Metasets, Intuitionistic Fuzzy Sets and Uncertainty

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Abstract. Metaset is a new concept of set with partial membership relation. It is directed towards computer implementations and applications. The degrees of membership for metasets are expressed as binary sequences and they may be evaluated as real numbers too.
The forcing mechanism discussed in this paper is used to assign certainty values to sentences involving metasets. It turns out, that for a sentence involving finite first order metasets only its certainty value complements the certainty value of its negation. This is not true in general: sentences expressing properties of metasets may have positive uncertainty value. We supply an example of a sentence which is totally uncertain.

Keywords: metaset, partial membership, classical set theory, intuitionistic fuzzy set

1 Introduction

Metaset is a new concept of set with partial membership relation [6]. It is based on the classical set theory [3], [4] and it is directed towards computer implementations. Its scope of practical applications [5] are similar to intuitionistic fuzzy sets [1].

In this paper we investigate certainty values of sentences expressing facts concerning metasets. In particular we focus on their significant feature which is the capability of expressing uncertainty. We present the example of the sentence whose certainty value and the certainty value of its negation are equal 0. The uncertainty value of such sentence is equal 1. We then show that for sentences involving finite first order metasets only, the certainty value complements the certainty value of its negation, i.e., they sum up to unity – the truth value. This means that such sentences admit no uncertainty.

The capability of expressing uncertainty allows for representing intuitionistic fuzzy sets [1] by metasets [7]. By the main result of this paper – which says that the uncertainty value vanishes for finite first order metasets – we claim that we cannot directly represent arbitrary intuitionistic fuzzy sets by metasets in computers, i.e., using finite metasets. However, we can represent [8] finite fuzzy sets [11].

2 Metasets

Informally, a metaset is a set whose elements have associated degrees of membership. We formalize this idea by means of ordered pairs. Each member of a metaset – viewed as a classical set – is encapsulated in an ordered pair. The first element of the pair is the member and the second element is a node of the binary tree, which specifies its degree of membership. For simplicity, we present results for first order metasets here. A generalization is outlined in the section 6.

Definition 1. *A set which is either the empty set $\emptyset$ or which has the form:*

$$\tau = \{ \langle \sigma, p \rangle : \sigma \text{ is a set, } p \in \mathbb{T} \}$$

is called a first order metaset (fo-metaset).

The binary tree $\mathbb{T}$ is the set of all finite binary sequences, i.e., functions whose domains are finite ordinals, valued in 2:[1]

$$\mathbb{T} = \bigcup_{n \in \mathbb{N}} 2^n \ . \tag{1}$$

We define the ordering $\leq$ in the tree $\mathbb{T}$ to be the reverse inclusion of functions seen as sets. Thus, for $p, q \in \mathbb{T}$ such, that $p \colon n \mapsto 2$ and $q \colon m \mapsto 2$, we have $p \leq q$ whenever $p \supseteq q$, i.e., $n \geq m$ and $p_{\restriction m} = q$. The root $\mathbb{1}$ is the largest element of $\mathbb{T}$ in this ordering: it is included in each function and for all $p \in \mathbb{T}$ we have $p \leq \mathbb{1}$.

A *level* in $\mathbb{T}$ is the set of all sequences with the same length. Each level has a number. The level with the number n is the set 2^n. The level 0 consists of the empty sequence $\mathbb{1}$ only.

A *branch* in $\mathbb{T}$ is an infinite binary sequence, i.e., a function $\mathbb{N} \mapsto 2$. A branch intersects all levels in $\mathbb{T}$, and each of them only once.

Nodes of the tree $\mathbb{T}$ are sometimes called *conditions*. If $p \leq q \in \mathbb{T}$, then we say that the condition p is *stronger* than the condition q, and q is *weaker* than p. A stronger condition is meant to designate a stipulation which is harder to satisfy than the one described by a weaker condition. For instance, "very cold" and "slightly cold" are stronger conditions than just "cold", since they carry more information concerning the temperature.

The class of first order metasets is denoted by $\mathfrak{M}^1$. The first element σ of an ordered pair $\langle \sigma, p \rangle$ contained in a fo-metaset τ is called a *potential element* of τ, since it is a member of τ to a degree p which usually is less than certainty. A potential element may be simultaneously paired with multiple different conditions which taken together comprise its membership degree in the fo-metaset. From the point of view of the set theory a fo-metaset is a relation between a crisp set and a set of nodes of the binary tree. Therefore, we adopt the following terms and notation concerning relations. For the given metaset τ, the set of its potential elements:

$$\operatorname{dom}(\tau) = \{ \sigma \colon \exists_{p \in \mathbb{T}} \ \langle \sigma, p \rangle \in \tau \} \tag{2}$$

[1] For $n \in \mathbb{N}$, let $2^n = \{ f \colon n \mapsto 2 \}$ denote the set of all functions with the domain n and the range $2 = \{ 0, 1 \}$ – they are binary sequences of the length n.

is called the *domain* of the metaset τ, and the set:

$$\operatorname{ran}(\tau) = \left\{\, p \colon \exists_{\sigma \in \operatorname{dom}(\tau)} \ \langle \sigma, p \rangle \in \tau \,\right\} \tag{3}$$

is called the *range* of the metaset τ.

The class of finite first ordered metasets is denoted by $\mathfrak{MF}^1$. Metasets in this class are particularly important for computer applications, where representable entities are naturally finite. Thus,

$$\tau \in \mathfrak{MF}^1 \quad \text{iff} \quad |\operatorname{dom}(\tau)| < \aleph_0 \wedge |\operatorname{ran}(\tau)| < \aleph_0 \,. \tag{4}$$

3 Interpretations

An interpretation of a first order metaset is a crisp set extracted out of the metaset by means of a branch in the binary tree. For the given fo-metaset, each branch in $\mathbb{T}$ determines a different interpretation. All the interpretations taken together make up a collection of sets with specific internal dependencies, which represents the fo-metaset by means of its crisp views. In practical applications these particular views are treated as various experts' opinions on some vague term represented by the fo-metaset.

Properties of crisp sets which are interpretations of the given first order metaset determine the properties of the fo-metaset itself. We use the forcing mechanism (sec. 4) for transferring relationships between sets which are interpretations onto the fo-metaset. A good example is the definition of the membership relation which relies on membership among interpretations (sec. 4.2).

Definition 2. *Let τ be a first order metaset and let $\mathcal{C} \subset \mathbb{T}$ be a branch. The set*

$$\tau_{\mathcal{C}} = \{\, \sigma \in \operatorname{dom}(\tau) \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \,\}$$

is called the interpretation of the first order metaset τ given by the branch $\mathcal{C}$.

Any interpretation of the empty fo-metaset is the empty set, independently of the branch. The process of producing the interpretation of a fo-metaset consists in two stages. In the first stage we remove all the ordered pairs whose second elements are conditions which do not belong to the branch $\mathcal{C}$. The second stage replaces the remaining pairs – whose second elements lie on the branch $\mathcal{C}$ – with their first elements. As the result we obtain a crisp set.

A fo-metaset may have multiple different interpretations – each branch in the tree determines one. Usually, many of them are pairwise equal, so the number of different interpretations is much less than the number of branches. Finite fo-metasets always have a finite number of different interpretations. There are metasets whose interpretations are all equal, even when they are not finite.

In this paper we deal with finite first order metasets. For such metasets we consider the greatest level number of the level whose conditions may affect interpretations.

Definition 3. *The deciding level for a finite first order metaset τ, denoted by $\mathfrak{l}_\tau$, is the greatest level number of conditions in* $\operatorname{ran}(\tau)$*:*

$$\mathfrak{l}_\tau = \max \{ \, |p| \colon p \in \operatorname{ran}(\tau) \, \} \ .$$

If $\tau = \emptyset$, then we take $\mathfrak{l}_\tau = 0$.

Since $p \in \mathbb{T}$ is a function, then $|p|$ is its cardinality – the number of ordered pairs which is just the length of the binary sequence p. It is also equal to the level number to which it belongs. Thus, $\mathfrak{l}_\tau$ is the length of the longest sequence in $\operatorname{ran}(\tau)$. Conditions on levels below $\mathfrak{l}_\tau$ do not affect interpretations of τ.

Lemma 1. *Let τ be a finite first order metaset and let $\mathcal{C}'$ and $\mathcal{C}''$ be branches. If initial segments of size $\mathfrak{l}_\tau$ of $\mathcal{C}'$ and $\mathcal{C}''$ are equal:*

$$\forall_{n \leq \mathfrak{l}_\tau} \, \mathcal{C}'(n) = \mathcal{C}''(n) \ ,$$

then $\tau_{\mathcal{C}'} = \tau_{\mathcal{C}''}$.

Proof. Since there are no conditions on levels below $\mathfrak{l}_\tau$ in $\operatorname{ran}(\tau)$, and by the assumption, we obtain $\{ \, \langle \sigma, p \rangle \in \tau \colon p \in \mathcal{C}' \, \} = \{ \, \langle \sigma, p \rangle \in \tau \colon p \in \mathcal{C}'' \, \}$. Therefore, $\tau_{\mathcal{C}'} = \{ \, \sigma \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C}' \, \} = \{ \, \sigma \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C}'' \, \} = \tau_{\mathcal{C}''}$.

4 Forcing

In this section we define and investigate a relation between a condition and a sentence. This relation, called *forcing* relation [2], is designed to describe the level of confidence or certainty assigned to the sentence. The level is evaluated by means of nodes of $\mathbb{T}$. The root condition $\mathbb{1}$ specifies the absolute certainty, whereas its descendants represent less certain degrees. The sentences are classical set theory formulas, where free variables are substituted by fo-metasets and bound variables range over the class of first order metasets.

Given a branch $\mathcal{C}$, we may substitute particular fo-metasets in the sentence $\sigma \in \tau$ with their interpretations which are ordinary crisp sets, e.g.: $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$. The resulting sentence is a set-theory sentence expressing some property of the sets $\tau_{\mathcal{C}}$ and $\sigma_{\mathcal{C}}$, the membership relation in this case. Such sentence may be either true or false, depending on $\tau_{\mathcal{C}}$ and $\sigma_{\mathcal{C}}$.

For the given fo-metaset τ each condition $p \in \mathbb{T}$ specifies a family of interpretations of τ: they are determined by all the branches $\mathcal{C}$ containing this particular condition p. If for each such branch the resulting sentence – after substituting fo-metasets with their interpretations – has the same logical value, then we may think of conditional truth or falsity of the given sentence, which is qualified by the condition p. Therefore, we may consider p as the certainty degree for the sentence.

Let Φ be a formula built using some of the following symbols: variables $(x^1, x^2, \ldots)$, the constant symbol $(\emptyset)$, the relational symbols $(\in, =, \subset)$, logical connectives $(\wedge, \vee, \neg, \rightarrow)$, quantifiers $(\forall, \exists)$ and parentheses. If we substitute each free variable x^i ($i = 1 \dots n$) with some metaset ν^i, and restrict the range of each quantifier to the class of first order metasets $\mathfrak{M}^1$, then we get as the result the sentence $\Phi(\nu^1, \dots, \nu^n)$ of the metaset language, which states some property of the metasets $\nu^1, \dots, \nu^n$. By the *interpretation* of this sentence, determined by the branch $\mathcal{C}$, we understand the sentence $\Phi(\nu^1_{\mathcal{C}}, \dots, \nu^n_{\mathcal{C}})$ denoted shortly with $\Phi_{\mathcal{C}}$. The sentence $\Phi_{\mathcal{C}}$ is the result of substituting free variables of the formula Φ with the interpretations $\nu^i_{\mathcal{C}}$ of the metasets ν^i, and restricting the range of bound variables to the class of all sets $\mathbf{V}$. In other words, we replace the metasets in the sentence Φ with their interpretations. The only constant $\emptyset$ in Φ as well as in $\Phi_{\mathcal{C}}$ denotes the empty set which is the same set in both cases: as a crisp set and as a metaset.

Definition 4. *Let $x^1, x^2, \dots x^n$ be all free variables of the formula Φ and let $\nu^1, \nu^2, \dots \nu^n$ be first order metasets. We say that the condition $p \in \mathbb{T}$ forces the sentence $\Phi(\nu^1, \nu^2, \dots \nu^n)$, whenever for each branch $\mathcal{C} \subset \mathbb{T}$ containing the condition p, the sentence $\Phi(\nu^1_{\mathcal{C}}, \nu^2_{\mathcal{C}}, \dots \nu^n_{\mathcal{C}})$ is true. We denote the forcing relation with the symbol $\Vdash$. Thus,*

$$p \Vdash \Phi(\nu^1, \dots \nu^n) \quad \textit{iff} \quad \textit{for each branch } \mathcal{C} \ni p \textit{ holds } \Phi(\nu^1_{\mathcal{C}}, \dots \nu^n_{\mathcal{C}}) \; .$$

We use the abbreviation $p \nVdash \Phi$ for expressing the negation $\neg(p \Vdash \Phi)$. In such case, not for each branch $\mathcal{C}$ containing p the sentence $\Phi_{\mathcal{C}}$ holds, however, such branches may exist. Furthermore, the symbol $\notin$ in the formula $\mu \notin \tau$ will stand for $\neg(\mu \in \tau)$, and similarly, $\mu \neq \tau$ will stand for $\neg(\mu = \tau)$.

The key idea of the forcing relation lies in transferring properties from crisp sets onto fo-metasets. Let a property described by a formula $\Phi(x)$ be satisfied by all crisp sets of the form $\nu_{\mathcal{C}}$, where ν is a metaset and $\mathcal{C}$ is a branch in $\mathbb{T}$. In other words, $\Phi(\nu_{\mathcal{C}})$ holds for all the sets which are interpretations of the metaset ν given by all branches $\mathcal{C}$ in $\mathbb{T}$. Then we might think that this property also "holds" for the metaset ν, and we formulate this fact by saying that $\mathbb{1}$ forces $\Phi(\nu)$. If $\Phi(\nu_{\mathcal{C}})$ holds only for branches $\mathcal{C}$ containing some condition p, then we might think that it "holds to the degree p" for the metaset ν; we say that p forces $\Phi(\nu)$ in such case. Since we try to transfer – or force – satisfiability of some property from crisp sets onto fo-metasets, we call this mechanism *forcing*.[2] The next example shows how to transfer the property of being equal onto two specific fo-metasets.

The following two lemmas expose the most fundamental and significant features of the forcing relation. The first says that forcing is propagated down the branch, i.e., if a condition p forces Φ, then stronger conditions force Φ too. However, weaker conditions do not have to force it. It should be understood that the stronger conditions carry more detailed information above the weaker ones.

Lemma 2. *Let $p, q \in \mathbb{T}$ and let Φ be a sentence. If p forces Φ and q is stronger than p, then q forces Φ too:*

$$p \Vdash \Phi \wedge q \leq p \quad \rightarrow \quad q \Vdash \Phi \; .$$

[2] This mechanism is similar to, and in fact was inspired by the method of forcing in the classical set theory [2]. It has not much in common with the original.

Proof. If $q \leq p$, then each branch containing q also contains p. If $\mathcal{C}$ is any such branch and $p \Vdash \Phi$, then $\Phi_{\mathcal{C}}$ holds. Because it is true for all $\mathcal{C} \ni q$, then we have $q \Vdash \Phi$.

A finite maximal antichain of conditions stronger than $p \in \mathbb{T}$ propagates forcing upwards to the condition p. A set $R \subset \mathbb{T}$ is called an antichain when all its members are pairwise incomparable. It is a maximal antichain in $\mathbb{T}$, when each $q \in \mathbb{T}$ is comparable to some element of R. It is a maximal antichain below p, when each $q \leq p$ is comparable to some element of R and all the members of R are stronger than p.

Lemma 3. *Let $p \in \mathbb{T}$, $R \subset \mathbb{T}$ and let Φ be a sentence. If R is a finite maximal antichain below p and each $q \in R$ forces Φ, then p also forces Φ.*

Proof. $p \Vdash \Phi$ whenever for each branch $\mathcal{C} \ni p$ holds $\Phi_{\mathcal{C}}$. Since R is a finite maximal antichain whose elements are stronger than p, then each branch containing p must also contain some element $q \in R$. Each such q forces Φ, so for any branch $\mathcal{C} \ni p$ we have $\Phi_{\mathcal{C}}$.

4.1 Forcing and Certainty Degrees

If we treat conditions as certainty degrees for sentences, then the stronger condition specifies the degree which is less than the degree specified by the weaker one (assuming the conditions are different). Indeed, by the above lemmas $r \Vdash \Psi$ is equivalent to the conjunction $r \cdot 0 \Vdash \Psi \wedge r \cdot 1 \Vdash \Psi$ (where $r \cdot 0$ and $r \cdot 1$ denote the direct descendants of r) meaning that the certainty degree specified by r is equal to the "sum" of certainty degrees specified by both $r \cdot 0$ and $r \cdot 1$ taken together. But if it happens that $r \cdot 0 \Vdash \Psi$ and $r \cdot 1 \nVdash \Psi$, then also $r \nVdash \Psi$. In such case the $r \cdot 0$ contributes only a half of the certainty degree specified by r – another half of it could be contributed by $r \cdot 1$, but is not in this case. The root $\mathbb{1}$, being the largest element in $\mathbb{T}$, specifies the highest certainty degree. The ordering of certainty degrees is consistent with the ordering of conditions in $\mathbb{T}$. We stress that the term certainty degree is used informally in this paper. We define now other precise terms for measuring the certainty of sentences.

For the given sentence Φ, the following set $\mathcal{T}_{\Phi}$ is called the *certainty set* for Φ.

$$\mathcal{T}_{\Phi} = \{\, p \in \mathbb{T} \colon p \Vdash \Phi \,\} \ . \tag{5}$$

It contains all the conditions which force the given sentence and it gives a measure of certainty that the sentence is true. Members of this set are called *certainty factors* for Φ. Each certainty factor contributes to the overall degree of certainty that the sentence is true, which is represented by the certainty set.

By the lemma 2, if there exists a $p \in \mathbb{T}$ which forces Φ, then there exist infinitely many other conditions which force Φ too. Among them are all those stronger than p. Therefore, the whole certainty set is equivalent to the set of its maximal elements. Since,

$$p \Vdash \Phi \quad \rightarrow \quad \exists_{q \geq p}\, q \in \max\{\mathcal{T}_{\Phi}\} \wedge q \Vdash \Phi \ , \tag{6}$$

then each $p \in \mathcal{T}_\Phi \backslash \max\{\mathcal{T}_\Phi\}$ is redundant. The substantial information concerning the conditions which force Φ is contained in $\max\{\mathcal{T}_\Phi\}$ exclusively. Forcing of Φ by any stronger conditions may be concluded by applying the lemma 2. Thus we come to the following concept of certainty degree for sentences.

Definition 5. *Let Φ be a sentence. The set of maximal elements of the certainty set for Φ:*

$$\|\Phi\| = \max\{\, p \in \mathbb{T} \colon p \Vdash \Phi \,\}$$

is called the certainty grade for Φ. If the certainty set is empty, then the certainty grade is empty too.

When the certainty set is equal to the whole tree $\mathbb{T}$, then the certainty grade is the singleton containing only the root: $\|\Phi\| = \{\, \mathbb{1} \,\}$. We may evaluate certainty of sentences numerically too.

Definition 6. *Let Φ be a sentence. The following value is called the certainty value for Φ:*

$$|\Phi| = \sum_{p \in \|\Phi\|} \frac{1}{2^{|p|}} \ .$$

One may easily see that whenever no p forces Φ, then $|\Phi| = 0$ and if each $p \in \mathbb{T}$ forces Φ, then $|\Phi| = 1$. Therefore, $|\Phi| \in [0, 1]$.

4.2 Membership and Non-membership

We do not give thorough presentation of relations for metasets in this paper. For completeness, we supply only the definitions of conditional membership and non-membership. Other relations, like conditional equality and non-equality, are defined similarly – by means of the forcing mechanism.

In fact, we define an infinite number of membership relations. Each of them designates the membership satisfied to some degree specified by a node of the binary tree. Moreover, any two fo-metasets may be simultaneously in multiple membership relations qualified by different conditions.

Definition 7. *We say that the metaset μ belongs to the metaset τ under the condition $p \in \mathbb{T}$, whenever $p \Vdash \mu \in \tau$. We use the notation $\mu \epsilon_p \tau$.*

In other words, $\mu \epsilon_p \tau$ whenever for each branch $\mathcal{C} \subset \mathbb{T}$ containing p holds $\mu_\mathcal{C} \in \tau_\mathcal{C}$. The conditional membership reflects the idea that a metaset μ belongs to a metaset τ whenever some conditions are fulfilled. The conditions are represented by nodes of $\mathbb{T}$.

Each $p \in \mathbb{T}$ specifies another relation ϵ_p. Different conditions specify membership relations which are satisfied with different certainty factors. The lemmas 2 and 3 prove that the relations are not independent. For instance, $\mu \epsilon_p \tau$ is equivalent to $\mu \epsilon_{p \cdot 0} \tau \wedge \mu \epsilon_{p \cdot 1} \tau$, i.e., being a member under the condition p is equivalent to being a member under both conditions $p \cdot 0$ and $p \cdot 1$ which are the direct descendants of p.

We introduce another set of relations for expressing non-membership. The reason for this is due to the fact that $p \nVdash \mu \in \tau$ is not equivalent to $p \Vdash \mu \notin \tau$. Indeed, $p \nVdash \mu \in \tau$ means, that it is not true that for each branch $\mathcal{C}$ containing p holds $\mu_{\mathcal{C}} \in \tau_{\mathcal{C}}$, however such branches may exist. On the other hand, $p \Vdash \mu \notin \tau$ means that for each $\mathcal{C} \ni p$ holds $\mu_{\mathcal{C}} \notin \tau_{\mathcal{C}}$. That is why we need another relation "is not a member under the condition p".

Definition 8. *We say that the metaset μ does not belong to the metaset τ under the condition $p \in \mathbb{T}$, whenever $p \Vdash \mu \notin \tau$. We use the notation $\mu \notin_p \tau$.*

Thus, $\mu \notin_p \tau$, whenever for each branch $\mathcal{C}$ containing p the set $\mu_{\mathcal{C}}$ is not a member of the set $\tau_{\mathcal{C}}$. Contrary to the classical case, where a set is either a member of another or it is not at all, for two fo-metasets it is possible that they are simultaneously in different membership and non-membership relations.

For metasets σ, τ, the membership grade of σ in τ is just the certainty grade of the sentence $\sigma \in \tau$, represented by the set $\|\sigma \in \tau\|$. The membership value is $|\sigma \in \tau|$. Similarly, the non-membership grade is $\|\sigma \notin \tau\|$ and non-membership value is $|\sigma \notin \tau|$. The membership and non-membership values, when considered as functions of σ, resemble membership and non-membership functions of an intuitionistic fuzzy set [1]. We now investigate the problem of uncertainty, in particular uncertainty of membership, which is the core of intuitionistic fuzzy set idea.

5 Certainty and Uncertainty

Let $\Phi(x_1, \ldots, x_n)$ be a formula with all free variables shown and let $\mu_1, \ldots, \mu_n$ be finite first order metasets. If we substitute each free variable x_i in the formula Φ with the corresponding metaset μ_i and restrict the range of each quantifier to the class $\mathfrak{MF}^1$ then we call the resulting sentence $\Phi(\mu_1, \ldots, \mu_n)$ a $\mathfrak{MF}^1$-*sentence*.

If a sentence involves metasets which are not finite, then it is possible, that neither the sentence nor its negation is forced by any condition. The following example demonstrates fo-metasets σ, τ such, that both $p \nVdash \sigma \in \tau$ and $p \nVdash \sigma \notin \tau$, for all $p \in \mathbb{T}$. Of course, each interpretation of the sentence is either true or false.

Example 1. Let $\sigma = \{ \langle n, p \rangle : p \in \mathbb{T} \wedge n = \Sigma_{i \in \mathrm{dom}(p)}\, p(i) \}$, $\tau = \{ \langle \mathbb{N}, \mathbb{1} \rangle \}$. Recall, that conditions are functions $p \colon m \mapsto 2$ with domains in $\mathbb{N}$. Each ordered pair in σ is comprised of an arbitrary condition $p \in \mathbb{T}$ and the natural number $n \in \mathbb{N}$, which is equal to the number of occurrences of 1 in the binary representation of p: $n = \Sigma_{i \in \mathrm{dom}(p)}\, p(i)$. In other words

$$\sigma = \{ \langle n, p_n \rangle : n \in \mathbb{N} \text{ and } p_n \text{ has exactly } n \text{ occurrences of } 1 \} .$$

For instance: p_0 may be [0], [00], etc., p_1 may be of form [100], [01], [0010].

If $\mathcal{C}$ is a branch containing a finite number of 1s and infinite number of 0s, i.e., $\Sigma_{i \in \omega} \mathcal{C}(i) = n < \infty$, then $\sigma_{\mathcal{C}} = \{ 0, \ldots, n \}$, so $\sigma_{\mathcal{C}} \notin \tau_{\mathcal{C}} = \{ \mathbb{N} \}$. If, on the other hand, $\mathcal{C}$ contains infinite number of 1s, then $\sigma_{\mathcal{C}} = \mathbb{N}$, since for any $n \in \mathbb{N}$

there exists at least one condition $p_n \in \mathcal{C}$ such, that $n = \Sigma_{i \in \mathrm{dom}(p_n)} p_n(i)$ and $\langle n, p_n \rangle \in \sigma$. In such case we have $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$. Thus, for an arbitrary $p \in \mathbb{T}$ holds $p \nVdash \sigma \in \tau$ as well as $p \nVdash \sigma \notin \tau$, since for $\mathcal{C}$ containing infinitely may 1s the membership holds in interpretations, whereas for the remaining ones – it does not hold.

Let Φ denote the sentence $\sigma \in \tau$. The example shows that although for each branch $\mathcal{C}$ either $\Phi_{\mathcal{C}}$ or $\neg\Phi_{\mathcal{C}}$ holds, the certainty sets for both Φ and $\neg\Phi$ are empty. Therefore, also certainty values $|\Phi|$ and $|\neg\Phi|$ are equal 0. The difference $1 - (|\Phi| + |\neg\Phi|)$ is the measure of uncertainty of the sentence Φ. Since it is equal to 1 in this case, then we say that Φ is totally uncertain – we cannot say anything about truth or falsity of Φ. The example 1 may be modified so, that both certainty values $|\Phi|$, $|\neg\Phi|$, as well as the uncertainty value $1 - (|\Phi| + |\neg\Phi|)$ are positive [7].

We now show that for any $\mathfrak{M}\mathfrak{F}^{\mathbf{1}}$-sentence Φ the certainty value for Φ complements the certainty value for $\neg\Phi$, i.e., their sum is equal to 1. It means that $\mathfrak{M}\mathfrak{F}^{\mathbf{1}}$-sentences admit no uncertainty.

Let $\Phi(x^1, \ldots, x^n)$ be a formula with all free variables shown and let $\tau^i \in \mathfrak{M}\mathfrak{F}^{\mathbf{1}}$, for $i = 1, \ldots, n$. Let $\mathfrak{l}_\Phi$ denote the greatest of the deciding levels of all τ^i:

$$\mathfrak{l}_\Phi = \max \{ \mathfrak{l}_{\tau^i} : i = 1, \ldots, n \} \ . \tag{7}$$

We call $\mathfrak{l}_\Phi$ the *deciding level* for the $\mathfrak{M}\mathfrak{F}^{\mathbf{1}}$-sentence Φ. It has the following property.

Theorem 1. *If Φ is a $\mathfrak{M}\mathfrak{F}^{\mathbf{1}}$-sentence and $\mathfrak{l}_\Phi$ is the deciding level for Φ, then the following holds*

$$p \in 2^{\mathfrak{l}_\Phi} \quad \to \quad p \Vdash \Phi \vee p \Vdash \neg\Phi \ .$$

Proof. Let $\tau^1, \ldots, \tau^n \in \mathfrak{M}\mathfrak{F}^{\mathbf{1}}$ be all fo-metasets occurring in Φ (not bound by quantifiers). Take arbitrary $p \in 2^{\mathfrak{l}_\Phi}$ and let us assume that $p \nVdash \Phi$. By the definition there exists a branch $\mathcal{C} \ni p$ such, that $\neg\Phi_{\mathcal{C}}$ is true. Let $\mathcal{C}'$ be another branch containing p. There are no elements which are less than p in any of the sets $\mathrm{ran}(\tau^i)$, $i = 1, \ldots, n$. Therefore, $\mathcal{C} \cap \mathrm{ran}(\tau^i) = \mathcal{C}' \cap \mathrm{ran}(\tau^i)$ and by the lemma 1 we conclude $\tau^i_{\mathcal{C}} = \tau^i_{\mathcal{C}'}$ for each τ^i. Clearly, $\neg\Phi(\tau^1_{\mathcal{C}}, \ldots, \tau^n_{\mathcal{C}}) \wedge \bigwedge_{i=1}^{i=n} \tau^i_{\mathcal{C}} = \tau^i_{\mathcal{C}'}$ implies $\neg\Phi(\tau^1_{\mathcal{C}'}, \ldots, \tau^n_{\mathcal{C}'})$. Since for each branch $\mathcal{C}' \ni p$ holds $\neg\Phi(\tau^1_{\mathcal{C}'}, \ldots, \tau^n_{\mathcal{C}'})$, then $p \Vdash \neg\Phi$.

Lemma 4. *Let Φ be a $\mathfrak{M}\mathfrak{F}^{\mathbf{1}}$-sentence and let $\mathfrak{l}_\Phi$ be the deciding level for Φ. Let $F_\Phi = \left\{ p \in 2^{\mathfrak{l}_\Phi} : p \Vdash \Phi \right\}$. The following holds:*

$$|\Phi| = \sum_{p \in F_\Phi} \frac{1}{2^{|p|}} \ .$$

Proof. By the definition 6 we have $|\Phi| = \sum_{p \in \|\Phi\|} \dfrac{1}{2^{|p|}}$. If $p \in F_\Phi$, then there exists a $q \in \|\Phi\|$ such, that $p \leq q$. Let $F_\Phi{\restriction}_q = \{ p \in F_\Phi : p \leq q \}$. We claim, that

$$\frac{1}{2^{|q|}} = \sum_{p \in F_\Phi \restriction_q} \frac{1}{2^{|p|}} \ . \tag{8}$$

Indeed, by the lemma 2, $F_\Phi\restriction_q$ contains all the conditions in the deciding level $2^{\mathfrak{l}_\Phi}$, which are stronger than q, since all of them force Φ. Applying the formula $\frac{1}{2^{|p|}} = \frac{1}{2^{|p \cdot 0|}} + \frac{1}{2^{|p \cdot 1|}}$ appropriate number of times we obtain (8). To complete the proof note, that $F_\Phi = \bigcup_{q \in \|\Phi\|} F_\Phi\restriction_q$.

Corollary 1. *If Φ is a $\mathfrak{MF}^1$-sentence, then $|\Phi| + |\neg\Phi| = 1$.*

We may easily calculate certainty values for $\mathfrak{MF}^1$-sentences applying the theorem 1. Let $T_\Phi = \left\{ p \in 2^{l_\Phi} : p \Vdash \Phi \right\}$ and $N_\Phi = \left\{ p \in 2^{l_\Phi} : p \Vdash \neg\Phi \right\}$. By the theorem we have $T_\Phi \cup N_\Phi = 2^{\mathfrak{l}_\Phi}$ – these sets fill the whole deciding level. Since there are $2^{\mathfrak{l}_\Phi}$ elements on the $\mathfrak{l}_\Phi$-th level, then

$$|\Phi| = \frac{|T_\Phi|}{2^{\mathfrak{l}_\Phi}} \quad \text{and} \quad |\neg\Phi| = \frac{|N_\Phi|}{2^{\mathfrak{l}_\Phi}} \ . \tag{9}$$

We apply here lemmas 2, 3 and take into account that $\frac{1}{2^{|p|}} = \frac{1}{2^{|p \cdot 0|}} + \frac{1}{2^{|p \cdot 1|}}$ for any $p \in \mathbb{T}$.

6 Generalization

For the sake of simplicity, we presented results for the class of first order metasets. However, they are valid for metasets in general. Details, as well as other generalizations of these results can be found in [10]. For completeness, we mention the general definition of metaset and interpretation.

Definition 9. *A set which is either the empty set $\emptyset$ or which has the form:*

$$\tau = \{ \langle \sigma, p \rangle : \sigma \textit{ is a metaset, } p \in \mathbb{T} \}$$

is called a metaset.

Formally, this is a definition by induction on the well founded relation $\in$. By the Axiom of Foundation in the Zermelo-Fraenkel set theory (ZFC) there are no infinite branches in the recursion as well as there are no cycles.[3] Therefore, no metaset is a member of itself. From the point of view of ZFC a metaset is a particular case of a $\mathbb{P}$-name (see also [4, Ch. VII, §2] for justification of such type of definitions).

The definition of interpretation for general metasets is recursive too.

Definition 10. *Let τ be a metaset and let $\mathcal{C} \subset \mathbb{T}$ be a branch. The set*

$$\operatorname{int}(\tau, \mathcal{C}) = \{ \operatorname{int}(\sigma, \mathcal{C}) \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \}$$

is called the interpretation of the metaset τ given by the branch $\mathcal{C}$.

[3] The Axiom of Foundation in ZFC says that every non-empty set x contains an element y which is disjoint from x:

$$\forall_{x \neq \emptyset}\, \exists_{y \in x}\, \neg \exists z\, (z \in x \wedge z \in y) \ .$$

The definition 4 of forcing applies without change to metasets in general – the restriction to first order metasets was not really necessary.

With the above general definitions we prove in [10], that for a $\mathfrak{MF}$-sentence Φ the union $\|\Phi\| \cup \|\neg\Phi\|$ is a maximal finite antichain in $\mathbb{T}$. A $\mathfrak{MF}$-sentence differs from a $\mathfrak{MF}^1$-sentence in that all metasets involved are hereditarily finite sets[4] instead of just first order finite. Note, that a maximal finite antichain in $\mathbb{T}$ intersects all branches in the tree, so in such case each branch contains a condition which either forces Φ or $\neg\Phi$. This result is more general and it implies the theorem 1.

7 Metasets and Intuitionistic Fuzzy Sets

If $\sigma, \tau \in \mathfrak{MF}^1$, then the membership value of σ in τ is equal to $|\sigma \in \tau|$ and the non-membership value of σ in τ is equal to $|\sigma \notin \tau|$. By the corollary 1 we know that $|\sigma \in \tau| + |\sigma \notin \tau| = 1$. However, if any of σ, τ is not a finite fo-metaset, then this sum may be less than 1, or even equal 0, like in the example 1. The complement to 1 of this sum: $1 - |\sigma \in \tau| - |\sigma \notin \tau|$, is called the *uncertainty value* of membership. This resembles intuitionistic fuzzy sets [1]. An intuitionistic fuzzy set is a triple $\langle X, \mu, \nu \rangle$, where $\mu\colon X \mapsto [0,1]$ is the membership function and $\nu\colon X \mapsto [0,1]$ is the non-membership function. They satisfy requirement $\mu(x) + \nu(x) \leq 1$, for each $x \in X$. The difference $1 - (\mu(x) + \nu(x))$ is called the *hesitancy degree*. In [7] we demonstrate the method for representing intuitionistic fuzzy sets by means of metasets. For the given intuitionistic fuzzy set $\langle X, \mu, \nu \rangle$ we construct a sequence of metasets $\{\, \rho_x \,\}_{x \in X}$ and an additional metaset Ω such, that $|\rho_x \in \Omega| = \mu(x)$ and $|\rho_x \notin \Omega| = \nu(x)$, for each $x \in X$. We also show how to evaluate the uncertainty grade to obtain the uncertainty value of membership for the metasets $\{\, \rho_x \,\}_{x \in X}$ and Ω. We conclude, that the uncertainty value of membership of ρ_x in Ω is equal $1 - (\mu(x) + \nu(x))$, for each $x \in X$.

By the corollary 1, the metasets ρ_x and Ω cannot be finite first order metasets. Indeed, the uncertainty of membership vanishes for such metasets. Therefore, we conclude that intuitionistic fuzzy sets cannot be directly represented by metasets in computers, where all representable entities are naturally finite.

On the other hand, it is possible to represent ordinary finite fuzzy sets [11] by means of metasets either using the method outlined above [7] and assuming that the hesitancy degree is 0, or with another method introduced in [8].

8 Summary

We have introduced the concept of metaset – set with partial membership relation. We have defined the fundamental techniques of interpretation and forcing and we have shown how to evaluate certainty values for sentences of the metaset language, in particular certainty values of membership and non-membership.

[4] A set is hereditarily finite whenever it is a finite set and all its members are hereditarily finite sets.

We have proved, that for sentences involving finite first order metasets exclusively, the certainty value of a sentence complements the certainty value of its negation. We have demonstrated the example showing, that it is not true in general: a sentence involving infinite metasets may have positive uncertainty value. For sentences expressing membership this resembles the hesitancy degree of intuitionistic fuzzy sets [1].

The class of finite metasets is especially important due to the fact, that metasets implementable in computers are naturally finite. Therefore, the presented results are significant for computer applications of metasets [5].

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