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Fuzzy Sets as Metasets

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Abstract

Fuzzy sets are well known means for expressing partial membership of an element to a set. As opposed to the classical set theory, where sets have crisp boundaries and other sets either entirely belong to the given set or are completely outside of it, the fuzzy set theory admits cases when an element is a member of a fuzzy set to some degree and -- at the same time -- it is not a member of the fuzzy set to another degree. Fuzzy sets play an important role in computer science and its applications nowadays, despite of the fact that the concept of fuzzy set does not seem to be perfectly suited for direct computer implementations because of necessity of handling real-valued functions.

The metaset theory is a new set theory, based on the classical set theory, which is designed to handle partial membership and non-membership of elements to a metaset. It is meant as an alternative to the fuzzy set theory. One of its most significant advantages are computer oriented definitions of basic relations and algebraic operations. This means that they are easily and efficiently implementable using programming languages and, consequently, they allow for effective computer applications.

Metasets generalize fuzzy sets. In this paper we construct a metaset that has similar properties to the given fuzzy set. In particular, membership degrees of corresponding elements of the fuzzy set and the metaset are equal. The construction takes into account computer limitations by making some assumptions about the membership function of the fuzzy set. Thus, although the construction is limited to some particular class of fuzzy sets, it suffices for computer applications and allows for replacing fuzzy sets with metasets there. As the result we should obtain faster, more efficient programs operating on sets with partial membership relation.

1. Introduction

Many real-world problems cannot be described by means of the classical two-valued logic and the classical set theory ([1]). Such problems occur frequently in industry and computer applications. This is the reason for growing popularity of alternative theories able to handle other logical values than truth or falsity, and set theories able to express membership degrees other than crisp, full membership or non-membership.

The most significant theory in this area nowadays is the fuzzy set theory ([6]). It is successfully applied in a wide range of areas. However, it has some disadvantages. One of them is the fact, that operations on fuzzy sets usually involve processing of real-valued functions by computers, what is inefficient and might involve subtle calculation errors.

As a step towards the solution of this and other problems, the theory of metasets was invented. Metasets, similarly to fuzzy sets, are means for expressing partial membership of elements to the metaset. However, as opposed to fuzzy sets, the definitions of set-theoretic relations and algebraic operations for metasets are computer oriented. This means that they are easily and efficiently implementable using modern programming languages. This should lead to productive industry applications.

Metasets may generalize fuzzy sets in various ways. In this paper we will present one of the methods of representation of fuzzy sets by means of metasets, which is particularly applicable in computer applications. More precisely, we will construct a metaset which will have analogous properties to some given fuzzy set. For the sake of our goal we will make some computer specific assumptions on the membership function of the fuzzy set. Although these assumptions make the construction valid only in particular circumstances and therefore it is not completely general in a mathematical sense, they are naturally satisfied in computer implementations. This suffices for replacing fuzzy sets with metasets in computer applications

2. Fuzzy Sets

Fuzzy set theory was developed to describe partial membership of an element to a fuzzy set. Using fuzzy sets it is possible to express such vague terms like "tall", "warm", "young", etc. This type of notions is hardly expressible within the classical logic or classical set theory, which are designed to describe world with crisp boundaries. On the contrary, fuzzy sets enable expressing gradual transition from "being a member" to "not being a member", or between satisfying or not some property. It is done by introducing the concept of membership function, which assigns membership degrees to elements of some universe.

Assuming X is a crisp set, the fuzzy set A in X is defined as the set of ordered pairs: $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to [0\ldots 1]$ is the membership function which maps each element in X to its membership grade. To stress similarities between fuzzy sets and metasets we will call the set X the *domain* of the fuzzy set A.

Usually X is an infinite set and the membership function may acquire arbitrary real values. Of course, we cannot represent neither infinite sets nor real numbers accurately in computers. Therefore, we will assume that X is a finite set and also that the membership function acquires only rational values, whose denominators are powers of 2. Such entities (and no other) are representable in computers without any approximations. Such restriction should also make applications run faster.

Based on this idea we construct a metaset corresponding to a fuzzy set which satisfies the above mentioned assumptions.

3. Metasets

Metasets – similarly to fuzzy sets – are means for representing rough, inaccurate data or collections. Also, they allow for expressing a degree to which some property is satisfied or not. This degree does not necessarily have to be a number – it might be an element of some lattice or a partial order. As opposed to fuzzy sets, and similarly to classical crisp sets, elements of metasets are also metasets. Thanks to this property we may model an imprecise collection comprised of imprecise elements. One of the most significant characteristics of metasets is the fact, that the definitions of basic relations and algebraic operations for metasets are directed towards computer implementations.

We introduce now very briefly basic ideas related to metasets. We concentrate here only on these concepts which are required for further discussion and presentation of the main result. For the detailed treatment of fundamentals of the metaset theory the reader is referred to [4], [5] and also [3].

We start with establishing some well known terms concerning binary trees.

3.1. The Binary Tree

We denote the full and infinite binary tree (see Fig. 1) with the symbol *T,* and its largest element (root) with the symbol $\mathfrak{R}$. The elements of *T* are ordered in such way that larger ones are closer to the root. Nodes of the tree *T* are denoted using sequences of 0 and 1 surrounded with square brackets, with the exception of the root which is the empty sequence. A *branch* in the tree *T* is a maximal chain, i.e. a maximal set of mutually comparable nodes. For instance, the elements $\mathfrak{R}$, [0], [01], [010] on the Fig. 1 form an initial segment of a sample branch (which always has infinite number of elements). We say that the branch *C* contains a node *p* whenever $p \in C$.

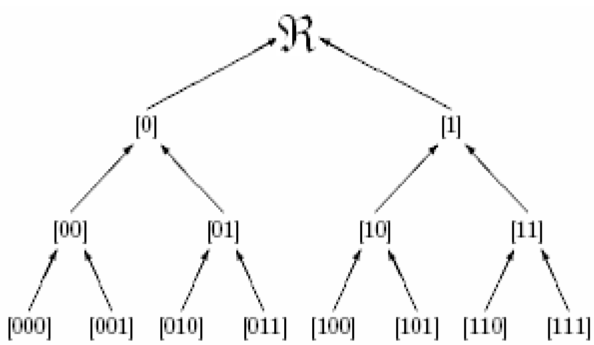

Fig. 1. The binary tree T.

The level n in the tree *T*, denoted with the symbol T_n , is the set of all binary sequences of the same length n. For instance, the level 0 contains only the root $\mathfrak{R}$, the level 1 consists of two sequences: [0] and [1]. Nodes within a level are naturally ordered "from left to right". To be more precise, if we interpret binary sequences as numbers, then the ordering of these sequences is induced by the natural ordering of natural numbers. For instance nodes on the level 2 are ordered as follows: [00], [01], [10], [11]. We will refer to this ordering as *level ordering*.

3.2. Fundamental Concepts

Informally, a metaset is a collection of other metasets, where each element is decorated with a label in form of a node of the binary tree *T*. This collection does not necessarily have to be a set, since it might contain multiple occurrences of elements. If an element occurs more than once in the collection, then each occurrence must be labeled with different label. More precisely, a metaset is a relation between a crisp set of other metasets and the set of nodes of the binary tree *T*. Here follows the formal definition.

Definition 1. A *metaset* is the empty set $\varnothing$ or a set of form: $\tau = \{\langle\sigma, p\rangle : \sigma \text{ is a metaset} \wedge p \in T\}$, where *T* is the binary tree and $\langle\sigma, p\rangle$ denotes ordered pair.

The definition is recursive, however, recursion stops at the empty set, just like it is the case for crisp sets. Since a metaset is a relation it is natural to consider its domain and range.

Definition 2. The *domain* of the metaset τ is the following set: $dom(\tau) = \{\sigma : \langle\sigma, p\rangle \in \tau\}$.

Elements of $dom(\tau)$ are called *potential elements* of the metaset τ. They are called potential, since they belong to the metaset to some degree which usually is less than certainty. The degree of membership of a potential element may be represented by the following set of nodes.

Definition 3. If τ and σ are metasets such that $\sigma \in dom(\tau)$, then the set $\tau[\sigma] = \{p \in T : \langle\sigma, p\rangle \in \tau\}$ is called the *image of* τ *at* σ.

Of course, if $\sigma \in dom(\tau)$ then $\tau[\sigma]$ is never empty. The image of τ at σ is similar to the value of a function at some point of its domain. The difference is that the image might contain multiple elements, whereas the value of a function is a single element.

We will not precisely define here the membership relation for metasets, however, intuitively, the nodes of the tree *T* represent membership degrees of potential elements in the metaset. Larger nodes correspond to higher membership degrees, the root represents the full membership, similar to the membership relation for crisp sets. Potential elements may occur in a metaset paired with multiple different nodes. If this is the case, then multiple nodes determine the membership degree as follows. Assume two different nodes $q \neq p$ belong to the image of τ *at* σ. If q and p are incomparable, then they both supply independent membership information to the degree of membership of σ in τ. On the other hand, if they are comparable, then the larger supplies more membership information and thus, the lesser one is redundant. The following example tries to explain this.

Example 1. If $p \in T$, then $\tau = \{\langle\varnothing, p\rangle\}$ is the simplest example of a non-trivial metaset. It has the single potential element which is the empty set. Given $q \in T$ such, that $q \neq p$, we may build another metaset: $\sigma = \{\langle\varnothing, p\rangle, \langle\varnothing, q\rangle\}$. Note, that $dom(\tau) = dom(\sigma) = \{\varnothing\}$. It is clear that $\tau[\varnothing] = \{p\}$ and $\sigma[\varnothing] = \{p, q\}$. If $p \geq q$, then the membership degree of $\varnothing$ in τ is equal to its membership degree in σ and the pair $\langle\varnothing, q\rangle$ does not supply any additional membership degree factor. If p and q are incomparable, then the membership degree of $\varnothing$ in σ is larger than in τ, and it is comprised of the degrees represented by p and q together and independently.

We distinguish the very important class of metasets which correspond to crisp sets.

Definition 4. A *canonical* metaset is the empty set $\varnothing$ or a set of form:
$\check{\tau} = \{\langle\check{\sigma}, \mathfrak{R}\rangle : \check{\sigma} \text{ is a canonical metaset} \wedge p \in T\}$.

The internal structure of a canonical metaset resembles the structure of a crisp set. Indeed, if we get rid of second elements of each ordered pair (and the pair itself), on each level of membership hierarchy, leaving only the first elements, then we obtain a crisp set. Similarly, given a crisp set x, we may construct a canonical metaset corresponding to it by decorating each element on each level of membership hierarchy with the root of the tree *T*. The first process (i.e. extracting first elements of pairs or removing pairs together with second elements) is fundamental for the metaset theory. We will not discuss it thoroughly here, however we give the definition since it is necessary in further proofs. For the details the reader is referred to [4] and [5].

Definition 5. Let τ be a meta set and let *C* be a branch in the binary tree *T*. The *interpretation* of the meta set τ, given by the branch *C*, is the following crisp set: $\tau_C = \{\sigma_C : \langle\sigma, p\rangle \in \tau \wedge p \in C\}$.

So given a meta set τ and the branch *C*, the crisp set τ_C is obtained by removing those pairs from τ whose second elements do not belong to the branch *C*, and then removing labels from remaining pairs (or rather stripping nodes from potential elements of τ), on each level of the membership hierarchy. This procedure is applied recursively to potential elements of τ, their potential elements and so on, giving the crisp set τ_C as the result.

The idea behind the interpretation technique is that it allows to view a metaset as a family of crisp sets. The family consists of all interpretations of the metaset and it is indexed with branches: $\{\tau_C : C \text{ is a branch in } T\}$. Interpretations allow to define basic set-theoretic relations and other properties for metasets so that they are consistent with similar relations and properties for crisp sets.

Example 2. In the classical set theory natural numbers are defined as follows: $0 = \varnothing$, $1 = \{0\} = \{\varnothing\}$,

$2 = \{0,1\} = \{\emptyset, \{\emptyset\}\}$, and so on: $n = \{0,1,\ldots n-1\}$. We may construct canonical metasets corresponding to natural numbers (left hand side of each equality defines a new symbol corresponding to a canonical metaset representing a natural number): $\check{0} = \emptyset$, $\check{1} = \{\langle \check{0}, \mathfrak{R} \rangle\}$, $\check{2} = \{\langle \check{0}, \mathfrak{R} \rangle, \langle \check{1}, \mathfrak{R} \rangle\}$, and so on: $\check{n} = \{\langle \check{0}, \mathfrak{R} \rangle, \langle \check{1}, \mathfrak{R} \rangle, \ldots \langle (\check{n-1}), \mathfrak{R} \rangle\}$. The interpretations of these canonical metasets are equal to natural numbers they represent, independently of the chosen branch *C*: $\check{0}_C = \emptyset$, $\check{1}_C = \{\emptyset\} = 1$, $\check{2}_C = \{\check{0}_C, \check{1}_C\} = 2$.

The following obvious proposition follows from the fact, that the root $\mathfrak{R}$ is included in each branch.

Proposition 1. All interpretations of a canonical metaset are equal.

4. Fuzzy Sets and Metasets

We are now ready to describe the correspondence between fuzzy sets and metasets. For the given fuzzy set, fulfilling some special assumptions, we will construct a metaset whose potential elements will have the same membership degrees as their corresponding elements in the domain of the fuzzy set. Of course, sharing the same membership degrees is not enough for the construction to consider metasets a replacement for fuzzy sets. Therefore, below we briefly mention some other common properties of a fuzzy set and the constructed metaset.

4.1 The Representation

Let A be a fuzzy set with domain X and membership function $\mu_A : X \to [0 \ldots 1]$. We assume, that the domain of A is a finite set: $|X| < \aleph_0$ and its membership function μ_A acquires only rational values whose denominators are powers of 2, i.e. they have form $a/2^b$, for some natural numbers a, b. We want to represent such a fuzzy set by means of some metaset ρ_A. Therefore, first we must establish the domain for the metaset. To simplify formulas let us assume, that elements of the set X are all natural numbers which are less than the cardinality of X, including 0: $X = \{0, 1, \ldots, |X|-1\}$. Of course, this stipulation does not affect the generality of reasoning. Now, we take as the domain for the metaset ρ_A the set of all canonical counterparts (see Ex. 2) of natural numbers from the set X: $dom(\rho_A) = \{\check{0}, \check{1}, \ldots, \check{m}\}$, where $m = |X|-1$. Thus, we obtain a natural isomorphism between the sets X and $dom(\rho_A)$, which assigns to each $n \in X$ its canonical counterpart $\check{n} \in dom(\rho_A)$.

Let $k \in \mathbb{N}$ be such, that all the values of the membership function μ_A may be represented as rationals with denominators of form 2^k, *i.e.:* $\mu_A(n) = l_n / 2^k$ for each natural number $n \in X$. Such k must exist, since X is finite and μ_A takes only rational values whose denominators are powers of two, by the assumptions. It is clear that for each $n \in X$ holds $0 \le l_n \le 2^k$.

Let $p_j \in T_k$ for $0 \le j < 2^k$ denote elements of the k-th level of the binary tree *T* ordered in the level ordering. We define the metaset ρ_A as follows:

$$\rho_A = \{\langle \check{n}, p_j \rangle : n \in X \wedge 0 \le j < \mu_A(n) \cdot 2^k\}.$$

Note, that first elements of ordered pairs are not natural numbers, but canonical metasets that correspond to natural numbers. We may rewrite the above formula as follows:

$$\rho_A = \{\langle \check{n}, p_j \rangle : \check{n} \in dom(\rho_A) \wedge 0 \le j < l_n\},$$

where l_n is the numerator of the rational number $\mu_A(n)$ by the denominator equal to 2^k, for each $n \in X$.

As we see, each potential element $\check{n} \in dom(\rho_A)$ is paired with all the initial nodes from T_k, which are less than $l_n = \mu_A(n) \cdot 2^k$ in the level ordering. Thus we have $|\rho_A[\check{n}]| = l_n$, or in another formulation:

$$\frac{|\rho_A[\check{n}]|}{2^k} = \mu_A(n).$$

The set of nodes $\rho_A[\check{n}]$, i.e. the image of ρ_A at $\check{n}$, specifies the degree of membership of the potential element $\check{n}$ to the metaset ρ_A. Note, that all nodes in $\rho_A[\check{n}]$ are pairwise incomparable. The numerical value of this degree is the quotient of the number of elements in $\rho_A[\check{n}]$ and the number of all the nodes in T_k, which is 2^k. For instance, when $|\rho_A[\check{n}]| = 2^k$, then the degree of membership is equal to certainty, and $\mu_A(n) = 1$ in such case.

We have defined the metaset ρ_A which is equivalent to the fuzzy set A. The equivalence assures that the degrees of membership of corresponding elements from the domains of ρ_A and A, are equal. The degree of membership for the fuzzy set is given by its membership function. The membership degree for the metaset is given by the cardinalities of images of the metaset at its potential elements, divided by 2^k (in this particular case, when elements of T_k are pairwise incomparable).

4.2 Other Common Properties

The correspondence between fuzzy sets and metasets defined above has many interesting properties. We prove now one such simple property. Recall that the α-cut of a fuzzy set A in X is defined as follows: $S_\alpha = \{x : \mu_A(x) \ge \alpha\}$.

Lemma 1. Let A be a fuzzy set with the domain X which consists of all natural numbers less than m: $X = \{0, 1, \ldots, m-1\}$, and whose membership function μ_A acquires only rational values with denominator of form 2^k, for some given m and k. Let $p_j \in T_k$ for $0 \le j < 2^k$ be consecutive nodes from the k-th level of the binary tree T (enumerated according to level ordering). Let ρ_A be the following metaset:

$$\rho_A = \{\langle \check{n}, p_j \rangle : 0 \leq n < m \wedge 0 \leq j < \mu_A(n) \cdot 2^k\}.$$

If C_i is a branch containing an element $p_i \in T_k$ for some $i < 2^k$, then the interpretation $int(\rho_A, C_i)$ is the α-cut of the fuzzy set A, for $\alpha = (i+1)/2^k$. Moreover, each α-cut of the fuzzy set A is equal to some interpretation of the metaset ρ_A.

Proof. To begin with, note that each α-cut of A is equal to one of α-cuts of form

$$S_{i/2^k} = \{n \in X : \mu_A(n) \geq i/2^k\}$$

for $0 < i \leq 2^k$. This is a consequence of the assumption on the values of the membership function – they are rationals with denominators equal 2^k. For some i, let $p_i \in T_k$ and let C_i be an arbitrary branch containing p_i. Each such branch gives the same interpretation of the metaset ρ_A:

$$p_i \in C_i^1 \wedge p_i \in C_i^2 \rightarrow int(\rho_A, C_i^1) = int(\rho_A, C_i^2).$$

Indeed, potential elements of ρ_A are canonical metasets, so their interpretations are always equal, independent of the branch chosen (cf. Prop. 1). On the other hand the range of ρ_A (i.e. the set $\{p : \langle \sigma, p \rangle \in \rho_A \text{ for some } \sigma\}$) is entirely included in T_k, so it does not contain any node below p_i (i.e. further from the root), which might affect the interpretation of ρ_A.

Directly from the definition of ρ_A it follows that its interpretations are sets of natural numbers. Thus, let us take $n \in int(\rho_A, C_i)$. According to the definitions of metaset and interpretation, the pair $\langle \check{n}, p_i \rangle$ belongs to ρ_A. This means, that $i < l_n$, where l_n is the numerator of the fraction $\mu_A(n)$ by the denominator equal to 2^k. In other words $i/2^k < \mu_A(n)$. Since μ_A acquires only rational values with denominator of form 2^k, then we also have $(i+1)/2^k \leq \mu_A(n)$. Therefore, n belongs to the α-cut of A for $\alpha = (i+1)/2^k$, and, consequently, $int(\rho_A, C_i) \subset S_{(i+1)/2^k}$.

Now let $n \in S_{(i+1)/2^k}$ for $0 \leq i < 2^k$. This means that $\mu_A(n) \geq (i+1)/2^k$ what is equivalent to $\mu_A(n) > i/2^k$. Therefore, $i < \mu_A(n) \cdot 2^k$ and – by the definition – the metaset ρ_A includes the pair $\langle \check{n}, p_i \rangle$.

If C_i is a branch containing p_i, then $n \in int(\rho_A, C_i)$ and finally $S_{(i+1)/2^k} \subset int(\rho_A, C_i)$. This finishes the proof.

Additionally, it is worth noting that if C_0 is a branch containing p_0 (e.g. the left-most node on the level T_k), then $int(\rho_A, C_0) = supp(A)$ is the support of the fuzzy set A, i.e. the set $\{x \in X : \mu_A(x) > 0\}$. On the other hand, if C_{2^k-1} is a branch containing p_{2^k-1} (e.g. the right-most node), then $int(\rho_A, C_{2^k-1}) = kern(A)$ is the kernel of the fuzzy set A, i.e. the set $\{x \in X : \mu_A(x) = 1\}$.

5. Conclusions

There are several other results in this area. The basic set-theoretic relations for fuzzy sets and metasets coincide. Also the algebraic operations of sum and intersection do agree, whereas the complement does not ([2]).

The described above method of representing fuzzy sets as metasets is substantial for computer representations of both. It is because of assumptions on the values acquired by the membership function. Such limitations are not restrictive in the world of computer applications. However, the presented method might be generalized to arbitrary fuzzy sets or even to intuitionistic fuzzy sets.

We stress that the presented construction allows for replacing fuzzy sets with metasets in – among others – computer applications, which should lead to increased efficiency of operation. It should be mentioned, though, that no efficiency comparison of both method (fuzzy sets and metasets) was done yet: neither theoretical nor experimental. This is scheduled as a future work and should be done soon. Anyway, the already implemented experimental version of library of metasets operations gives strong indications as to its high efficiency.

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