_{i=1}^{k} \bigcap_{j=1}^{m} p_{ij}^{\leq}$$

for some k ($k = n_1 \cdot \ldots \cdot n_m$). Taking into account the fact, that:

$$q^{\leq} \cap r^{\leq} = \begin{cases} \emptyset & \text{if } \; q \perp r \; , \\ q^{\leq} & \text{if } \; q \leq r \; , \\ r^{\leq} & \text{if } \; q \geq r \end{cases}$$

we have:

$$P^{\perp} = \bigcup_{i=1}^{k} \bigcap_{j=1}^{m} p_{ij}^{\leq} = \bigcup_{i=1}^{k} P_i \,, \quad \text{where} \quad P_i = \begin{cases} p_{ij_i}^{\leq} & \text{for} \quad \bigcap_{j=1}^{m} p_{ij}^{\leq} = p_{ij_i}^{\leq} \,, \\ \emptyset & \text{for} \quad \bigcap_{j=1}^{m} p_{ij}^{\leq} = \emptyset \,. \end{cases}$$

Thus, $\max(P^{\perp})$ is a finite set, because $\max(P^{\perp}) \subset \{ p_1, \ldots, p_k \}$, where each $p_i = p_{ij_i}$ from the above formula, for some j_i, in the cases, when the intersections are not empty.

Note, that for $P \neq \emptyset$, the set $\max(P^{\top})$ is always finite, as it contains the single element: $\mathbb{1}$. In the general case, for an arbitrary P, the set $\max(P)$ may be infinite. Consider for example the infinite antichain: $P = \{ [0], [10], [110], [1110], \ldots \}$. Clearly, $\max(P) = P$.

We now introduce the definition of the boundary. It represents a "complement" of σ to τ in the case when their domains are equal.

Definition 19. *Let $\tau, \eta \in \mathfrak{MF}^{\mathbf{1}}$. The boundary of the meta set η in the meta set τ is the set:*

$$\tilde{\eta}^{\tau} = \left\{ \langle \xi, p \rangle : \xi \in \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta) \wedge p \in \tau[\xi] \cap \eta[\xi]^{\perp} \cup \max(\eta[\xi]^{\perp}) \lhd \tau[\xi] \right\} \,.$$

If $\mathrm{dom}(\tau) \cap \mathrm{dom}(\eta) = \emptyset$, then, of course, $\tilde{\eta}^{\tau} = \emptyset$. If $\xi \in \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta)$, then $\tilde{\eta}^{\tau}[\xi]$ consists of:

- those conditions from $\tau[\xi]$ which are incomparable to conditions from $\eta[\xi]$ (i.e. elements of $\tau[\xi] \cap \eta[\xi]^{\perp}$), and
- those maximal elements in $\eta[\xi]^{\perp}$, which have some weaker condition from $\tau[\xi]$ above. In other words, they are conditions incomparable to conditions from $\eta[\xi]$, for which there exists no weaker condition incomparable to any element of $\eta[\xi]$, but there exists a weaker condition from $\tau[\xi]$.

As an example explaining the above definition let us consider meta sets $\tau = \{ \langle \check{\sigma}, p \rangle \}$ and $\eta = \{ \langle \check{\sigma}, q \rangle \}$ for some canonical $\check{\sigma} \in \mathfrak{MF}^{\mathbf{1}}$, and conditions $p = [1]$ and $q = [11]$. The meta set $\check{\sigma}$ belongs to τ "to a higher degree" than to η, as $\check{\sigma} \ \epsilon_p \ \tau$, $\check{\sigma} \ \epsilon_q \ \eta$ and $q \leq p$. In other words, for each branch such that $\check{\sigma}_{\mathcal{C}} \in \eta_{\mathcal{C}}$ we also have $\check{\sigma}_{\mathcal{C}} \in \tau_{\mathcal{C}}$. We want to define the boundary of η in τ in such a manner, that it will be not empty in this case, and will behave like the set-theoretic difference of τ and η in interpretations. To be more precise: for $\mathcal{C}$ such, that $\check{\sigma}_{\mathcal{C}} \in \eta_{\mathcal{C}}$ and $\check{\sigma}_{\mathcal{C}} \in \tau_{\mathcal{C}}$ hold (i.e. $[11] \in \mathcal{C}$), or $\check{\sigma}_{\mathcal{C}} \notin \eta_{\mathcal{C}}$ and $\check{\sigma}_{\mathcal{C}} \notin \tau_{\mathcal{C}}$ hold (in this case $[0] \in \mathcal{C}$), the interpretations determined by $\mathcal{C}$, of the boundary $\tilde{\eta}^{\tau}$ should be the empty set. On the other hand, for $\mathcal{C}$ such, that $\check{\sigma}_{\mathcal{C}} \notin \eta_{\mathcal{C}}$ and $\check{\sigma}_{\mathcal{C}} \in \tau_{\mathcal{C}}$, ($[10] \in \mathcal{C}$) any interpretation of the boundary should contain $\check{\sigma}_{\mathcal{C}}$. But this precisely means that $\tilde{\eta}^{\tau}[\check{\sigma}]$ contains a maximal finite antichain below $[10]$. This is how we define the boundary of η in τ. Indeed, $\tau[\check{\sigma}] \cap \eta[\check{\sigma}]^{\perp} = \{ [1] \} \cap \{ [11] \}^{\perp} = \emptyset$, as $\{ [11] \}^{\perp}$ is the sum of two subtrees rooted at $[0]$ and $[10]$. In this case $\tau[\check{\sigma}] \subset \eta[\check{\sigma}]^{\top}$, because p is comparable to q. On the other hand $\max(\eta[\check{\sigma}]^{\perp}) = \{ [0], [10] \}$. But only the element $r = [10]$ has an element from $\tau[\check{\sigma}]$ above it (it is $p = [1]$, of course). Thus, the half convolution $\max(\eta[\check{\sigma}]^\perp) \lhd \tau[\check{\sigma}]$ contains only the condition r, and finally $\tilde{\eta}^\tau = \{ \langle \check{\sigma}, r \rangle \}$. Now, if $\mathcal{C}$ is a branch such, that $r = [10] \in \mathcal{C}$, then $\tilde{\eta}^\tau_\mathcal{C} = \{ \check{\sigma}_\mathcal{C} \}$, $\tau_\mathcal{C} = \{ \check{\sigma}_\mathcal{C} \}$ and $\eta_\mathcal{C} = \emptyset$. If $[0] \in \mathcal{C}$, then $\tilde{\eta}^\tau_\mathcal{C} = \tau_\mathcal{C} = \eta_\mathcal{C} = \emptyset$. If $[11] \in \mathcal{C}$, then $\tilde{\eta}^\tau_\mathcal{C} = \emptyset$ and $\tau_\mathcal{C} = \eta_\mathcal{C} = \{ \check{\sigma}_\mathcal{C} \}$. Therefore, $\tilde{\eta}^\tau_\mathcal{C} = \tau_\mathcal{C} \setminus \sigma_\mathcal{C}$ for all $\mathcal{C}$.

We see, that the boundary of η in τ behaves like the difference of τ and η in the case, when their domains are equal. We consider the general case further. Prior to this we state two important properties of the boundary.

Proposition 7. *If* $\tau, \eta \in \mathfrak{MF}^\mathbf{1}$, *then* $\mathrm{dom}(\tilde{\eta}^\tau) \subset \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta)$.

The above proposition follows directly from the definition. It is worth noting that we can't have equality here instead of inclusion, as for some $\xi \in \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta)$ there may occur simultaneously $\tau[\xi] \cap \eta[\xi]^\perp = \emptyset$ and $\max(\eta[\xi]^\perp) \lhd \tau[\xi] = \emptyset$. In such a case $\xi \notin \mathrm{dom}(\tilde{\eta}^\tau)$.

If $\mathrm{dom}(\tau) = \mathrm{dom}(\eta)$, then it is possible that $\tilde{\eta}^\tau = \emptyset$ (e.g. $\tilde{\tau}^\tau = \emptyset$), but it is also possible that $\tilde{\eta}^\tau = \tau$. Consider for example $\tau = \{ \langle \check{\sigma}, [0] \rangle \}$ and $\eta = \{ \langle \check{\sigma}, [1] \rangle \}$ for some canonical $\check{\sigma}$. We have $\mathrm{dom}(\tau) = \mathrm{dom}(\eta) = \{ \check{\sigma} \}$, and $\tau[\check{\sigma}] = \{ [0] \}$ and $\eta[\check{\sigma}] = \{ [1] \}$. It is easy to see that $\eta[\check{\sigma}]^\top$ contains the root $\mathbb{1}$ and the subtree rooted at [1], whereas $\eta[\check{\sigma}]^\perp$ consists of the subtree rooted at [0]. Therefore $\tau[\check{\sigma}] \cap \eta[\check{\sigma}]^\perp = \{ [0] \}$. In this case also $\max(\eta[\check{\sigma}]^\perp) = \{ [0] \}$, so $\max(\eta[\check{\sigma}]^\perp) \lhd \tau[\check{\sigma}] = \{ [0] \}$. This implies $\tilde{\eta}^\tau = \tau$, because $\tilde{\eta}^\tau[\check{\sigma}] = \{ [0] \} = \tau[\check{\sigma}]$. It is never possible that $\tilde{\eta}^\tau = \eta$, as for $\xi \in \mathrm{dom}(\tilde{\eta}^\tau) \cap \mathrm{dom}(\eta)$ there is always $\tilde{\eta}^\tau[\xi] \cap \eta[\xi] = \emptyset$, because $\tilde{\eta}^\tau[\xi] \subset \eta[\xi]^\perp$.

Lemma 4. *If* $\tau, \eta \in \mathfrak{MF}^\mathbf{1}$, *then* $\tilde{\eta}^\tau \in \mathfrak{MF}^\mathbf{1}$.

Proof. It is enough to show that for $\xi \in \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta)$ the sets $\tau[\xi] \cap \eta[\xi]^\perp$ and $\max(\eta[\xi]^\perp) \lhd \tau[\xi]$ are finite. The finiteness of the former one is implied by the assumption, as $\tau[\xi] \cap \eta[\xi]^\perp \subset \tau[\xi]$, and $\tau \in \mathfrak{MF}^\mathbf{1}$. The finiteness of the latter set follows from the fact that $\max(\eta[\xi]^\perp) \lhd \tau[\xi]$ is included in $\max(\eta[\xi]^\perp)$, which is finite by the lemma 3 and by the assumption that $\eta \in \mathfrak{MF}^\mathbf{1}$.

If $\mathrm{dom}(\tau) \subset \mathrm{dom}(\eta)$, then the boundary $\tilde{\eta}^\tau$ is the meta difference of τ and η. In the general case we must add something to $\tilde{\eta}^\tau$ to obtain their meta difference.

Definition 20. *Let* $\tau, \eta \in \mathfrak{MF}^\mathbf{1}$. *The difference of the meta sets* τ *and* η *is the meta set:*

$$\tau \mathbin{\tilde{\setminus}} \eta \;=\; \tau\restriction_{\mathrm{dom}(\tau) \setminus \mathrm{dom}(\eta)} \cup \tilde{\eta}^\tau \;.$$

The expression $\tau\restriction_{\mathrm{dom}(\tau) \setminus \mathrm{dom}(\eta)}$ denotes the restriction of the domain of the relation τ (a meta set is a relation) to the set $\mathrm{dom}(\tau) \setminus \mathrm{dom}(\eta)$, i.e. $\mathrm{dom}(\tau\restriction_{\mathrm{dom}(\tau) \setminus \mathrm{dom}(\eta)}) = \mathrm{dom}(\tau) \setminus \mathrm{dom}(\eta)$.

Let us have a look at the above definition. If $\mathrm{dom}(\tau) \cap \mathrm{dom}(\eta) = \emptyset$, then $\tau \mathbin{\tilde{\setminus}} \eta = \tau$. It is clear that $\tau \mathbin{\tilde{\setminus}} \tau = \emptyset$. Indeed, $\tau\restriction_{\mathrm{dom}(\tau) \setminus \mathrm{dom}(\tau)} = \emptyset$ and $\tilde{\tau}^\tau = \emptyset$. If $\mathrm{dom}(\tau) = \mathrm{dom}(\eta)$, then the first operand to the sum is empty and then $\tau \mathbin{\tilde{\setminus}} \eta = \tilde{\eta}^\tau$. In such a case it is possible that $\tau \mathbin{\tilde{\setminus}} \eta = \emptyset$ even if $\tau \neq \eta$.

Let $\sigma = \tau \mathbin{\tilde{\setminus}} \eta$. If $\xi \notin \mathrm{dom}(\tau)$, then $\xi \notin \mathrm{dom}(\sigma)$ independently of the fact that $\xi \in \mathrm{dom}(\eta)$ holds or not. If $\xi \in \mathrm{dom}(\tau)$ and $\xi \notin \mathrm{dom}(\eta)$, then $\xi \in \mathrm{dom}(\sigma)$ always holds. If $\xi \in \mathrm{dom}(\tau) \cap \mathrm{dom}(\eta)$, then $\xi \in \mathrm{dom}(\sigma)$, whenever $\xi \in \mathrm{dom}(\tilde{\eta}^{\tau})$, i.e. at least one of the sets $\tau[\xi] \cap \eta[\xi]^{\perp}$, $\max(\eta[\xi]^{\perp}) \lhd \tau[\xi]$ is not empty. The above, together with the proposition 7 imply:

Proposition 8. *If* $\tau, \eta \in \mathfrak{MF}^{\mathbf{1}}$*, then* $\mathrm{dom}(\tau \mathbin{\tilde{\setminus}} \eta) \subset \mathrm{dom}(\tau)$.

The lemma 4 implies the following important property.

Lemma 5. $\tau, \eta \in \mathfrak{MF}^{\mathbf{1}} \;\rightarrow\; \tau \mathbin{\tilde{\setminus}} \eta \in \mathfrak{MF}^{\mathbf{1}}$.

1.5 The Boolean Algebra of Meta Sets

In this section we will prove that algebraic operations for meta sets satisfy the axioms of Boolean algebra.

Note, that by lemmas 1, 2 and 5, for the given first order hereditarily finite meta sets $\tau, \eta \in \mathfrak{MF}^{\mathbf{1}}$, the results of operations $\tau \mathbin{\tilde{\cup}} \eta$, $\tau \mathbin{\tilde{\cap}} \eta$ and $\tau \mathbin{\tilde{\setminus}} \eta$ are also first order hereditarily finite meta sets.

1.5.1 Some Properties of the Convolution

We start with some technical lemmas. First, note that the convolution operation is commutative.

Proposition 9. *If* $P, Q \subset \mathbb{T}$*, then* $P \diamond Q \;=\; Q \diamond P$.

The obvious proof follows directly from the definition 17. The half convolution and the convolution are distributive over the sum, what proves the next proposition.

Proposition 10. *Let* $P, Q, S \subset \mathbb{T}$*. The following equalities hold:*

$$P \lhd (Q \cup S) \;=\; P \lhd Q \cup P \lhd S\,, \tag{1.13}$$

$$P \rhd (Q \cup S) \;=\; P \rhd Q \cup P \rhd S\,, \tag{1.14}$$

$$P \diamond (Q \cup S) \;=\; P \diamond Q \cup P \diamond S\,. \tag{1.15}$$

Proof. To begin with, note, that if $P = \emptyset$, then $P \diamond (Q \cup S) = \emptyset$, as well as $P \diamond Q = P \diamond S = \emptyset$. If $Q = \emptyset$, then $P \diamond Q = \emptyset$ and the first equality is satisfied. Similarly for $S = \emptyset$. The same rule applies for the operators $\lhd$ and $\rhd$. Thus, we may assume that all the sets P, Q, S are not empty.

To prove (1.13) pick up $s \in P \lhd (Q \cup S)$. By the definition $s \in P$ and there exists $t \geq s$ such, that $t \in Q \cup S$. If $t \in Q$, then $s \in P \lhd Q$, and if $t \in S$, then $s \in P \lhd S$. Therefore, $P \lhd (Q \cup S) \subset P \lhd Q \cup P \lhd S$. On the other hand, if $t \in P \lhd Q$, then, of course, $P \lhd (Q \cup S)$, and similarly for $P \lhd S$. This way we obtain $P \lhd Q \cup P \lhd S \;\subset\; P \lhd (Q \cup S)$.

Analogously we prove the second equality (1.14). To prove the third one we display the convolution (applying the definition) as the sum of half convolutions, assuming the following notation:

$$\overbrace{P \diamond (Q \cup S)}^{L} \; = \; \overbrace{P \lhd (Q \cup S)}^{L_L} \cup \overbrace{P \rhd (Q \cup S)}^{L_R} \,,$$

$$\overbrace{P \lhd Q}^{R_{LL}} \cup \overbrace{P \rhd Q}^{R_{LR}} \cup \overbrace{P \lhd S}^{R_{RL}} \cup \overbrace{P \rhd S}^{R_{RR}} \; = \; \overbrace{P \diamond Q}^{R_L} \cup \overbrace{P \diamond S}^{R_R} \;.$$

We must show that $L = R_L \cup R_R$, i.e.:

$$L_L \cup L_R \; = \; R_{LL} \cup R_{LR} \cup R_{RL} \cup R_{RR} \;.$$

We obtain this equality by adding both sides of equalities (1.13) and (1.14).

The convolution is associative, what will be shown in the lemma 6. We will need the following properties of convolution and half convolution to prove it.

Proposition 11. *For arbitrary $P, Q, R \subset \mathbb{T}$:*

$$(P \lhd Q) \lhd R \; = \; (P \lhd R) \lhd Q \;.$$

Proof. If any of the sets P, Q, R is empty, then the left hand side and the right hand side of the equality is also the empty set. Therefore, we assume that P, Q, R are not empty.

If $p \in (P \lhd Q) \lhd R$, then $p \in P$, as well as $\exists_{q \in Q}\, q \geq p$ and $\exists_{r \in R}\, r \geq p$. The fact, that $p \in (P \lhd R) \lhd Q$ also means that $p \in P$, as well as $\exists_{r \in R}\, r \geq p$ and $\exists_{q \in Q}\, q \geq p$. Thus, the left hand side and the right hand side of the equality represent the same subset of P.

Proposition 12. *For arbitrary $P, Q, R \subset \mathbb{T}$:*

$$(P \lhd Q) \lhd R \; = \; P \lhd (Q \diamond R) \,.$$

Proof. Similarly as before we may assume, that P, Q, R are not empty.

If $p \in (P \lhd Q) \lhd R$, then $p \in P$ and $\exists_{q \in Q}\, q \geq p$ and $\exists_{r \in R}\, r \geq p$. Two cases are possible: $p \leq q \leq r$ and $p \leq r \leq q$. In the first case we have $p \in P \lhd (Q \lhd R) \subset P \lhd (Q \diamond R)$. Similarly, in the second case holds $p \in P \lhd (R \lhd Q) \subset P \lhd (Q \diamond R)$.

If $p \in P \lhd (Q \diamond R)$, then $p \in P$, and there exists $s \in Q \diamond R$ such, that $p \leq s$. On the other hand, $s \in Q \diamond R$ means, that $\exists_{q \in Q}\, s = q$ or $\exists_{r \in R}\, s = r$. The first case implies the existence of $r \in R$ such, that $s = q \leq r$, so we have $p \leq q \leq r$. In the second case $\exists_{q \in Q}\, s = r \leq q$ and $p \leq r \leq q$ holds. The first part of the proof implies that in both cases $p \in (P \lhd Q) \lhd R$.

Lemma 6. *The convolution is associative, i.e. for any $P, Q, S \subset \mathbb{T}$ the following equality holds:*

$$(P \diamond Q) \diamond S \; = \; P \diamond (Q \diamond S) \;.$$

Proof. If any of the sets P, Q, S is empty, then both sides of the equality represent the empty set, so we further assume $P, Q, S \neq \emptyset$.

Let us display the convolution as the sum of half convolutions, assuming the following notation (we apply the proposition 10):

$$\begin{aligned}
\overbrace{(P \diamond Q) \diamond S}^{L} &= \overbrace{(P \lhd Q \cup P \rhd Q) \lhd S}^{L_L} \cup \overbrace{(P \diamond Q) \rhd S}^{L_R} , \\
&= \overbrace{(P \lhd Q) \lhd S}^{L_{LL}} \cup \overbrace{(Q \lhd P) \lhd S}^{L_{LR}} \cup \overbrace{S \lhd (P \diamond Q)}^{L_{RR}} , \\
\overbrace{P \diamond (Q \diamond S)}^{R} &= \overbrace{P \lhd (Q \diamond S)}^{R_L} \cup \overbrace{P \rhd (Q \lhd S \cup Q \rhd S)}^{R_R} , \\
&= \overbrace{P \lhd (Q \diamond S)}^{R_L} \cup \overbrace{(Q \lhd S) \lhd P}^{R_{RL}} \cup \overbrace{(S \lhd Q) \lhd P}^{R_{RR}} .
\end{aligned}$$

By the proposition 12, we have $L_{LL} = R_L$. Further, the proposition 11 gives us $L_{LR} = R_{RL}$. Combining the propositions 9 and 12 obtain we $L_{RR} = R_{RR}$.

1.5.2 The Field of Meta Sets

Analogously to the field of sets in the crisp set theory we define the field of meta sets. This structure will form the basis for the Boolean algebra of meta sets.

Definition 21. *Let $\delta \in \mathfrak{MF}^1$ be a non-empty meta set and let $\mathcal{D} \subset \mathfrak{MF}^1$ be a non-empty family of meta subsets of δ (i.e. $\lambda \in \mathcal{D} \to \lambda \mathrel{\tilde{\subseteq}} \delta$). The family $\mathcal{D}$ is called the field of meta sets on δ, when the following axioms are satisfied:*

$$\lambda \in \mathcal{D} \ \to \ \delta \mathrel{\tilde{\setminus}} \lambda \in \mathcal{D} \, , \tag{1.16}$$
$$\lambda \in \mathcal{D} \wedge \rho \in \mathcal{D} \ \to \ \lambda \mathbin{\tilde{\cup}} \rho \in \mathcal{D} \, , \tag{1.17}$$
$$\lambda \in \mathcal{D} \wedge \rho \in \mathcal{D} \ \to \ \lambda \mathbin{\tilde{\cap}} \rho \in \mathcal{D} \, . \tag{1.18}$$

Usually, the definition of the field of sets involves only the first axiom together with one of the second or the third, as another is implied by de Morgan's laws. In the world of meta sets these laws do not hold with the strict equality, however they do hold with the meta equality:[3]

$$\delta \mathrel{\tilde{\setminus}} (\alpha \mathbin{\tilde{\cup}} \beta) \ \approx \ (\delta \mathrel{\tilde{\setminus}} \alpha) \mathbin{\tilde{\cap}} (\delta \mathrel{\tilde{\setminus}} \beta) \ , \tag{1.19}$$
$$\delta \mathrel{\tilde{\setminus}} (\alpha \mathbin{\tilde{\cap}} \beta) \ \approx \ (\delta \mathrel{\tilde{\setminus}} \alpha) \mathbin{\tilde{\cup}} (\delta \mathrel{\tilde{\setminus}} \beta) \ . \tag{1.20}$$

As this is not enough to make the axioms 1.17 and 1.18 equivalent, we need both in the definition.

We now prove two simple and well known properties of algebraic operations for crisp sets in the case of meta sets.

[3] It will follow from the theorem 1.

Lemma 7. *If $\alpha, \delta \in \mathfrak{MF}^1$ and $\alpha \Subset \delta$, then $\alpha \mathbin{\tilde{\cap}} (\delta \mathbin{\tilde{\setminus}} \alpha) = \emptyset$.*

Proof. Because $\alpha \Subset \delta$, then from the propositions 2, 4 and 8 follows:

$$\operatorname{dom}(\alpha \mathbin{\tilde{\cap}} (\delta \mathbin{\tilde{\setminus}} \alpha)) \subset \operatorname{dom}(\alpha) \cap \operatorname{dom}(\delta \mathbin{\tilde{\setminus}} \alpha) \subset \operatorname{dom}(\alpha) \cap \operatorname{dom}(\delta) = \operatorname{dom}(\alpha) .$$

Let then $\xi \in \operatorname{dom}(\alpha)$. We will show, that $\alpha[\xi] \diamond (\delta \mathbin{\tilde{\setminus}} \alpha)[\xi] = \emptyset$, that is $\alpha[\xi] \lhd \tilde{\alpha}^\delta[\xi] = \emptyset$ and $\alpha[\xi] \rhd \tilde{\alpha}^\delta[\xi] = \emptyset$ (because $(\delta \mathbin{\tilde{\setminus}} \alpha)\restriction_{\operatorname{dom}(\alpha)} = \tilde{\alpha}^\delta$). The definition 19 of the boundary implies the following:

$$\tilde{\alpha}^\delta[\xi] = \left\{ p \in \mathbb{T} \colon p \in \delta[\xi] \cap \alpha[\xi]^\perp \vee p \in \max(\alpha[\xi]^\perp) \lhd \delta[\xi] \right\} \subset \alpha[\xi]^\perp .$$

Moreover, $\alpha[\xi] \lhd \alpha[\xi]^\perp = \emptyset$, as no element from the set of conditions incomparable to $\alpha[\xi]$ may occur above any condition from $\alpha[\xi]$. Therefore $\alpha[\xi] \lhd \tilde{\alpha}^\delta[\xi] = \emptyset$. Similarly, $\alpha[\xi] \rhd \alpha[\xi]^\perp = \emptyset$, because when $p \in \alpha[\xi]^\perp$, then p is incomparable to any condition from $\alpha[\xi]$, so it cannot have any condition from $\alpha[\xi]$ above itself. This implies $\alpha[\xi] \rhd \tilde{\alpha}^\delta[\xi] = \emptyset$.

Lemma 8. *If $\mathcal{D}$ is a field of meta sets on δ, then $\emptyset \in \mathcal{D}$ and $\delta \in \mathcal{D}$.*

Proof. A field of meta sets is not empty by the definition. Let then $\xi \in \mathcal{D}$. In that case also $\delta \mathbin{\tilde{\setminus}} \xi \in \mathcal{D}$. The lemma 7 implies ($\xi \Subset \delta$, as $\xi \in \mathcal{D}$), that $\xi \mathbin{\tilde{\cap}} (\delta \mathbin{\tilde{\setminus}} \xi) = \emptyset$. The family $\mathcal{D}$ is closed with respect to $\tilde{\cap}$ operation, so $\emptyset \in \mathcal{D}$. That is why also $\delta \mathbin{\tilde{\setminus}} \emptyset = \delta \in \mathcal{D}$.

1.5.3 The Main Theorem

In this section we will prove that the algebraic operations for the first order meta sets satisfy the well known axioms of Boolean algebra. The theorem 1 presents all these axioms adopted to the meta sets notation.

Theorem 1. *Let $\delta \in \mathfrak{MF}^1$ be a non-empty first order meta set, and let $\mathcal{D}$ be a field of meta sets on δ. If $\alpha, \beta, \gamma \in \mathcal{D}$ then the following equalities hold:*

$$\alpha \mathbin{\tilde{\cup}} (\beta \mathbin{\tilde{\cup}} \gamma) \approx (\alpha \mathbin{\tilde{\cup}} \beta) \mathbin{\tilde{\cup}} \gamma , \tag{1.21}$$
$$\alpha \mathbin{\tilde{\cap}} (\beta \mathbin{\tilde{\cap}} \gamma) \approx (\alpha \mathbin{\tilde{\cap}} \beta) \mathbin{\tilde{\cap}} \gamma , \tag{1.22}$$
$$\alpha \mathbin{\tilde{\cup}} \beta \approx \beta \mathbin{\tilde{\cup}} \alpha , \tag{1.23}$$
$$\alpha \mathbin{\tilde{\cap}} \beta \approx \beta \mathbin{\tilde{\cap}} \alpha , \tag{1.24}$$
$$\alpha \mathbin{\tilde{\cup}} (\alpha \mathbin{\tilde{\cap}} \beta) \approx \alpha , \tag{1.25}$$
$$\alpha \mathbin{\tilde{\cap}} (\alpha \mathbin{\tilde{\cup}} \beta) \approx \alpha , \tag{1.26}$$
$$\alpha \mathbin{\tilde{\cup}} (\beta \mathbin{\tilde{\cap}} \gamma) \approx (\alpha \mathbin{\tilde{\cup}} \beta) \mathbin{\tilde{\cap}} (\alpha \mathbin{\tilde{\cup}} \gamma) , \tag{1.27}$$
$$\alpha \mathbin{\tilde{\cap}} (\beta \mathbin{\tilde{\cup}} \gamma) \approx (\alpha \mathbin{\tilde{\cap}} \beta) \mathbin{\tilde{\cup}} (\alpha \mathbin{\tilde{\cap}} \gamma) , \tag{1.28}$$
$$\alpha \mathbin{\tilde{\cup}} (\delta \mathbin{\tilde{\setminus}} \alpha) \approx \delta , \tag{1.29}$$
$$\alpha \mathbin{\tilde{\cap}} (\delta \mathbin{\tilde{\setminus}} \alpha) \approx \emptyset . \tag{1.30}$$

Thus, $\mathcal{D}$ is a Boolean algebra.

We will split the proof into a number of lemmas.

Lemma 9. *If* $\alpha, \beta, \gamma \in \mathfrak{MF}^1$*, then* $\alpha \,\tilde{\cap}\, (\beta \,\tilde{\cap}\, \gamma) = (\alpha \,\tilde{\cap}\, \beta) \,\tilde{\cap}\, \gamma$ *.*

Proof. Let $\eta = \beta \,\tilde{\cap}\, \gamma$ and let $\xi = \alpha \,\tilde{\cap}\, \beta$. As we may easily see, the equality $\mathrm{dom}(\eta) = \mathrm{dom}(\beta) \cap \mathrm{dom}(\gamma)$ holds, as well as $\mathrm{dom}(\xi) = \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta)$. Moreover, let $\tau = \alpha \,\tilde{\cap}\, (\beta \,\tilde{\cap}\, \gamma) = \alpha \,\tilde{\cap}\, \eta$ and $\sigma = (\alpha \,\tilde{\cap}\, \beta) \,\tilde{\cap}\, \gamma = \xi \,\tilde{\cap}\, \gamma$. We have:

$$\begin{aligned}\mathrm{dom}(\tau) &= \mathrm{dom}(\alpha) \cap \mathrm{dom}(\eta) \ , \\ &= \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta) \cap \mathrm{dom}(\gamma) \ , \\ &= \mathrm{dom}(\xi) \cap \mathrm{dom}(\gamma) \ , \\ &= \mathrm{dom}(\sigma) \ .\end{aligned}$$

For $\mu \in \mathrm{dom}(\tau)$ the formula $\tau[\mu] = \alpha[\mu] \diamond \eta[\mu]$ holds. The lemma 6 implies:

$$\begin{aligned}\tau[\mu] &= \alpha[\mu] \diamond \eta[\mu] \ , \\ &= \alpha[\mu] \diamond (\beta[\mu] \diamond \gamma[\mu]) \ , \\ &= (\alpha[\mu] \diamond \beta[\mu]) \diamond \gamma[\mu] \ , \\ &= \xi[\mu] \diamond \gamma[\mu] \ , \\ &= \sigma[\mu] \ .\end{aligned}$$

We have shown that domains of the sets represented by the left and the right hand sides are equal, and that the images of appropriate potential elements are also equal. Thus both sides of the equality are equal.

Note, that we have proved the strong (crisp) equality $=$, not the meta equality $\approx$ required by the theorem 1. The above lemma allows for omitting parentheses and using the notation:

$$\alpha \,\tilde{\cap}\, \beta \,\tilde{\cap}\, \gamma \ = \ \alpha \,\tilde{\cap}\, (\beta \,\tilde{\cap}\, \gamma) \ = \ (\alpha \,\tilde{\cap}\, \beta) \,\tilde{\cap}\, \gamma \ . \tag{1.31}$$

Lemma 10. *If* $\alpha, \beta \in \mathfrak{MF}^1$*, then* $\alpha \,\tilde{\cup}\, (\alpha \,\tilde{\cap}\, \beta) \approx \alpha$*.*

Proof. Let $\tau = \alpha \,\tilde{\cup}\, (\alpha \,\tilde{\cap}\, \beta)$. From the propositions 3 and 4 follows that $\mathrm{dom}(\tau) = \mathrm{dom}(\alpha)$. By the definition 14 we need to show that for $\mu \in \mathrm{dom}(\alpha)$ holds $\tau[\mu] \parallel \alpha[\mu]$. In other words, for $p \in \alpha[\mu]$ we must show $\tau[\mu] \mid p$, and similarly, for $q \in \tau[\mu]$ the relation $\alpha[\mu] \mid q$ must hold.

$\tau[\mu] \mid p$ means, that $\tau[\mu]$ contains a maximal finite antichain below p, or $\tau[\mu]$ contains a condition weaker than p. If $p \in \alpha[\mu]$, then this is obvious, as $\alpha[\mu] \subset \tau[\mu]$, so $p \in \tau[\mu]$, and for any $a \in A$ always holds $A \mid a$, because $\{ a \} \mid a$ for any a.

Now, let us consider $q \in \tau[\mu]$. We will show $\alpha[\mu] \mid q$. If $q \in \alpha[\mu]$, then, of course, $\alpha[\mu] \mid q$. In the converse case, when $q \in \tau[\mu] \setminus \alpha[\mu]$, we have

$$q \in (\alpha \,\tilde{\cup}\, (\alpha \,\tilde{\cap}\, \beta))\,[\mu] \setminus \alpha[\mu] \ = \ \alpha[\mu] \cup (\alpha \,\tilde{\cap}\, \beta)[\mu] \setminus \alpha[\mu] \ \subset \ (\alpha \,\tilde{\cap}\, \beta)[\mu] \ .$$

Thus, by the definitions 18 and 17,

$$q \in \alpha[\mu] \diamond \beta[\mu] \; = \; \alpha[\mu] \lhd \beta[\mu] \cup \alpha[\mu] \rhd \beta[\mu] \; .$$

If it were that $q \in \alpha[\mu] \lhd \beta[\mu]$, then $q \in \alpha[\mu]$, what would contradict the assumption that $q \in \tau[\mu] \smallsetminus \alpha[\mu]$. Therefore $q \in \beta[\mu] \lhd \alpha[\mu]$, which means, that $q \in \beta[\mu]$ and there exists $r \in \alpha[\mu]$ such, that $q \leq r$. This implies $\alpha[\mu] \mid q$ and finally $\tau[\mu] \parallel \alpha[\mu]$, what gives $\tau \approx \alpha$.

Lemma 11. *If $\alpha, \beta, \gamma \in \mathfrak{MF}^1$, then: $\alpha \mathbin{\tilde{\cap}} (\beta \mathbin{\tilde{\cup}} \gamma) = (\alpha \mathbin{\tilde{\cap}} \beta) \mathbin{\tilde{\cup}} (\alpha \mathbin{\tilde{\cap}} \gamma)$.*

Proof. If $\alpha = \emptyset$, then both sides of the equality represent empty sets. If $\beta = \emptyset$ or $\gamma = \emptyset$, then we get the identity. Further we assume that all the sets are not empty.

Let $\lambda = \alpha \mathbin{\tilde{\cap}} (\beta \mathbin{\tilde{\cup}} \gamma)$, and let $\rho = (\alpha \mathbin{\tilde{\cap}} \beta) \mathbin{\tilde{\cup}} (\alpha \mathbin{\tilde{\cap}} \gamma)$. Also, let $\langle \xi, p \rangle \in \lambda$. By propositions 3 and 4 we obtain:

$$\begin{aligned} \xi \; \in \; \mathrm{dom}(\alpha \mathbin{\tilde{\cap}} (\beta \mathbin{\tilde{\cup}} \gamma)) \; &\subset \; \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta \mathbin{\tilde{\cup}} \gamma) \; , \\ &= \; \mathrm{dom}(\alpha) \cap (\mathrm{dom}(\beta) \cup \mathrm{dom}(\gamma)) \; , \\ &= \; \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta) \cup \mathrm{dom}(\alpha) \cap \mathrm{dom}(\gamma) \; . \end{aligned}$$

Additionally, the definition of the meta sum implies, that $p \in \alpha[\xi] \diamond (\beta \cup \gamma)[\xi]$. The proposition 10 implies that:

$$\alpha[\xi] \diamond (\beta \cup \gamma)[\xi] \; = \; \alpha[\xi] \diamond \beta[\xi] \; \cup \; \alpha[\xi] \diamond \gamma[\xi] \; .$$

If $\xi \in \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta)$, then $p \in \alpha[\xi] \diamond \beta[\xi]$. Directly from the definition of the meta intersection follows, that in this case $\langle \xi, p \rangle \in \alpha \mathbin{\tilde{\cap}} \beta$. Similarly, if $\xi \in \mathrm{dom}(\alpha) \cap \mathrm{dom}(\gamma)$, then $p \in \alpha[\xi] \diamond \gamma[\xi]$, and this case $\langle \xi, p \rangle \in \alpha \mathbin{\tilde{\cap}} \gamma$. Thus, we have, $\langle \xi, p \rangle \in \rho$, and consequently $\lambda \subset \rho$.

Now let $\langle \zeta, q \rangle \in \rho$. We see that $\zeta \in \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta) \cup \mathrm{dom}(\alpha) \cap \mathrm{dom}(\gamma)$, so $\zeta \in \mathrm{dom}(\alpha) \cap (\mathrm{dom}(\beta \mathbin{\tilde{\cup}} \gamma))$. Similarly as before, there are two cases possible for q: if $\zeta \in \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta)$, then $q \in \alpha[\xi] \diamond \beta[\xi] \subset \alpha[\xi] \diamond (\beta \mathbin{\tilde{\cup}} \gamma)[\xi]$, and if $\zeta \in \mathrm{dom}(\alpha) \cap \mathrm{dom}(\gamma)$, then $q \in \alpha[\xi] \diamond \gamma[\xi] \subset \alpha[\xi] \diamond (\beta \mathbin{\tilde{\cup}} \gamma)[\xi]$. Thus $\langle \zeta, q \rangle \in \lambda$, and finally $\rho \subset \lambda$.

Lemma 12. *If $\alpha, \delta \in \mathfrak{MF}^1$ and $\alpha \Subset \delta$, then $\alpha \mathbin{\tilde{\cup}} (\delta \mathbin{\tilde{\smallsetminus}} \alpha) \approx \delta$.*

Proof. Assume the following notation: $\lambda = \alpha \mathbin{\tilde{\cup}} (\delta \mathbin{\tilde{\smallsetminus}} \alpha)$. First, note that $\mathrm{dom}(\lambda) = \mathrm{dom}(\delta)$. Indeed, because $\alpha \Subset \delta$, so the propositions 2, 3 and 8 imply:

$$\mathrm{dom}(\lambda) = \mathrm{dom}(\alpha) \cup \mathrm{dom}(\delta \mathbin{\tilde{\smallsetminus}} \alpha) \subset \mathrm{dom}(\delta) \; .$$

On the other hand, from the definition 20 of the meta difference $\tilde{\smallsetminus}$ follows:

$$\begin{aligned} \mathrm{dom}(\delta) \; &= \; (\mathrm{dom}(\delta) \smallsetminus \mathrm{dom}(\alpha)) \cup \mathrm{dom}(\alpha) \\ &\subset \; \mathrm{dom}(\delta \mathbin{\tilde{\smallsetminus}} \alpha) \cup \mathrm{dom}(\alpha) \\ &= \; \mathrm{dom}(\lambda) \; . \end{aligned}$$

Let $\mu \in \mathrm{dom}(\lambda)$. To show the equality $\lambda \approx \delta$, we must prove $\lambda[\mu] \parallel \delta[\mu]$, i.e. the equivalence of images $\lambda[\mu]$ and $\delta[\mu]$. By the definition 10 of the equivalence this means, that for $p \in \lambda[\mu]$ must hold $\delta[\mu] \mid p$, and for $q \in \delta[\mu]$ must hold $\lambda[\mu] \mid q$. According to the definition 9 of the covering relation we must show, that $\delta[\mu]$ contains a finite maximal antichain below p or it contains some condition above p. Similarly for the set $\lambda[\mu]$ and the condition q.

Let $p \in \lambda[\mu]$. Note, that $\lambda[\mu] = \alpha[\mu] \cup (\delta \mathbin{\tilde{\smallsetminus}} \alpha)[\mu]$. If $p \in \alpha[\mu]$, then $\delta[\mu] \mid p$, as $\alpha \Subset \delta$ (see definition 13). In the converse case $p \in (\delta \mathbin{\tilde{\smallsetminus}} \alpha)[\mu] \smallsetminus \alpha[\mu]$. If $\mu \in \mathrm{dom}(\delta) \smallsetminus \mathrm{dom}(\alpha)$, then $\tilde{\alpha}^{\delta}[\mu] = \emptyset$, and because in this case holds

$$(\delta \mathbin{\tilde{\smallsetminus}} \alpha)[\mu] = \delta{\restriction}_{\mathrm{dom}(\delta)\smallsetminus\mathrm{dom}(\alpha)}[\mu] \subset \delta[\mu] \ ,$$

so $p \in \delta[\mu]$ and, of course, $\delta[\mu] \mid p$. However, if $p \in \mathrm{dom}(\delta) \cap \mathrm{dom}(\alpha)$, then because $(\delta \mathbin{\tilde{\smallsetminus}} \alpha)[\mu] = \tilde{\alpha}^{\delta}[\mu]$ holds in this case, so $p \in \tilde{\alpha}^{\delta}[\mu]$ and by the definition 19, $p \in \delta[\mu]$ or $\exists_{q \geq p}\, q \in \delta[\mu]$. In both cases $\delta[\mu] \mid p$.

Now, let $q \in \delta[\mu]$. We show, that $\lambda[\mu] \mid q$. By the definition of the difference we obtain:

$$\lambda[\mu] \ = \ \alpha[\mu] \cup (\delta \mathbin{\tilde{\smallsetminus}} \alpha)[\mu] \ = \ \alpha[\mu] \cup \delta{\restriction}_{\mathrm{dom}(\delta)\smallsetminus\mathrm{dom}(\alpha)}[\mu] \cup \tilde{\alpha}^{\delta}[\mu] \ .$$

If $\mu \notin \mathrm{dom}(\alpha)$, then $\alpha[\mu] = \tilde{\alpha}^{\delta}[\mu] = \emptyset$, so $\lambda[\mu] = \delta[\mu]$ and we get $\lambda[\mu] \mid p$. Therefore, we assume that $\mu \in \mathrm{dom}(\alpha)$, and in such case $\lambda[\mu] = \alpha[\mu] \cup \tilde{\alpha}^{\delta}[\mu]$. If $q \in \alpha[\mu]^{\perp}$, then also $q \in \delta[\mu] \cap \alpha[\mu]^{\perp}$, and by the definition 19 of the boundary and the above equality we have $q \in \tilde{\alpha}^{\delta}[\mu] \subset \lambda[\mu]$, which implies $\lambda[\mu] \mid q$. Let then $q \in \alpha[\mu]^{\top}$, i.e. q is comparable to some condition from $\alpha[\mu]$. If there exists $r \geq q$ such, that $r \in \alpha[\mu]$, then clearly $\lambda[\mu] \mid q$, as $\alpha[\mu] \subset \lambda[\mu]$, so $\lambda[\mu]$ contains r. In the converse case there must exist $r < q$ such, that $r \in \alpha[\mu]$. Thus, we have a condition from $\lambda[\mu] \supset \alpha[\mu]$, which lies below q and we have no conditions from $\lambda[\mu]$ above q (as $\tilde{\alpha}^{\delta}[\mu] \subset \alpha[\mu]^{\perp}$, and $q \in \alpha[\mu]^{\top}$). We will prove, that $R = \{\, r \leq q \colon r \in \lambda[\mu] \,\}$ contains a finite maximal antichain below q. This will imply that $\lambda[\mu] \mid q$.

Let $S = \left\{\, s \leq q \colon s \in \alpha[\mu]^{\perp} \,\right\}$. If $S = \emptyset$, then each condition stronger than q is comparable to some element of $\alpha[\mu]$, which – by the assumption – lies below q. The set $\max(\alpha[\mu])$ contains an antichain below q, which is maximal below q (by the previous sentence) and finite, as $\alpha \in \mathfrak{MF}^{1}$. Similarly, the set $R \cap \max(\alpha[\mu])$, and, consequently, R have this property. In the case when $S = \emptyset$, the above implies $\lambda[\mu] \mid q$.

So, assume that $S \neq \emptyset$. We see that $\max(S) \subset \tilde{\alpha}^{\delta}[\mu]$, as for $s \in \max(S)$ holds $s \in \max(\alpha[\mu]^{\perp}) \lhd \delta[\mu]$, because $s \leq q$ and $q \in \delta[\mu]$. Thus, $\max(S) \subset R$ and $\max(S)$ is a finite antichain (the lemma 3). The set $R \cap \max(\alpha[\mu])$ is also a finite antichain, and the sum $R \cap \max(\alpha[\mu]) \cup \max(S)$ contains a maximal antichain below q, because each condition stronger than q, either is comparable to some element from $\alpha[\mu]$ (and then also it is comparable to some element from $R \cap \max(\alpha[\mu])$), or it is not (and then it is comparable to some element of $\max(S)$). Because $R \cap \max(\alpha[\mu]) \cup \max(S) \subset R$, then R includes a finite maximal antichain below q, so it covers q and, consequently, $\lambda[\mu] \mid q$.

Now we are ready to prove the main theorem 1.

Proof. Recall, that $=$ implies $\approx$.

The axioms 1.21, 1.23 are obvious, 1.24 follows from the proposition 9.

The axiom 1.22 follows from the lemma 9.

The axiom 1.25 follows from the lemma 10.

The axiom 1.26 follows from 1.25 and 1.28 and from the fact, that $\alpha \,\tilde{\cap}\, \alpha = \alpha$ (as $P \diamond P = P$), in the following way:

$$\begin{aligned} \alpha \,\tilde{\cap}\, (\alpha \,\tilde{\cup}\, \beta) &\approx (\alpha \,\tilde{\cap}\, \alpha) \,\tilde{\cup}\, (\alpha \,\tilde{\cap}\, \beta) \ , && \text{(from 1.28)} \\ &= \alpha \,\tilde{\cup}\, (\alpha \,\tilde{\cap}\, \beta) \ , && (\text{since } \alpha \,\tilde{\cap}\, \alpha = \alpha) \\ &\approx \alpha \ . && \text{(from 1.25)} \end{aligned}$$

The distributive law 1.27 follows easily from other axioms:

$$\begin{aligned} \alpha \,\tilde{\cup}\, (\beta \,\tilde{\cap}\, \gamma) &\approx \alpha \,\tilde{\cup}\, (\alpha \,\tilde{\cap}\, \beta) \,\tilde{\cup}\, (\beta \,\tilde{\cap}\, \gamma) \ , && \text{(by 1.25)} \\ &\approx [\alpha \,\tilde{\cap}\, (\alpha \,\tilde{\cup}\, \gamma)] \,\tilde{\cup}\, [(\beta \,\tilde{\cap}\, \alpha) \,\tilde{\cup}\, (\beta \,\tilde{\cap}\, \gamma)] \ , && \text{(by 1.26, 1.24)} \\ &\approx [\alpha \,\tilde{\cap}\, (\alpha \,\tilde{\cup}\, \gamma)] \,\tilde{\cup}\, [\beta \,\tilde{\cap}\, (\alpha \,\tilde{\cup}\, \gamma)] \ , && \text{(by 1.28)} \\ &\approx (\alpha \,\tilde{\cup}\, \beta) \,\tilde{\cap}\, (\alpha \,\tilde{\cup}\, \gamma) \ . && \text{(by 1.24, 1.28)} \end{aligned}$$

The distributive law 1.28 follows from the lemma 11.

The axiom 1.29 is a consequence of the lemma 12.

The axiom 1.30 is a consequence of the lemma 7.

This ends the proof of the theorem.

1.6 Conclusions and Further Work

We have explained a basic idea of a meta set and have defined fundamental concepts related to them, in particular the interpretation of a meta set. For the important subclass $\mathfrak{MF}^1$ we have defined set-theoretic relations and algebraic operations. These relations coincide [4] with the relations defined in the general case for arbitrary meta sets [3] by means of the interpretations.

We have focused on $\mathfrak{M}^1$ meta sets here, as they are most common in applications. Their theory is the simplest to comprehend and they are closest to the well known fuzzy sets. The first order meta sets represent fuzzy collections of entities which may be described by means of ordinary crisp sets, i.e. the "elements" of such collections are constant and precisely defined. Moreover, as in computer applications we mostly deal with finite collections of data, then further restricting ourselves to the class $\mathfrak{MF}^1$ of the first order hereditarily finite meta sets does not really seem a drawback.

The way we have defined relations and operations for $\mathfrak{MF}^1$ meta sets allow for straightforward and efficient computer implementations. The appropriate algorithms will operate on subsets of the binary tree, or – using another representation – on binary sequences that arise due to encoding of elements of the binary tree in a programming language. Although the sequences will be finite due to computer limitations, we do not consider it a shortcoming, since data we deal with in applications have finite nature.

The fact that the operations for meta sets satisfy the Boolean algebra axioms is significant, as it allows for using them in contexts, where traditional crisp sets do not apply, because some kind of fuzziness is required. Note also, that "elements" of meta sets are other meta sets, what makes them applicable in situations, where fuzzy sets are not enough, because the structure of elements is important.

The meta sets theory is under development. The interpretation technique plays the key role in understanding meta sets as well as in defining their properties. For instance, we have managed to define the cardinality of a meta set as well as equinumerability of $\mathfrak{MF}^1$ meta sets [5].

List of Symbols

$\mathbb{T}$	the binary tree, p. 2
$\mathbb{1}$	the root of the tree $\mathbb{T}$, p. 2
$p \perp q$	incomparable conditions, p. 2
$p \top q$	comparable conditions, p. 2
$\mathrm{dom}(\tau)$	the domain of the meta set τ, p. 4
$\mathrm{ran}(\tau)$	the range of the meta set τ, p. 4
$\tau[\sigma]$	the image of the meta set τ at the meta set σ, p. 4
$\mathfrak{M}$	the class of meta sets, p. 3
$\mathfrak{M}^{\mathfrak{c}}$	the class of canonical meta sets, p. 5
$\mathfrak{M}^1$	the class of the first order meta sets, p. 7
$\mathfrak{MF}$	the class of hereditarily finite meta sets, p. 5
$\mathfrak{MF}^{\mathfrak{c}}$	the class of hereditarily finite, canonical meta sets, p. 7
$\mathfrak{MF}^1$	the class of the first order, hereditarily finite meta sets, p. 7
$\check{\tau}$	a canonical meta set, p. 5
$\tau_{\mathcal{C}}$	the interpretation of the meta set τ given by the branch $\mathcal{C}$, p. 5
$R \mid p$	the set R covers the condition p, p. 7
$Q \parallel R$	the sets Q and R are equivalent, p. 8
$\tau \epsilon \sigma$	τ is a meta member of σ, p. 8
$\tau \epsilon_p \sigma$	τ belongs to σ under the condition p, p. 8
$\tau \Subset \sigma$	τ is a meta subset of σ, p. 8
$\tau \approx \sigma$	τ is meta equal σ, p. 9
$\tau \mathbin{\tilde{\cup}} \sigma$	the meta sum of τ and σ, p. 9
$\tau \mathbin{\tilde{\cap}} \sigma$	the meta intersection of τ and σ, p. 10
$P \lhd R$	the half convolution of P below R, p. 9
$P \rhd R$	the half convolution of P over R, p. 9
$P \diamond R$	the convolution of P and R, p. 10

$\max(P)$	the set of maximal elements in P, p. 11
$P^{\top}$	the set of conditions comparable to any condition in P, p. 11
$P^{\perp}$	the set of conditions incomparable to all condition in P, p. 11
$\tilde{\eta}^{\tau}$	the boundary of η in τ, p. 13
$\tau \mathbin{\tilde{\setminus}} \sigma$	the meta difference of τ and σ, p. 14

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