Set-theoretic Relations for Metasets

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Abstract
The paper introduces partial set-theoretic relations for metasets and investigates their properties. Metaset is a concept of imprecise set designed to represent vague notions in Artificial Intelligence, whose idea stems from Boolean-valued techniques in classical set theory. The basic set relations are extended to functions valued in a Boolean algebra or unit interval. Important classical set theory axioms: extensionality and comprehension are formulated for metasets. The latter enables formal definitions of collections with blurred boundaries by using set-theory formulae. This facilitates representing and reasoning about imprecise data.

KEYWORDS
metaset, partial membership, vagueness, imprecise set, set-theoretic relation, knowledge representation

1. Introduction

We propose a new concept of set with partial membership relation, called metaset. It admits other possibilities of belonging than the classical binary membership / non-membership. In the most general case the membership degrees are evaluated in a non-trivial Boolean algebra. For common Artificial Intelligence (AI) applications the unit interval may be chosen as the evaluation space. We also introduce partial equality for metasets and negations of basic relations: non-membership and unequality. We conclude with metaset formulation of basic classical set properties: extensionality and comprehension (separation).

The motivation behind this study is the necessity of a formalism for modelling vague notions, using a language similar to the language of the classical Zermelo-Fraenkel set theory (ZFC). The traditional binary approach failed to properly describe human perception and theories dealing with imprecise, vague concepts became fundamental problem of AI. However, our scholar intuition is shaped by the classical set theory based on binary logic and so are computing machines we use. A variety of techniques have been devised to build a path between two-valued mathematics underlying contemporary machine data representation, and vague, many-valued human reasoning and perception. The most successful seem to be fuzzy sets and rough sets. These theories impose different semantics of membership from the one defined by ZFC. They model gradedness, certainty, similarity or incompleteness of information. The main advantage

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of the presented approach, as opposed to these well known solutions, is maintaining the classical semantics of membership of an element to a set, as expressed by the Axiom Schema of Comprehension[1] in ZFC. It states that a set is a collection of other sets satisfying some predicate. Preservation of the original semantics is possible through extending the classical membership relation to a Boolean-valued function.

Many-valued view on subset separation introduced in the paper enables formal representation of collections of objects which satisfy non-binary predicates. As a simple example consider a selection of 'sunny days' during holidays. If there were mixed instances: 'on day X the weather was sunny in the morning and cloudy in the evening', was the day 'X' sunny, partly sunny or rather cloudy and not sunny? The classical, binary approach fails to properly classify such ambiguous cases without making rough approximations. They can be naturally represented with metasets.

We start with short review of other set based approaches to modelling of vague terms and we discuss the background for metasets in Section 2. In Section 3 we recall some well known terms and notation. Section 4 contains the definition of metaset and the techniques of interpretation and forcing. In Section 5 they are used to define the set-theoretic relations for metasets and investigate their properties. Section 6 contains summary.

2. Modelling Vague Concepts with Sets

The problem of partial membership in a set has been studied actively for the past decades. The most successful approaches which find broad applications seem to fall into two categories depending on the scheme used for formalising vagueness. They are based on membership functions or families of approximation sets. We skip many-valued logics based solutions (Bolc & Borowik, 1992, 2003), since they treat the problem from a different perspective.

The first group includes fuzzy sets (Zadeh, 1965) and the derived concepts of intuitionistic fuzzy sets (Atanassov, 1986), L-fuzzy sets (Goguen, 1967), neutrosophic sets (Smarandache, 2005) or vague sets (Gau & Buehrer, 1993). Fuzzy sets formalise the concept of gradedness using membership functions valued in the unit interval. According to Zadeh (1965), 'the notion of "belonging", which plays a fundamental role in the case of ordinary sets, does not have the same role in the case of fuzzy sets. Thus, it is not meaningful to speak of a point x "belonging" to a fuzzy set A except in the trivial sense of $f_A(x)$[2] being positive' (see also Dubois (2011); Dubois and Prade (1997) for a detailed discussion of possible interpretations). L-fuzzy sets generalise fuzzy sets so that membership functions are valued in arbitrary lattice. Vague sets use an additional function for expressing non-membership. Intuitionistic fuzzy sets and neutrosophic sets enhance fuzzy sets with the notion of uncertainty by utilising three functions for describing the relation: membership, non-membership, and indeterminacy.

Rough sets (Pawlak, 1982) fall into the second group. They approximate crisp sets[3] with upper and lower boundaries. They focus on incompleteness of information rather than membership of elements: 'rough set theory is a new mathematical approach to imperfect knowledge' (Pawlak, 2004). Slightly modified approach is used in soft sets invented as a generic mathematical tool for modelling uncertainties (Molodtsov, 1999). A soft set approximates a crisp set U with its subsets: it is a mapping into the power

[1] Also known as Axiom Schema of Separation or Axiom Schema of Specification (Jech, 2006; Kunen, 1980).
[2] $f_A(x)$ is the membership function of the fuzzy set A.
[3] A *crisp set* is the common term used when referring to classical sets with sharp bounds.

set 2^U. Similarly, the concept of uncertain set is based on family of set approximations: an uncertain set is a function from an uncertainty space into a collection of subsets of $\mathbb{R}$ (Liu, 2010).

Metasets follow another path that originates from the concept of Rasiowa–Sikorski Boolean models (Rasiowa & Sikorski, 1963). It was applied first by Scott and Solovay (Scott, 1967a, 1967b) in their proof of independence of the Continuum Hypothesis from the axioms of the Zermelo-Fraenkel set theory (Cohen, 1963, 1964). They produced Boolean-valued models for set theory with the classical membership relation extended to a Boolean-valued function (Bell, 2007; Bell & Scott, 1981).

Subsequently, the Boolean-valued approach to vagueness gained recognition outside of the foundations of mathematics. Takeuti (1979, 2015) developed Boolean-valued analysis. Davis (1977) applied it in the interpretation of uncertainty in quantum theory. Akiba (2014) claims that 'Boolean-valued sets, and not fuzzy sets, are the vague sets we have been looking for as the denotations of vague predicates', since the latter exhibit properties which are difficult to accept from the philosophical point of view. It is worthwhile noting here the attempt at unification of treatment of fuzzy and Boolean-valued approaches proposed by Jin-Wen (1980). Jankowski and Skowron (2008) present a summary of research trends concerning the application of algebraic approach to logic in Artificial Intelligence and particularly in treatment of vague concepts.

In this paper, the Boolean-valued technique is used in the development of the concept of set with partial membership and equality relations, which satisfy extensionality principle and comprehension schema, adapted to many-valued relations. Completing other works on the subject, which deal with algebraic operations (Starosta & Kosiński, 2009) and cardinality of metasets (Starosta, 2014), it provides a formal tool for representing and processing of imprecise data in AI.

3. Preliminaries

We use standard set theory notation. Well-known terms used in the paper are defined here for clarity. Recall, that a natural number n is a finite ordinal of form $n = \{ 0, \ldots, n-1 \}$ or it is the empty set $\emptyset$ corresponding to the number 0. The set of all natural numbers is denoted by ω. For $n \in \omega$, let $2^n = \{ f \colon n \mapsto 2 \}$ denote the set of all functions with the domain n and the range 2; they are binary sequences of the length n.

Definition 3.1. Let T denote the set of all functions from finite ordinals into 2:

$$T = \bigcup_{n \in \omega} 2^n \, . \tag{1}$$

The pair $\mathbb{T} = \langle T, \supseteq \rangle$, where $\supseteq$ is the reverse inclusion ordering of functions (which are sets of ordered pairs), is called the *binary tree*.

The largest element (the *root*) of the ordering in $\mathbb{T}$ is denoted with the symbol $\mathbb{1}$.

Abusing notation we will refer to $\mathbb{T}$ as the set of all finite binary sequences. For $p \colon n \mapsto 2$ and $q \colon m \mapsto 2$, we have $p \leq q$, whenever $p \supseteq q$, i.e., $n \geq m$ and $p_{\restriction m} = q$. In other words, the binary sequence q is a prefix of the binary sequence p. The empty sequence $\mathbb{1}$ is the unique empty function $\emptyset \mapsto 2$.

Binary sequences which are nodes of $\mathbb{T}$ are depicted using square bracket notation: [0], [01], [00], etc. For $p \in \mathbb{T}$, let $p \cdot 0$ and $p \cdot 1$ denote its direct descendants, e.g.,

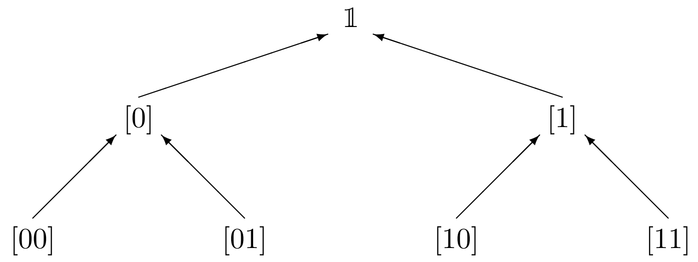

Figure 1. The binary tree $\mathbb{T}$ and its ordering: arrows point at the larger element

$[0] \cdot 0 = [00]$ and $[0] \cdot 1 = [01]$. Nodes of the tree $\mathbb{T}$ are called *conditions* in examples and applications, when they refer to circumstances which might be satisfied or not.

Definition 3.2. A set of nodes $C \subset \mathbb{T}$ is called a *chain* in $\mathbb{T}$, whenever all its elements are pairwise comparable:

$$\forall_{p,q \in C} \ (p \leq q \vee q \leq p) \tag{2}$$

A maximal chain in $\mathbb{T}$ is called a *branch*. Branches represent infinite binary sequences which are functions $\omega \mapsto 2$, so we sometimes write $\mathcal{C} \in 2^\omega$ (abusing the notation, since we actually mean $\bigcup \mathcal{C} \in 2^\omega$). We say that a branch $\mathcal{C}$ *contains* $p \in \mathbb{T}$, when the finite binary sequence p is a prefix of the infinite binary sequence $\mathcal{C}$ (i.e., $p \in \mathcal{C}$).

Definition 3.3. A set $A \subset \mathbb{T}$ is called *antichain* in $\mathbb{T}$, if it consists of pairwise incomparable elements:

$$\forall_{p,q \in A} \ (p \neq q \rightarrow \neg (p \leq q) \wedge \neg (p \geq q)) \tag{3}$$

For instance, in Figure 1, the set $\{ [00], [01] \}$ is an antichain. A *maximal antichain* is an antichain which cannot be extended by adding new elements; it is a maximal element with respect to inclusion of antichains. Examples of maximal antichains in Figure 1 are $\{ [0], [1] \}$ or $\{ [00], [01], [1] \}$ or even $\{ \mathbb{1} \}$.

Definition 3.4. Let R be antichain in $\mathbb{T}$ and let $p \in \mathbb{T}$. We say that R is an *antichain below* p, whenever

$$\forall_{q \in R} \ (q \leq p) \ . \tag{4}$$

A *maximal antichain below* p is an antichain below p, which cannot be extended by adding new element $q \leq p$; it is a maximal element with respect to inclusion of antichains below p. In Figure 1, the set $\{ [00], [01] \}$ is a maximal antichain below $[0]$.

The set of all infinite binary sequences 2^ω may be considered as a topological space, with product topology introduced by infinite product of discrete spaces $\{ 0, 1 \}$. It is called the *Cantor space*. For the given $p \in \mathbb{T}$, the set

$$\bar{p} = \{ \mathcal{C} \in 2^\omega \colon p \in \mathcal{C} \} \tag{5}$$

of all infinite branches containing p is closed-open (clopen) set in the Cantor space.

Definition 3.5 (Sikorski 1969)**.** The Boolean algebra $\mathbb{B}$ of all clopen sets in Cantor space 2^ω is called *Cantor algebra*.

The complement $\overline{\mathbb{B}}$ of this algebra is called *Cohen algebra* (Balcar, Jech, & Zapletal, 1997). It is closed under infinite meets and joins of conditions in $\mathbb{T}$.

Throughout the paper, the illustrative examples are based on the scheme introduced by the Example 3.6. A vague notion of *Day* understood as a time frame is modelled by splitting the interval 4 AM – 8 PM into the hierarchy of shorter subintervals (Figure 2). This allows for answering questions of the form 'What was the weather yesterday?' by providing precise details on particular time intervals. Generally, such questions have no simple and unambiguous answer when weather conditions were mixed (like 'it was sunny in the morning and it was cloudy in the evening'). By dividing day into smaller time frames, we encompass all the observed phenomena in a single object which records their durations. The examples provided refer to medical examinations, i.e., questions like: 'What was the blood pressure yesterday?' or 'Is your pulse stable today?'

Example 3.6. A patient undergoes a series of measurements every 2 hours during the day, between 4 AM and 8 PM. Several parameters are monitored: blood pressure, temperature, pulse, glucose level, etc. If any aberration occurs, it is recorded as a binary alert. The patient is qualified for further medical procedures based on alert status. Times of day are represented as a hierarchy of intervals as depicted in Figure 2.

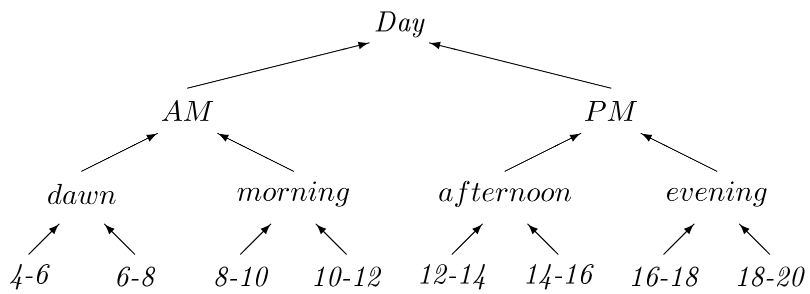

Figure 2. Splitting time intervals of *Day* into the binary tree

4. Metasets, Interpretations and Forcing

We introduce a notion of set with partial membership relation by imposing specific internal structure on a classical set. The structure determines membership characteristic of elements: they are associated with pieces of information describing their membership in the set. The membership description is represented by nodes of the binary tree $\mathbb{T}$.

Definition 4.1. A set which is either the empty set $\emptyset$ or which has the form:

$$\tau = \{ \langle \sigma, p \rangle : \sigma \text{ is a metaset}, p \in \mathbb{T} \} \tag{6}$$

is called *metaset* ($\langle \cdot, \cdot \rangle$ denotes ordered pair).

The class of all metasets is denoted with the letter $\mathfrak{M}$. Formally, this is a definition by induction on the well founded relation $\in$ ($\in$-induction)[4]. A metaset is a relation, so we adopt the following well known terms and notation. For the given metaset τ, the set $\operatorname{dom}(\tau) = \{\, \sigma \colon \langle \sigma, p \rangle \in \tau \,\}$ is called the *domain* of the metaset τ, the set $\operatorname{ran}(\tau) = \{\, p \colon \langle \sigma, p \rangle \in \tau \,\}$ is called the *range* of the metaset τ, and the set $\tau[\sigma] = \{\, p \in \mathbb{T} \colon \langle \sigma, p \rangle \in \tau \,\}$ is called the *image* of the metaset τ at the metaset σ.

Example 4.2. A patient π suffered high blood pressure in the morning and in the evening. It normalised during the day. Occurrences of abnormal pressure are encoded with the alert metaset $\mu = \{\, \langle \pi, \textit{8-10} \rangle, \langle \pi, \textit{10-12} \rangle, \langle \pi, \textit{16-18} \rangle, \langle \pi, \textit{18-20} \rangle \,\}$. The metaset μ represents the vague term of 'unusual blood pressure yesterday'. The membership information of π in μ is determined by the set $\operatorname{ran}(\mu) = \{\, \textit{8-10}, \textit{10-12}, \textit{16-18}, \textit{18-20} \,\}$. In this particular case it is interpreted as the level of deviation from normal blood pressure for π on that particular day.

A metaset encodes a family of classical sets. The process of decoding a set from a metaset is called *interpretation*. A 'key' for decoding is a branch in $\mathbb{T}$.

Definition 4.3. Let τ be a metaset and let $\mathcal{C} \subset \mathbb{T}$ be a branch. The set

$$\tau_{\mathcal{C}} = \{\, \sigma_{\mathcal{C}} \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \,\} \tag{7}$$

is called the *interpretation of the metaset* τ given by the branch $\mathcal{C}$.

The process of producing interpretation of the metaset involves two stages. First, all ordered pairs whose second elements do not belong to the branch $\mathcal{C}$ are removed. Then, the remaining pairs are replaced with interpretations of their first elements.

Informally, an interpretation of a metaset corresponds to one of the many possible precise formulations of some vague term represented by the metaset. From this perspective, a metaset is a collection of particular sharp views on some imprecise idea.

Let $\tau = \{\, \langle \emptyset, [0] \rangle \,\}$. If a branch $\mathcal{C}$ contains $[0]$, then $\tau_{\mathcal{C}} = \{\, \emptyset \,\}$; if it contains $[1]$, then $\tau_{\mathcal{C}} = \emptyset$. There are two different interpretations of τ: $\emptyset$ and $\{\, \emptyset \,\}$. If $\sigma = \{\, \langle \emptyset, \mathbb{1} \rangle \,\}$, then $\sigma_{\mathcal{C}} = \{\, \emptyset \,\}$, for each branch $\mathcal{C}$. A metaset whose range contains at most one element $\mathbb{1}$, and whose domain members hold this property too, has all interpretations equal. Such metasets correspond to classical sets: their elements are absolute (to the highest possible degree) members and they have unique interpretations.

Definition 4.4. A metaset $\check{\tau}$ is called a *canonical metaset*, if it is the empty set, or if it has the form:

$$\check{\tau} = \{\, \langle \check{\sigma}, \mathbb{1} \rangle : \check{\sigma} \text{ is a canonical metaset} \,\} \ . \tag{8}$$

The class of canonical metasets is denoted with $\mathfrak{M}^{\mathfrak{c}}$.

The one-to-one correspondence between the universe of all sets $\mathbf{V}$ and the class $\mathfrak{M}^{\mathfrak{c}}$ is called *canonical isomorphism* $\check{\ } : \mathbf{V} \mapsto \mathfrak{M}^{\mathfrak{c}}$:

$$\check{X} \ = \begin{cases} \emptyset & \text{iff} \quad X = \emptyset \ , \\ \{\, \langle \check{x}, \mathbb{1} \rangle : x \in X \,\} & \text{iff} \quad X \neq \emptyset \ \cdot \end{cases} \tag{9}$$

Note, that if $\mathcal{C}$ is any branch, then $\check{X}_{\mathcal{C}} = X$.

[4]See Kunen (1980, Ch. VII, §2) for the justification of such type of induction.

If $\tau = \{ \langle \emptyset, [0] \rangle, \langle \emptyset, [1] \rangle \}$, then for any branch $\mathcal{C}$ we have $\tau_{\mathcal{C}} = \{ \emptyset \}$. Being canonical is not a mandatory requirement for a metaset to have all interpretations equal. Note, that $\tau[\emptyset] = \{ [0], [1] \}$ is maximal antichain.

Example 4.5. Let $\zeta = \{ \langle \check{\imath}^1, \textit{4-6} \rangle, \langle \check{\imath}^1, \textit{6-8} \rangle, \langle \check{\imath}^2, \textit{4-6} \rangle \}$ represent glucose level alerts on days $\check{\imath}^1, \check{\imath}^2$. There are the following cases for a branch $\mathcal{C}$:

$$\textit{4-6} \in \mathcal{C} \quad \rightarrow \quad \zeta_{\mathcal{C}} = \{ \check{\imath}^1_{\mathcal{C}}, \check{\imath}^2_{\mathcal{C}} \} \ , \tag{10}$$

$$\textit{6-8} \in \mathcal{C} \quad \rightarrow \quad \zeta_{\mathcal{C}} = \{ \check{\imath}^1_{\mathcal{C}} \} \ , \tag{11}$$

$$\textit{morning}, \textit{PM} \in \mathcal{C} \quad \rightarrow \quad \zeta_{\mathcal{C}} = \emptyset \ . \tag{12}$$

Thus, non-empty interpretations refer to times of day with abnormal glucose levels. There were no alerts in the *morning* and after noon.

A branch $\mathcal{C}$ in $\mathbb{T}$ corresponds here to a particular moment during the day. An interpretation of ζ given by $\mathcal{C}$ is the glucose status in this particular moment (we assume that the measurement is valid for the covered interval of two hours). A vague notion of glucose status on a given day ('Was glucose fine yesterday?') has precise interpretations which gathered together make up the metaset ζ.

Note, that we use a canonical $\check{\imath}^1 \neq \check{\imath}^2$ for members to assure their unique, different interpretations $\check{\imath}^1_{\mathcal{C}} \neq \check{\imath}^2_{\mathcal{C}}$ for any branch $\mathcal{C}$.

The properties of interpretations $\mu_{\mathcal{C}}$ of the given metaset μ determine the properties of μ itself. We introduce a method for transferring set-theoretic relations from classical sets onto metasets. We define a relation between conditions and sentences[5]. It will be applied to assign degrees of certainty, represented by sets of conditions, to sentences. The sentences are formulae of the classical set theory with free variables substituted by metasets and bound variables ranging over the class $\mathfrak{M}$.

Given a branch $\mathcal{C}$ and a sentence (e.g., $\sigma \in \tau$), we substitute metasets in the sentence with their interpretations ($\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$). The result is the sentence of the classical set theory stating some property of the sets $\tau_{\mathcal{C}}$ and $\sigma_{\mathcal{C}}$, the membership relation in this case. It may be true or false, depending on $\tau_{\mathcal{C}}$ and $\sigma_{\mathcal{C}}$.

For the given metaset τ, a condition $p \in \mathbb{T}$ specifies a family of interpretations of τ determined by branches $\mathcal{C}$ containing this p. If for each such branch the resulting sentence has a constant logical value, then we take it as the conditional truth or falsity of the given sentence, qualified by the condition p.

Let Φ be a formula built using some of the following symbols: variables $(x^1, x^2, \ldots)$, the constant symbol $(\emptyset)$, the relational symbols $(\in, =, \subset)$, logical connectives $(\wedge, \vee, \neg, \rightarrow)$, quantifiers $(\forall, \exists)$ and parentheses. If we substitute each free variable x^i $(i = 1 \ldots n)$ with some metaset ν^i, and restrict the range of each quantifier to the class of metasets $\mathfrak{M}$, then we get the sentence $\Phi(\nu^1, \ldots, \nu^n)$ which states some property of the metasets $\nu^1, \ldots, \nu^n$. By the *interpretation* of this sentence, determined by the branch $\mathcal{C}$, we understand the sentence $\Phi(\nu^1_{\mathcal{C}}, \ldots, \nu^n_{\mathcal{C}})$ denoted shortly with $\Phi_{\mathcal{C}}$. The sentence $\Phi_{\mathcal{C}}$ is the result of substituting free variables of the formula Φ with the interpretations $\nu^i_{\mathcal{C}}$ of the metasets ν^i, and restricting the range of bound variables to the universe of all sets $\mathbf{V}$. In other words, we replace the metasets in the sentence Φ with their interpretations. The only constant $\emptyset$ in Φ as well as in $\Phi_{\mathcal{C}}$ is the empty set.

Definition 4.6. Let $x^1, x^2, \ldots x^n$ be all free variables of the formula Φ and let $\nu^1, \nu^2, \ldots \nu^n \in \mathfrak{M}$ be metasets. The condition $p \in \mathbb{T}$ is said to *force* the sen-

[5]This idea may be made rigorous by formalising logic within the set theory using Gödelisation (Kunen, 1980).

tence $\Phi(\nu^1, \nu^2, \dots \nu^n)$, whenever for each branch $\mathcal{C} \subset \mathbb{T}$ containing p the sentence $\Phi(\nu^1_{\mathcal{C}}, \nu^2_{\mathcal{C}}, \dots \nu^n_{\mathcal{C}})$ is true:

$$p \Vdash \Phi(\nu^1, \dots \nu^n) \quad \text{iff} \quad \forall_{\mathcal{C}\subset\mathbb{T}} \left(\mathcal{C} \text{ is a branch } \wedge p \in \mathcal{C} \rightarrow \Phi(\nu^1_{\mathcal{C}}, \dots \nu^n_{\mathcal{C}})\right) . \tag{13}$$

The symbol $\Vdash$ denotes the *forcing* relation.

We use standard abbreviation $p \nVdash \Phi$ for $\neg(p \Vdash \Phi)$. Let a property described by a formula $\Phi(x)$ be satisfied by all sets of form $\nu_{\mathcal{C}}$, where ν is a metaset and $\mathcal{C}$ is a branch. Thus, $\Phi(\nu_{\mathcal{C}})$ holds for all the sets which are interpretations of the metaset ν given by all branches $\mathcal{C}$ in $\mathbb{T}$. We might think that this property also 'holds' for the metaset ν, and we formulate this fact by saying that $\mathbb{1}$ forces $\Phi(\nu)$. If $\Phi(\nu_{\mathcal{C}})$ holds for branches $\mathcal{C}$ containing condition p only, then we might think that it 'holds to the degree p' for the metaset ν; we say that p forces $\Phi(\nu)$ in such case.

The following lemmas demonstrate important features of the forcing relation. The first states that forcing is propagated down the branch, the second formulates the premises for propagating forcing upwards.

Lemma 4.7. *Let $p, q \in \mathbb{T}$ and let Φ be a sentence. The following holds:*

$$p \Vdash \Phi \wedge q \leq p \quad \rightarrow \quad q \Vdash \Phi . \tag{14}$$

Proof. If $q \leq p$, then each branch containing q also contains p. If $\mathcal{C}$ is any such branch and $p \Vdash \Phi$, then $\Phi_{\mathcal{C}}$ holds. Since it is true for all $\mathcal{C} \ni q$, then we have $q \Vdash \Phi$. □

It should be understood that conditions below $p \in \mathbb{T}$ carry more detailed information than p. On the other hand, a finite maximal antichain of conditions below p encompasses complete information contained in p, hence it propagates forcing upwards to p.

Lemma 4.8. *Let $p \in \mathbb{T}$, $R \subset \mathbb{T}$ be a finite maximal antichain below p and let Φ be a sentence. The following implication holds:*

$$\forall_{q\in R}\, q \Vdash \Phi \quad \rightarrow \quad p \Vdash \Phi . \tag{15}$$

Proof. $p \Vdash \Phi$ whenever for each branch $\mathcal{C} \ni p$ holds $\Phi_{\mathcal{C}}$. Since R is a finite maximal antichain whose elements are below p, then each branch containing p must also contain some element $q \in R$. Each such q forces Φ, so for any branch $\mathcal{C} \ni p$ we have $\Phi_{\mathcal{C}}$. □

If $p \nVdash \Phi$ then there might still exist a branch $\mathcal{C}$ such, that $\Phi_{\mathcal{C}}$ holds, so it is not true that $p \Vdash \neg\Phi$ in such case.

Lemma 4.9. *Let $p \in \mathbb{T}$ and let Φ be a sentence.*

$$p \Vdash \Phi \quad \rightarrow \quad p \nVdash \neg\Phi , \tag{16}$$

$$p \nVdash \Phi \quad \nrightarrow \quad p \Vdash \neg\Phi . \tag{17}$$

Proof. $p \Vdash \Phi$ implies that for each branch $\mathcal{C} \ni p$ holds $\Phi_{\mathcal{C}}$. Therefore, for no such $\mathcal{C}$ it is possible that $\neg\Phi_{\mathcal{C}}$. Consequently, $p \Vdash \neg\Phi$ cannot be true.

For (17), let $p = \mathbb{1}$, $\mu = \{\, \langle\emptyset, [1]\rangle \,\}$ and let Φ be the sentence: $\mu = \emptyset$. We have $[0] \Vdash \mu = \emptyset$ and $[1] \Vdash \mu \neq \emptyset$, so $\mathbb{1} \nVdash \Phi$ and $\mathbb{1} \nVdash \neg\Phi$. □

The set of conditions which force the given sentence $\mathcal{T}_\Phi = \{ p \in \mathbb{T} \colon p \Vdash \Phi \}$ gives a measure of certainty that the sentence is 'true'. It contains all the information needed to evaluate Φ. By Lemma 4.7, the equivalent certainty information is contained in the set of its maximal elements. Thus, if $\mathcal{T}_\Phi \neq \emptyset$, then the set

$$\|\Phi\| = \max \{ p \in \mathbb{T} \colon p \Vdash \Phi \} \tag{18}$$

is called the *certainty grade* for Φ. If $\mathcal{T}_\Phi = \emptyset$, then we take $\|\Phi\| = \emptyset$.

The certainty grade $\|\Phi\|$ determines a member $\|\Phi\|_{\mathbb{B}}$ of the Cohen algebra $\overline{\mathbb{B}}$.

$$\|\Phi\|_{\overline{\mathbb{B}}} = \bigvee_{p \in \|\Phi\|} \bar{p} \,, \tag{19}$$

where $\bar{p}$ is the clopen set in the Cantor space 2^ω consisting of all branches containing p. Thus, metaset sentences are valued in the Boolean algebra $\overline{\mathbb{B}}$. For an atomic sentence, say $\tau \in \sigma$, where all metasets involved (τ, σ) are hereditarily finite sets[6], this is actually the Cantor algebra $\mathbb{B}$, since $\|\Phi\|$ is finite set in such case. If Φ includes metasets which are not hereditarily finite sets, then $\|\Phi\|$ might be infinite (Starosta & Kosiński, 2013). In such case $\|\Phi\|_{\overline{\mathbb{B}}} \in \overline{\mathbb{B}}$ holds, but not necessarily $\|\Phi\|_{\overline{\mathbb{B}}} \in \mathbb{B}$.

Certainty grades for sentences can be evaluated numerically, what seems more vital to applications than the Boolean evaluation.

Definition 4.10. Let Φ be a sentence. The following number $|\Phi|$ is called the *certainty value* for Φ:

$$|\Phi| = \sum_{p \in \|\Phi\|} \frac{1}{2^{|p|}} \,, \tag{20}$$

where $|p|$ is the cardinality of function p, i.e., the length of the binary sequence p (with $|\mathbb{1}| = 0$). If $\|\Phi\| = \emptyset$, then we take $|\Phi| = 0$.

One may verify that if each $p \in \mathbb{T}$ forces Φ, or equivalently $\mathbb{1} \Vdash \Phi$, then $|\Phi| = 1$. Therefore, $|\Phi| \in [0, 1]$.

5. Relations on Metasets

The basic set-theoretic relations for metasets are defined by forcing atomic sentences. We define a countable number of relations parameterised by conditions in $\mathbb{T}$, for expressing different degrees of certainty.

Definition 5.1. A metaset μ is said to be a *member of a metaset τ under the condition $p \in \mathbb{T}$*, whenever $p \Vdash \mu \in \tau$. We write $\mu \,\epsilon_p\, \tau$.

In other words, $\mu \,\epsilon_p\, \tau$ whenever for each branch $\mathcal{C}$ containing p holds $\mu_{\mathcal{C}} \in \tau_{\mathcal{C}}$. If $\mu \,\epsilon_p\, \tau$, then by Lemma 4.7, we also have $\mu \,\epsilon_q\, \tau$, for $q \leq p$. On the other hand, $\mu \,\epsilon_{p\cdot 0}\, \tau$ and $\mu \,\epsilon_{p\cdot 1}\, \tau$ imply $\mu \,\epsilon_p\, \tau$, by Lemma 4.8. If $\mu \,\epsilon_{\mathbb{1}}\, \tau$, then we have $|\mu \in \tau| = 1$.

[6] A set is hereditarily finite, when the transitive closure of set membership relation for this set is finite, i.e., the set itself is finite and its members are hereditarily finite.

Besides the partial membership, we introduce its negation as a separate relation. The reason is given by Lemma 4.9, which forbids to conclude $p \Vdash \sigma \in \tau$ from the falsity of $p \Vdash \sigma \notin \tau$.

Definition 5.2. A metaset μ is said to be *a non-member of a metaset τ under the condition $p \in \mathbb{T}$*, whenever $p \Vdash \mu \notin \tau$. We write $\mu \not\epsilon_p \tau$.

For $\mu = \emptyset$ and $\tau = \{ \langle \mu, [0] \rangle \}$ we have $\mu \epsilon_{[0]} \tau$ and $\mu \not\epsilon_{[1]} \tau$, since $\tau_{\mathcal{C}} = \{ \emptyset \}$ for $\mathcal{C} \ni [0]$, and $\tau_{\mathcal{C}} = \emptyset$ for $\mathcal{C} \ni [1]$. Hence, for incomparable p, q it is possible that membership and non-membership hold simultaneously:

$$\mu \not\epsilon_p \tau \quad \wedge \quad \mu \epsilon_q \tau \,. \tag{21}$$

Note that $\mu \epsilon_{\mathbb{1}} \tau$ and $\mu \not\epsilon_{\mathbb{1}} \tau$ are both false here. We see that $\neg\, \mu \epsilon_{\mathbb{1}} \tau$ does not exclude existence of some $p \in \mathbb{T}$ such, that $\mu \epsilon_p \tau$. Furthermore, $|\mu \in \tau| = 0.5$ and $|\mu \notin \tau| = 0.5$.

Proposition 5.3. *Let μ and τ be metasets. If $p, q \in \mathbb{T}$ are comparable, then*

$$\mu \epsilon_p \tau \;\rightarrow\; \neg \left(\mu \not\epsilon_q \tau \right) \,, \tag{22}$$

$$\mu \not\epsilon_p \tau \;\rightarrow\; \neg \left(\mu \epsilon_q \tau \right) \,. \tag{23}$$

Proof. We prove (22), proof of (23) is analogous. Assume $\mu \epsilon_p \tau$. When $p = q$, then $\neg \left(\mu \not\epsilon_q \tau \right)$ follows directly from Lemma 4.9. If $q \leq p$, then by Lemma 4.7 also $\mu \epsilon_q \tau$, and applying Lemma 4.9 again we obtain $\neg \left(\mu \not\epsilon_q \tau \right)$. Let $q \geq p$ and assume $\mu \not\epsilon_q \tau$. By Lemma 4.7 also $\mu \not\epsilon_p \tau$. Since by Lemma 4.9 we have $\neg \left(\mu \not\epsilon_p \tau \right)$ we get contradiction. $\square$

The schema of partial equality relations for metasets is defined analogously to the membership relations.

Definition 5.4. A metaset σ is said to be *equal to a metaset τ under the condition $p \in \mathbb{T}$*, whenever $p \Vdash \sigma = \tau$. We write $\sigma \approx_p \tau$.

Definition 5.5. A metaset σ is said to be *unequal to a metaset τ under the condition $p \in \mathbb{T}$*, whenever $p \Vdash \sigma \neq \tau$. We write $\sigma \not\approx_p \tau$.

Conditional equality and unequality for metasets takes into account the elements of their domains (as in the classical case) and also the associated membership information.

Example 5.6. Let $\Lambda = \left\{ \left\langle \check{\imath}^1, AM \right\rangle, \left\langle \check{\imath}^2, morning \right\rangle \right\}$ be metaset representation of expected blood pressure alerts during some two-day treatment of a patient in a hospital. The metaset Λ encodes the expected pressure deviation which might be expressed in the natural language: 'The pressure is expected to improve in the afternoon of the first day and in the morning on the second day'. A patient χ undergoing the treatment has the following record: $\chi = \left\{ \check{\imath}^1 \right\} \times \{ \textit{4-6, 6-8, 10-12} \} \cup \left\{ \check{\imath}^2 \right\} \times \{ \textit{8-10} \}$, i.e., χ alerts deviated from the expected at measurements *8-10* (day $\check{\iota}^1$), *10-12* (day $\check{\iota}^2$). We write:

$$\check{\iota}^1 \,\epsilon_{AM}\, \Lambda \quad \wedge \quad \check{\iota}^1 \not\epsilon_{PM}\, \Lambda \;, \tag{24}$$

$$\check{\iota}^1 \,\epsilon_{dawn}\, \chi \quad \wedge \quad \check{\iota}^1 \,\epsilon_{\textit{10-12}}\, \chi \quad \wedge \quad \check{\iota}^1 \not\epsilon_{\textit{8-10}}\, \chi \quad \wedge \quad \check{\iota}^1 \not\epsilon_{PM}\, \chi \;. \tag{25}$$

Similarly for $\check{\iota}^2$.

We answer the question: 'Did the patient χ measurements match the expected textbook response during the treatment?' in metaset language as follows:

$$\chi \approx_{dawn} \Lambda \quad \wedge \quad \chi \approx_{PM} \Lambda \quad \wedge \quad \chi \not\approx_{morning} \Lambda \;. \tag{26}$$

Indeed, consider the following cases for a branch $\mathcal{C}$ (let ι^i denote $\check{\iota}^i_{\mathcal{C}}$):

$$dawn \in \mathcal{C} \quad \to \quad \chi_{\mathcal{C}} = \{\, \iota^1 \,\} \quad = \quad \Lambda_{\mathcal{C}} = \{\, \iota^1 \,\} \;, \tag{27}$$

$$\textit{8-10} \in \mathcal{C} \quad \to \quad \chi_{\mathcal{C}} = \{\, \iota^2 \,\} \quad \neq \quad \Lambda_{\mathcal{C}} = \{\, \iota^1, \iota^2 \,\} \;, \tag{28}$$

$$\textit{10-12} \in \mathcal{C} \quad \to \quad \chi_{\mathcal{C}} = \{\, \iota^1 \,\} \quad \neq \quad \Lambda_{\mathcal{C}} = \{\, \iota^1, \iota^2 \,\} \;, \tag{29}$$

$$PM \in \mathcal{C} \quad \to \quad \chi_{\mathcal{C}} = \emptyset \quad = \quad \Lambda_{\mathcal{C}} = \emptyset \;, \tag{30}$$

$\chi \approx_{dawn} \Lambda$ and $\chi \approx_{PM} \Lambda$ follow directly from the definition (see Theorem 5.8 for the general statement). $\chi \not\approx_{morning} \Lambda$ follows from Lemma 4.8, since $\{\, \textit{8-10}, \textit{10-12} \,\}$ is a finite maximal antichain below *morning*.

Among the most fundamental classical properties of membership and equality, there are the following:

$$x \in y \wedge x = z \quad \to \quad z \in y \;, \tag{31}$$

$$x \in y \wedge y = z \quad \to \quad x \in z \;, \tag{32}$$

$$x = y \quad \leftrightarrow \quad \forall_z \,(z \in x \leftrightarrow z \in y) \;. \tag{33}$$

They are preserved for metasets, for each pair of membership and equality relations separately, with a slight modification of extensionality (33).

Proposition 5.7. *Let σ, μ, τ be arbitrary metasets and let $p \in \mathbb{T}$.*

$$\sigma \,\epsilon_p\, \tau \wedge \sigma \approx_p \mu \quad \to \quad \mu \,\epsilon_p\, \tau \;, \tag{34}$$

$$\sigma \,\epsilon_p\, \tau \wedge \tau \approx_p \mu \quad \to \quad \sigma \,\epsilon_p\, \mu \;. \tag{35}$$

Proof. Assume $\sigma \,\epsilon_p\, \tau \wedge \sigma \approx_p \mu$. For any branch $\mathcal{C} \ni p$ holds $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$ and $\sigma_{\mathcal{C}} = \mu_{\mathcal{C}}$, therefore $\mu_{\mathcal{C}} \in \tau_{\mathcal{C}}$ holds too. This gives $\mu \,\epsilon_p\, \tau$. Analogously, if in each interpretation $\tau_{\mathcal{C}} = \mu_{\mathcal{C}}$ and $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$, then also $\sigma_{\mathcal{C}} \in \mu_{\mathcal{C}}$, and consequently $\sigma \,\epsilon_p\, \mu$. $\square$

If $p \neq q$, say $p < q$, and $\sigma \,\epsilon_p\, \tau \wedge \sigma \approx_q \mu$, then also $\sigma \approx_p \mu$ (by Lemma 4.7) , therefore $\sigma \,\epsilon_p\, \tau \wedge \sigma \approx_p \mu$ (by Lemma 4.9), and consequently $\mu \,\epsilon_p\, \tau$. Similarly for other cases.

Given two metasets μ, ν, their conditional equality ($\mu \approx_p \nu$) requires more than just having the same members under the condition p. Let $\delta = \emptyset$ and let $\gamma = \{\, \langle [0], \emptyset \rangle \,\}$. Since for each branch $\mathcal{C}$ containing $[1]$ we have $\gamma_{\mathcal{C}} = \emptyset$, then no μ can satisfy $\mu \,\epsilon_{\mathbb{1}}\, \gamma$. Therefore, it is true that $\forall_\mu \,(\mu \,\epsilon_{\mathbb{1}}\, \delta \leftrightarrow \mu \,\epsilon_{\mathbb{1}}\, \gamma)$. However, it is not true that $\gamma \approx_{\mathbb{1}} \delta$, since for a branch $\mathcal{C}$ containing $[0]$ we have $\{\, \emptyset \,\} = \gamma_{\mathcal{C}} \neq \delta_{\mathcal{C}} = \emptyset$, although for other branches holds $\gamma_{\mathcal{C}} = \delta_{\mathcal{C}} = \emptyset$. Therefore, the extensionality needs taking into account membership under all conditions below p.

Theorem 5.8 (Metaset Extensionality)**.** *Let σ, τ be arbitrary metasets and $p \in \mathbb{T}$.*

$$\sigma \approx_p \tau \quad \leftrightarrow \quad \forall_\mu \forall_{q \leq p} \left(\mu \epsilon_q \sigma \leftrightarrow \mu \epsilon_q \tau \right) \ . \tag{36}$$

Proof. Assume the left hand side. Let $\mathcal{C}$ be a branch containing p and let $x \in \sigma_{\mathcal{C}}$ be an arbitrary set from the interpretation $\sigma_{\mathcal{C}}$. By the assumption, we have $x \in \tau_{\mathcal{C}}$. Furthermore, for any $x \in \tau_{\mathcal{C}}$, we have $x \in \sigma_{\mathcal{C}}$. Take an arbitrary metaset μ and let $q \leq p$ be such that $\mu \epsilon_q \sigma$. For each branch $\mathcal{C}_q$ containing q holds $\mu_{\mathcal{C}_q} \in \sigma_{\mathcal{C}_q}$. However, each such branch contains also p, so by the assumption $\mu_{\mathcal{C}_q} \in \tau_{\mathcal{C}_q}$. Consequently, $\mu \epsilon_q \tau$, and conversely, if $\mu \epsilon_q \tau$, then $\mu \epsilon_q \sigma$.

Now assume the right hand side and take any $x \in \sigma_{\mathcal{C}}$, where $\mathcal{C}$ is a branch containing p. From Definition 4.3 of the interpretation, it follows that there exists a metaset $\xi \in \mathrm{dom}(\sigma)$, such that $\xi_{\mathcal{C}} = x$. Moreover, there exists $r \in \mathcal{C}$ such, that $\langle \xi, r \rangle \in \sigma$. However, then $\xi \epsilon_r \sigma$ holds. If $r > p$, then we also have $\xi \epsilon_p \sigma$, by Lemma 4.7. By the assumption we conclude $\xi \epsilon_r \tau$ (or $\xi \epsilon_p \tau$, if $r > p$), so $\xi_{\mathcal{C}} \in \tau_{\mathcal{C}}$. We have shown that if $x \in \sigma_{\mathcal{C}}$, then $x \in \tau_{\mathcal{C}}$. Analogous reasoning in the opposite direction gives $\sigma_{\mathcal{C}} = \tau_{\mathcal{C}}$. Since we chose an arbitrary branch $\mathcal{C} \ni p$, we finally get $\sigma \approx_p \tau$. □

The relation $=$ used with metasets denotes equality of sets, i.e., identity. The identity implies the conditional equality of metasets. On the other hand, $\tau \neq \eta$ does not imply $\tau \not\approx_{\mathbb{1}} \eta$. For instance, if $\tau = \{ \langle \emptyset, \mathbb{1} \rangle \}$ and $\eta = \{ \langle \emptyset, [0] \rangle , \langle \emptyset, [1] \rangle \}$, then $\tau \neq \eta$ and $\tau \approx_{\mathbb{1}} \eta$.

Definition 5.9. A metaset τ is said to be a *subset of a metaset σ under the condition* $p \in \mathbb{T}$, whenever $p \Vdash \sigma \subset \tau$. We write $\sigma \Subset_p \tau$.

In other words, $\sigma \Subset_p \tau$ means that for each branch $\mathcal{C} \ni p$ holds $\sigma_{\mathcal{C}} \subset \tau_{\mathcal{C}}$.

Example 5.10. For metasets $\chi = \left\{ \check{\imath}^1 \right\} \times \{ \textit{4-6}, \textit{6-8}, \textit{10-12} \} \cup \left\{ \check{\imath}^2 \right\} \times \{ \textit{8-10} \}$ and $\Lambda = \left\{ \left\langle \check{\imath}^1, AM \right\rangle , \left\langle \check{\imath}^2, morning \right\rangle \right\}$ as described in Example 5.6, we have $\chi \Subset_{\mathbb{1}} \Lambda$. Indeed, we can write $\subset$ instead of $\neq$ in equations (28, 29). Since $D = \{ \textit{dawn}, \textit{8-10}, \textit{10-12}, PM \}$ is a maximal antichain and $\chi \Subset_p \Lambda$, for $p \in W$, then $\chi \Subset_{\mathbb{1}} \Lambda$ follows from Lemma 4.8.

The inclusion of metasets may be characterised in terms of the conditional membership relation.

Proposition 5.11. *Let σ, τ be arbitrary metasets and $p \in \mathbb{T}$.*

$$\sigma \Subset_p \tau \quad \leftrightarrow \quad \forall_\mu \forall_{q \leq p} \left(\mu \epsilon_q \sigma \rightarrow \mu \epsilon_q \tau \right) \ . \tag{37}$$

Proof. See proof of Theorem 5.8. □

The following relationship between conditional equality and conditional inclusion, similar to the one known from classical set theory, holds as well.

Corollary 5.12. *Let σ, τ be arbitrary metasets and let $p \in \mathbb{T}$.*

$$\sigma \approx_p \tau \quad \leftrightarrow \quad \sigma \Subset_p \tau \wedge \tau \Subset_p \sigma \ . \tag{38}$$

Proof. Refer to interpretations. □

The Axiom of Extensionality (33) in the classical set theory guarantees that a set defined with Comprehension Axiom Schema (39) is unique: given a set X and a predicate $\Phi(z)$, we can determine a subset $Y \subset X$ whose members are precisely the members of X that satisfy Φ:

$$\forall_X \exists_Y \forall_z \ (z \in Y \ \leftrightarrow \ [z \in X \wedge \Phi(z)]) \ . \tag{39}$$

In order to formulate a corresponding property for metasets we need to distinguish a class of metasets with non-binary membership on the first level of membership hierarchy at most.

Definition 5.13. A metaset τ whose domain $\operatorname{dom}(\tau) \subset \mathfrak{M}^{\mathfrak{c}}$ consists of canonical metasets is called a *first order metaset*.

The class of first order metasets is denoted with $\mathfrak{M}^{\mathbf{1}}$. The membership for a first order metaset is crisp on all levels of membership hierarchy except possibly for the first one, since its first level members (i.e., elements of its domain) are canonical metasets, and they correspond to crisp sets by the canonical isomorphism (9). Note, that each canonical metaset is a first order metaset: $\mathfrak{M}^{\mathfrak{c}} \subset \mathfrak{M}^{\mathbf{1}}$.

Lemma 5.14. *Let $\check{\sigma} \in \mathfrak{M}^{\mathfrak{c}}$ and $\tau \in \mathfrak{M}^{\mathbf{1}}$. If $\check{\sigma} \epsilon_p \tau$, for some $p \in \mathbb{T}$, then $\check{\sigma} \in \operatorname{dom}(\tau)$.*

Proof. Let $\mathcal{C}$ be a branch and let $x = \check{\sigma}_{\mathcal{C}}$. Since $\check{\sigma} \in \mathfrak{M}^{\mathfrak{c}}$, then x does not depend on $\mathcal{C}$. By the assumption $\check{\sigma} \epsilon_p \tau$, if $p \in \mathcal{C}$, then $x \in \tau_{\mathcal{C}}$. Since $\operatorname{dom}(\tau) \subset \mathfrak{M}^{\mathfrak{c}}$, then there exists unique $\check{\eta} \in \operatorname{dom}(\tau)$ such, that $x = \check{\eta}_{\mathcal{C}} \in \tau_{\mathcal{C}}$, by Definition 4.3. From (9) it follows, that $\check{\eta} = \check{x} = \check{\sigma}$. Since $\check{\eta} \in \operatorname{dom}(\tau)$, then $\check{\sigma} \in \operatorname{dom}(\tau)$. □

The non-binary comprehension schema for metasets is stated as follows (refer to Definition 4.6 and the preceding discussion for the details concerning the formula Φ).

Theorem 5.15 (Metaset Comprehension Schema)**.** *Let $\Phi(x)$ be a set-theoretic formula with one free variable x and all quantifiers restricted to the class of metasets $\mathfrak{M}$.*

$$\forall_{\check{\Gamma} \in \mathfrak{M}^{\mathfrak{c}}} \exists_{\Lambda \in \mathfrak{M}^{\mathbf{1}}} \forall_{\check{\eta} \in \mathfrak{M}^{\mathfrak{c}}} \forall_{p \in \mathbb{T}} \ \big(\check{\eta} \epsilon_p \Lambda \ \leftrightarrow \ \big[\check{\eta} \epsilon_{\mathbb{1}} \check{\Gamma} \wedge p \Vdash \Phi(\check{\eta})\big]\big) \ . \tag{40}$$

Proof. If $\check{\eta}, \check{\Gamma} \in \mathfrak{M}^{\mathfrak{c}}$ and $\check{\eta} \epsilon_{\mathbb{1}} \check{\Gamma}$, then $\check{\eta} \in \operatorname{dom}(\check{\Gamma})$, by Lemma 5.14. Let $D \subset \operatorname{dom}(\check{\Gamma})$ consist of $\check{\eta}$ such, that $p \Vdash \Phi(\check{\eta})$, for some $p \in \mathbb{T}$. Take $\Lambda = \{ \langle \check{\eta}, p \rangle : \check{\eta} \in D \wedge p \Vdash \Phi(\check{\eta}) \}$. $\Lambda \in \mathfrak{M}^{\mathbf{1}}$, since $\operatorname{dom}(\Lambda) \subset \operatorname{dom}(\check{\Gamma}) \subset \mathfrak{M}^{\mathfrak{c}}$. □

In (39) the set Y is the crisp collection of items holding the property Φ. In (40) the metaset Λ is the vague collection, with non-binary membership of items holding the property Φ to various degrees. The first order metaset Λ is contained in the canonical $\check{\Gamma}$: $\Lambda \mathrel{\overset{\subset}{\sim}}_{\mathbb{1}} \check{\Gamma}$. A member of Λ is a member of $\check{\Gamma}$ too, with the degree of membership in Λ determined by forcing the predicate Φ.

The metaset Λ contains pairs of form $\langle \check{\eta}, p \rangle \in \operatorname{dom}(\Lambda) \times \mathbb{T}$. Λ is not unique with respect to identity $=$, since by Lemmas 4.7,4.8, redundant pairs with comparable conditions may be added or removed. However, it is unique with respect to conditional metaset equality $\approx_p$ for any p, by Theorem 5.8.

As an example, consider $\check{\Gamma} = \{ \check{\imath}^1, \check{\imath}^2 \} \times \{ \mathbb{1} \}$ representing two-day holiday. Let Φ assert that $\check{\imath}^1$ is a nice day after the noon and $\check{\imath}^2$ is nice in the morning (cf. Figure 2). The metaset of nice days, as described by Φ, is $\Lambda = \{ \langle \check{\imath}^1, PM \rangle, \langle \check{\imath}^2, morning \rangle \}$. If other pairs conforming to Φ, like $\langle \check{\imath}^1, afternoon \rangle$, are added to Λ, the resulting metaset is a different set, but it is conditionally equal (under the condition $\mathbb{1}$) to the original Λ, by Theorem 5.8.

The metaset formulation of comprehension schema (40) together with metaset extensionality (36) allow for defining vague sets using predicates satisfied with various degrees. These facts are important for applications of metasets in AI, particularly when large amount of data is generated automatically to produce a metaset based model.

We finish with a brief discussion of uncertainty of membership. Yatabe, Kakuda, and Kikuchi (2003) extended the classical Zermelo-Fraenkel set theory with additional relations for expressing uncertainty of membership and non-membership, however, they are still binary-valued. The metaset approach to uncertainty of membership is developed in Starosta (2010). Since it involves a countable number of membership and non-membership relations to enable gradedness, it is closer to the intuitionistic fuzzy sets approach (Atanassov, 1999), rather than to the mentioned above. It is worthwhile noting here that for the class of metasets which are hereditarily finite sets, the metaset membership complements the metaset non-membership, therefore cancelling the uncertainty (Starosta & Kosiński, 2013). Such metasets were used in examples and are natural in AI applications.

6. Summary

The paper introduced the concept of metaset and two associated techniques: interpretation and forcing. The latter enables evaluating metaset sentences and it was used to define set theoretic relations for metasets. There is an infinite, countable number of relations for denoting different degrees of membership, non-membership, equality, unequality and inclusion of metasets. They are valued in the Boolean algebra called Cohen algebra, or Cantor algebra for finitistic case, and can be evaluated in unit interval. The relations have similar properties to classical set relations. The extensionality principle and comprehension schema holds for metasets with slight adjustments necessary for expressing gradedness.

In Section 1 two categories of approaches to defining imprecise sets were proposed: membership functions and approximation sets. Metasets are defined with yet another method: by $\in$-induction. However, they provide membership functions and approximation sets too. The certainty value (Definition 4.10) of $\mu \in \tau$ determines a membership function on $\mathrm{dom}(\tau)$ valued in unit interval: $\mu \mapsto |\mu \in \tau|$, for $\mu \in \mathrm{dom}(\tau)$. Interpretations of the given metaset are its crisp set approximations. They stand for all the precise views on the imprecise idea represented by the metaset.

Metasets are intended for use in AI for representation of vague notions and reasoning about them. The starting point for the development of this approach was the observation of human perception of vague notions and how one scales their intensities. They are not binary, and rarely is their gradation linearly ordered. Mostly, we – as humans – produce a hierarchy of levels which might be formalised as a partial order, with the imposed formal structure of a lattice or even a Boolean algebra. A metaset facilitates such a structure, giving it a simple representation in the form of the binary tree and making it suitable for straightforward computer implementations (Starosta, 2011), leaving the Boolean approach hidden under the hood.

We demonstrated the decomposition of a vague notion of *Day* into a hierarchy of subconcepts (*morning*, *evening*, etc.), each of which bares a gradation (e.g., *AM* is half of *Day*). This facilitates formal answers to questions of form 'What was the weather yesterday?'. When applying the metaset idea in solving AI problems, one should start with decomposition of some term into a hierarchy of subterms. One of future research goals is to automate the process of splitting a vague notion into a hierarchy of subnotions by applying an algorithm of hierarchical clustering (Wierzchoń & Kłopotek, 2018) to a large scale data set. For further examples and applications the reader is referred to Kacprzak and Starosta (2014), Kacprzak, Starosta, and Węgrzyn-Wolska (2015a, 2015b).

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