Metasets.
A New Approach to Partial Membership

Bartłomiej Starosta
barstar@pjwstk.edu.pl

Polish-Japanese Institute of Information Technology,
ul. Koszykowa 86,
02-008 Warsaw, Poland

Abstract. Metaset is a new concept of set with partial membership relation. It is designed to represent and process vague, imprecise data – similarly to fuzzy sets. Metasets are based on the classical set theory primitive notions. At the same time they are directed towards efficient computer implementations and applications. The degrees of membership for metasets are expressed as finite binary sequences, they form a Boolean algebra and they may be evaluated as real numbers too. Besides partial membership, equality and their negations, metasets allow for expressing a hesitancy degree of membership – similarly to intuitionistic fuzzy sets. The algebraic operations for metasets satisfy axioms of Boolean algebra.

Keywords: metaset, partial membership, set theory, fuzzy set

1 Introduction

The paper gives a short overview of metaset theory – a new concept of set with fractional members. Contrary to classical sets and similarly to fuzzy sets [14] or rough sets [6], metasets are sets where an element may be a member of another to a variety of degrees, besides the full membership or non-membership. The mentioned above, traditional approaches to partial membership find broad applications nowadays in science and above all in industry. Unfortunately, they are not well suited for computer implementations. They also have other drawbacks, like the growth of fuzziness by multiple algebraic operations on fuzzy sets. Therefore, we tried to develop another idea of set with fractional members, which would be closer to classical Zermelo-Fraenkel Set Theory (ZFC) [5] and which would allow for efficient computer implementations. Another significant goal was to enable natural and straightforward modeling of vague terms as they are perceived and interpreted by a human. Thus, metasets are targeted at similar scope of applications as other traditional approaches. The theory of metasets is under development, however the results obtained so far indicate success. We point out the most significant of them.

2 Metasets

Informally, a metaset is a classical set whose elements are labeled with nodes of the binary tree. The nodes determine the membership degrees of elements in the metaset.

This point of view makes a metaset something similar to a fuzzy set, where the membership function assigns membership degrees to elements of its domain. The most noticeable difference at this point is that elements of a metaset are other metasets, like in the classical set theory, where elements of sets are other sets.

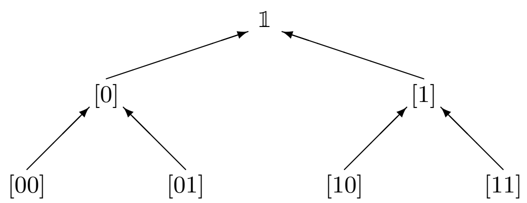

Fig. 1. Initial levels of the binary tree $\mathbb{T}$ and the ordering of nodes. Arrows point at the larger element.

The binary tree used in the definition of the metaset and throughout the paper is the full and infinite one and it is denoted with the symbol $\mathbb{T}$. Its elements are finite binary sequences denoted using square brackets, the root is the empty sequence denoted by $\mathbb{1}$ (see Fig. 1). They are ordered by reverse inclusion, so the root $\mathbb{1}$ is the largest element in $\mathbb{T}$. The nodes [0] and [1], which are direct descendants of the root form the first level of the tree, and so on.

Definition 1. *A set which is either the empty set* $\emptyset$ *or which has the form:*

$$\tau = \{ \langle \sigma, p \rangle : \sigma \text{ is a metaset, } p \in \mathbb{T} \}$$

is called metaset. The $\langle \cdot, \cdot \rangle$ *denotes an ordered pair.*

The definition of metaset is recursive, however, the Axiom of Foundation (Regularity) in ZFC guarantees that there are no infinite branches in the recursion tree – it is founded by the empty set, which is a metaset too.[1]

From the point of view of classical set theory a metaset is a relation, i.e., a set of ordered pairs. The first element of each pair is another metaset – a member, also called a *potential element*, and the second element is a node of the binary tree. A metaset σ which is a potential element of the metaset τ may be paired with several different nodes simultaneously, e.g. $\tau = \{ \langle \emptyset, p \rangle , \langle \emptyset, q \rangle \}$, for $p \neq q$ (cf. the example 1). Thus, a metaset is usually not a function.

[1] Formally, this is a definition by induction on the well founded membership relation $\in$, see [5, Ch. VII, §2] for a justification of such type of definitions.

Since a metaset is a relation, we may adopt some terms and notation connected to relations. For the given metaset τ, the set of its potential elements: $\mathrm{dom}(\tau) = \{\, \sigma \colon \langle \sigma, p \rangle \in \tau \,\}$ is called the *domain* of the metaset τ and the set $\mathrm{ran}(\tau) = \{\, p \colon \langle \sigma, p \rangle \in \tau \,\}$ is called the *range* of the metaset τ. For arbitrary metasets τ and σ the set $\tau[\sigma] = \{\, p \in \mathbb{T} \colon \langle \sigma, p \rangle \in \tau \,\}$ is called the *image* of the metaset τ at the metaset σ. The image $\tau[\sigma]$ is the empty set $\emptyset$, whenever σ is not a potential element of τ.

Example 1. The simplest metaset is the empty set $\emptyset$. It may be a potential element of other metasets:

$$\begin{array}{llll} \tau = \{\, \langle \emptyset, p \rangle \,\} \ , & \tau[\emptyset] = \{\, p \,\} \ , & \mathrm{dom}(\tau) = \{\, \emptyset \,\} \ , & \mathrm{ran}(\tau) = \{\, p \,\} \ , \\ \sigma = \{\, \langle \emptyset, p \rangle \,, \langle \emptyset, q \rangle \,\} \ , & \sigma[\emptyset] = \{\, p, q \,\} \ , & \mathrm{dom}(\sigma) = \{\, \emptyset \,\} \ , & \mathrm{ran}(\sigma) = \{\, p, q \,\} \ . \\ \eta = \{\, \langle \tau, p \rangle \,, \langle \sigma, q \rangle \,\} \ , & \eta[\emptyset] = \emptyset \ , & \mathrm{dom}(\eta) = \{\, \tau, \sigma \,\} \ , & \mathrm{ran}(\eta) = \{\, p, q \,\} \ . \end{array}$$

Clearly, $\eta[\tau] = p$, $\eta[\sigma] = q$ and since $\emptyset \notin \mathrm{dom}(\eta)$, then $\eta[\emptyset] = \emptyset$.

A classical, crisp set is called hereditarily finite when it is a finite set and all its members are hereditarily finite sets.

Definition 2. *A metaset τ is called a hereditarily finite metaset, if its domain and range are finite sets, and each potential element is also a hereditarily finite metaset.*

Hereditarily finite metasets are particularly important in computer applications, where representable entities are naturally finite. They also have some interesting properties indicated in section 5.

3 Interpretations

An interpretation of a metaset is a crisp set. It represents one of several possible crisp views on the metaset. An interpretation is determined by a branch in the tree $\mathbb{T}$. A branch in $\mathbb{T}$ is a maximal (with respect to inclusion) set of pairwise comparable nodes. Note, that p is comparable to q only, if there exists a branch containing p and q simultaneously. Similarly, p is incomparable to q whenever no branch contains both p and q.

Definition 3. *Let τ be a metaset and let $\mathcal{C} \subset \mathbb{T}$ be a branch. The set*

$$\tau_{\mathcal{C}} = \{\, \sigma_{\mathcal{C}} \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \,\}$$

is called the interpretation of the metaset τ given by the branch $\mathcal{C}$.

Any interpretation of the empty metaset is the empty set, independently of the branch: $\emptyset_{\mathcal{C}} = \emptyset$, for each $\mathcal{C} \subset \mathbb{T}$. The process of producing the interpretation of a metaset consists in two stages. In the first stage we remove all the ordered pairs whose second elements are nodes which do not belong to the branch $\mathcal{C}$. The second stage replaces the remaining pairs – whose second elements lie on the branch $\mathcal{C}$ – with interpretations of their first elements, which are other metasets. This two-stage process is repeated recursively on all the levels of the membership hierarchy. As the result we obtain a crisp set.

Example 2. Let $p \in \mathbb{T}$, and let $\tau = \{ \langle \emptyset, p \rangle \}$. If $\mathcal{C}$ is a branch, then

$$p \in \mathcal{C} \to \tau_{\mathcal{C}} = \{ \emptyset_{\mathcal{C}} \} = \{ \emptyset \} \ ,$$
$$p \notin \mathcal{C} \to \tau_{\mathcal{C}} = \emptyset \ .$$

Depending on the branch the metaset τ acquires different interpretations.

Each branch in the binary tree determines an interpretation of a metaset, so there may be infinitely many of them in general. Hereditarily finite metasets always have a finite number of different interpretations. There are metasets whose interpretations are all equal, even when they are not hereditarily finite.

When a metaset represents some vague, imprecise term, then its interpretations represent definite, precise approaches to the term. For instance, if we represent the term "warm temperature" as metaset, then its interpretations might be particular ranges of temperatures. Taken together they form the compound concept of "warm temperature".

The technique of interpretation introduces another point of view on metasets. A metaset may be perceived as a "fuzzy" family of crisp sets which are interpretations of the metaset. Here, the word "fuzzy" means that some of the members of the family – i.e., some interpretations of the metaset – occur more frequently than others. Those which appear frequently are better crisp approaches to metaset.

Properties of crisp sets which are interpretations of the given metaset determine its properties. Basic set-theoretic relations for metasets are defined by referring to the relations among interpretations of the metaset. When thinking about a metaset one has to bear in mind its interpretations.

4 Relations for Metasets

The membership relation for metasets is defined by referring to interpretations. In fact, we define an infinite number of relations, each specifying membership satisfied to another degree. The infinite number of relations allows for expressing a variety of different degrees to which membership may hold using classical two-valued logic.

Definition 4. *Let μ, τ be arbitrary metasets. We say that μ belongs to τ under the condition $p \in \mathbb{T}$, whenever for each branch $\mathcal{C} \subset \mathbb{T}$ containing p holds $\mu_{\mathcal{C}} \in \tau_{\mathcal{C}}$. We use the notation $\mu \epsilon_p \tau$.*

Thus, for each $p \in \mathbb{T}$ we define a separate relation ϵ_p. The root $\mathbb{1}$ specifies the highest possible membership degree. Since two metasets may be simultaneously in multiple membership relations specified by different nodes, then the overall membership degree is determined by a set of nodes of $\mathbb{T}$.

The conditional membership reflects the idea that a metaset μ belongs to a metaset τ whenever some conditions are fulfilled. Conditions correspond to nodes of the binary tree. In applications, they designate various circumstances affecting the degrees to which relations hold. For instance, consider the sentence: *John is happy when it is hot and when it is very cold.* In other words: *John* is a member of the metaset of happy people under the conditions *hot* and *very cold.*

Example 3. Let $\sigma = \emptyset$ and $\tau = \{ \langle\sigma, \mathbb{1}\rangle , \langle\sigma, [0]\rangle \}$. If $\mathcal{C}$ is any branch in $\mathbb{T}$, then $\sigma_{\mathcal{C}} = \emptyset$ and $\tau_{\mathcal{C}} = \{ \sigma_{\mathcal{C}} \} = \{ \emptyset \}$, so $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$. Therefore, $\sigma \;\epsilon_{\mathbb{1}}\; \tau$. Note, that the ordered pair $\langle\sigma, [0]\rangle$ is redundant in τ; it does not supply any additional membership information above the pair $\langle\sigma, \mathbb{1}\rangle$.

Besides the membership we define separate set of non-membership relations.

Definition 5. *We say that the metaset μ does not belong to the metaset τ under the condition $p \in \mathbb{T}$, whenever for each branch $\mathcal{C} \subset \mathbb{T}$ containing p holds $\mu_{\mathcal{C}} \notin \tau_{\mathcal{C}}$. We use the notation $\mu \not\epsilon_p \tau$.*

The reason for introducing independent non-membership relation follows from the fact that negation of conditional membership is not equivalent to conditional non-membership: $\neg\, \mu \;\epsilon_p\; \tau$ is not equivalent to $\mu \not\epsilon_p \tau$. Indeed, the former – by the definition – means that not for each branch $\mathcal{C}$ containing p holds $\mu_{\mathcal{C}} \in \tau_{\mathcal{C}}$. However, such branches may exist, so we cannot conclude that $\mu_{\mathcal{C}} \notin \tau_{\mathcal{C}}$ for each $\mathcal{C} \ni p$, i.e., $\mu \not\epsilon_p \tau$. Because of this $\neg\, \mu \;\epsilon_p\; \tau$ cannot be denoted with $\mu \not\epsilon_p \tau$, as it is in the classical case. Moreover, even though $\neg\, \mu \;\epsilon_p\; \tau$ holds, there still may exist $q \leq p$ such, that for each branch $\mathcal{C}' \ni q$ holds $\mu_{\mathcal{C}'} \in \tau_{\mathcal{C}'}$, so $\mu \;\epsilon_q\; \tau$.

Example 4. Let $\tau = \{ \langle\emptyset, [0]\rangle \}$. We check that $\emptyset \;\epsilon_{[0]}\; \tau \wedge \emptyset \not\epsilon_{[1]} \tau$. Indeed, if $\mathcal{C}^0$ is a branch containing [0], then $\emptyset_{\mathcal{C}^0} = \emptyset \in \{ \emptyset \} = \tau_{\mathcal{C}^0}$. Similarly, if $\mathcal{C}^1$ is a branch containing [1], then $\emptyset_{\mathcal{C}^1} = \emptyset \notin \emptyset = \tau_{\mathcal{C}^1}$. Also, $\neg\, \emptyset \;\epsilon_{\mathbb{1}}\; \tau \wedge \neg\, \emptyset \not\epsilon_{\mathbb{1}} \tau$, since it is not true, that for each branch $\mathcal{C}$ containing $\mathbb{1}$ holds $\emptyset_{\mathcal{C}} \in \tau_{\mathcal{C}}$ or $\emptyset_{\mathcal{C}} \notin \tau_{\mathcal{C}}$.

When $\sigma \;\epsilon_{\mathbb{1}}\; \tau$ (or $\sigma \not\epsilon_{\mathbb{1}} \tau$), then for any branch $\mathcal{C}$ holds $\mu_{\mathcal{C}} \in \tau_{\mathcal{C}}$ (or $\mu_{\mathcal{C}} \notin \tau_{\mathcal{C}}$). Since the membership here is independent of the branch and it holds always, then it naturally reflects the crisp, unconditional membership (or non-membership).

The two sets of conditional relations: membership and non-membership taken together realize fully the idea of "partial" membership; they enable formalization of simultaneous being a member and being not a member. Informally speaking, if some part of μ is outside of τ then – at the same time – another part of μ may be inside of τ. Formally we would write in such case $\mu \;\epsilon_p\; \tau \wedge \mu \not\epsilon_q \tau$, where p and q are some nodes. The above example shows that $\emptyset \;\epsilon_{[0]}\; \tau \wedge \emptyset \not\epsilon_{[1]} \tau$. Note, that $\mu \not\epsilon_p \tau \wedge \mu \;\epsilon_p\; \tau$ is false for any p.

The following two lemmas establish the relationships between different conditional membership (and non-membership) relations. They also enable evaluation of membership and non-membership degrees as real numbers. We must introduce some technical terms before.

A set $A \subset \mathbb{T}$ is called antichain in $\mathbb{T}$, if it consists of mutually incomparable elements: $\forall p, q \in A\ (p \neq q \rightarrow \neg\, (p \leq q) \wedge \neg\, (p \geq q))$. On the Fig. 1, the elements

$\{[00],[01],[10]\}$ form a sample antichain. A maximal antichain is an antichain which cannot be extended by adding new elements – it is a maximal element with respect to inclusion of antichains. Examples of maximal antichains on the Fig. 1 are $\{[0],[1]\}$ or $\{[00],[01],[1]\}$ or even $\{\mathbb{1}\}$. Let $R \subset \mathbb{T}$ and $p \in \mathbb{T}$. If R is an antichain A such that $\forall_{q \in A}\,(q \leq p)$, then we say, that R is an antichain below p.

Lemma 1. *Let σ, τ be arbitrary metasets and let $p, q \in \mathbb{T}$. If $p \leq q$ and $\sigma \epsilon_q \tau$ ($\sigma \not\epsilon_q \tau$), then $\sigma \epsilon_p \tau$ ($\sigma \not\epsilon_p \tau$).*

Lemma 2. *Let σ, τ be arbitrary metasets and let $p, q \in \mathbb{T}$. If $R \subset \mathbb{T}$ is a finite maximal antichain below p such, that for each $q \in R$ holds $\sigma \epsilon_q \tau$ ($\sigma \not\epsilon_q \tau$), then also $\sigma \epsilon_p \tau$ ($\sigma \not\epsilon_p \tau$).*

The lemmas follow directly from the definition of interpretation and membership. For the detailed proofs the reader is referred to [13].

We now show how to evaluate membership and non-membership degrees as numbers from the unit interval. Let σ, τ be metasets. The sets

$$\mathrm{M}(\sigma,\tau) = \max\{\, p \in \mathbb{T} \colon \sigma \epsilon_p \tau \,\}\ , \tag{1}$$

$$\mathrm{N}(\sigma,\tau) = \max\{\, p \in \mathbb{T} \colon \sigma \not\epsilon_p \tau \,\}\ . \tag{2}$$

are called membership and non-membership set, respectively. One may easily see that both M and N are antichains. By the above lemmas, the whole membership (non-membership) information for any two metasets is contained in these sets. Therefore, we may use them to evaluate relations numerically as follows:

$$\mathrm{m}(\sigma,\tau) = \sum_{p \in \mathrm{M}(\sigma,\tau)} \frac{1}{2^{|p|}}\ , \tag{3}$$

$$\mathrm{n}(\sigma,\tau) = \sum_{p \in \mathrm{N}(\sigma,\tau)} \frac{1}{2^{|p|}}\ , \tag{4}$$

where $|p|$ denotes the length of the binary sequence p. The value $\mathrm{m}(\sigma,\tau)$ ($\mathrm{n}(\sigma,\tau)$) is called the membership (non-membership) value for σ in τ. Clearly, the values fit into the unit interval.

Strangely enough, there exist metasets σ, τ such, that $\mathrm{m}(\sigma,\tau) + \mathrm{n}(\sigma,\tau) < 1$. The remaining difference $1 - \mathrm{m}(\sigma,\tau) - \mathrm{n}(\sigma,\tau)$ is interpreted as hesitancy degree of membership for metasets (cf. Th. 2). Such behavior resembles intuitionistic fuzzy sets, where besides membership and non-membership degrees we also have a hesitancy degree [1]. Also, this property allows for representing intuitionistic fuzzy sets as metasets [10].

Although a metaset is not a function, it determines a function which assigns membership degrees to elements of its domain, similarly to fuzzy sets. The range of this membership function is the Boolean algebra of closed-open sets in Cantor space 2^ω. Indeed, each node $p \in \mathbb{T}$ determines a set of branches containing it, which is a closed-open set in this Cantor space. For the given metasets τ and $\sigma \in \mathrm{dom}(\tau)$ the value of this membership function is the clopen set in 2^ω which is the union of the sets determined by elements of $\tau[\sigma]$ or – equivalently – by elements of the membership set $\mathrm{M}(\sigma, \tau)$. This function makes metasets similar to L-fuzzy sets whose membership functions are valued in lattices [3].

Analogously to membership and non-membership we define sets of conditional equality, unequality (i.e., negation of equality) and subset relations. They are consistent with partial membership and have similar properties to their classical counterparts (e.g., extensionality). They are investigated in [8].

5 Metasets and Computers

The concept of metaset is directed towards computer implementations and applications. The definitions of set-theoretic relations for computer representable metasets may be reformulated so that they are easily and efficiently implementable in computer languages. We now give an example of reformulation of the membership relation.

A metaset σ is called a *canonical* metaset if $\mathrm{ran}(\sigma) = \{ \mathbb{1} \}$ and its domain includes canonical metasets only. In other words, its range and the ranges of its members on all the levels of membership hierarchy contain at most the root $\mathbb{1}$. Such metasets correspond to crisp sets, since the membership relation is two-valued for them. Metasets, whose domains are comprised of canonical metasets only, but their ranges are arbitrary are called *first order* metasets. They correspond to fuzzy sets, where the structure of elements is irrelevant and only the membership of elements matters. Canonical metasets are members of first order metasets, to various degrees.

Theorem 1. *Let σ be a hereditarily finite canonical metaset and let τ be a hereditarily finite first order metaset. For any $p \in \mathbb{T}$, the following are equivalent:*

a) σ belongs to τ under the condition p ($\sigma \epsilon_p \tau$),
b) $\tau[\sigma]$ contains a finite maximal antichain below p, or it contains a node $q \geq p$.

Applying the above theorem we do not have to investigate all possible interpretations to verify the membership. The number of such interpretations may be infinite, what makes the process inapplicable for machines. The theorem delegates the membership problem to relationships between finite subsets of $\mathbb{T}$. Similarly, we may reformulate other relations. For the details, as well as the proof of the theorem, the reader is referred to [13].

It turns out, that metasets representable in machines have many additional interesting properties [12]. One of the most significant says that the membership degree complements the non-membership degree. In terms of real values it may be expressed as follows.

Theorem 2. *If σ and τ are hereditarily finite metasets, then*

$$\mathrm{m}(\sigma, \tau) + \mathrm{n}(\sigma, \tau) = 1 \, .$$

This means, that for such metasets the hesitancy degree disappears. In general, it is possible to construct metasets σ, τ such, that for all $p \in \mathbb{T}$ neither $\sigma \epsilon_p \tau$ nor $\sigma \notin_q \tau$ holds. In such case $\mathrm{m}(\sigma, \tau) + \mathrm{n}(\sigma, \tau) = 0$ and the hesitancy degree is equal to 1. Similarly, one may construct σ, τ such, that $\mathrm{m}(\sigma, \tau) + \mathrm{n}(\sigma, \tau)$ is equal to some arbitrary given value from the unit interval [10].

For the class of hereditarily finite first order metasets – the ones which are represented in computers and are sufficient for most applications – it is possible to define algebraic operations. The definitions rely on relationships between various subsets of $\mathbb{T}$ and they do not involve interpretations, like in Theorem 1. Therefore, they are easy to implement. Algebraic operations for the first order metasets satisfy axioms of Boolean algebra [13]. Contrary to algebraic operations for fuzzy sets, repeatedly applied operations do not increase fuzziness and their ordering does not matter.

The experimental implementation of relations and algebraic operations for metasets was carried out in Java programming language. It was then used in an application for character recognition which is available on-line as Java applet [7]. The mechanism used to match character samples against a defined character pattern is entirely based on metasets. It utilizes the concept of interpretation for representing several character samples as a single entity – a metaset. Membership relation is interpreted as similarity of characters. The application seems to reflect the human perception of similar characters. This construction may be further developed to recognition of arbitrary data with graphical representation [11], [9].

6 Summary

The paper presents the current state of development of the metaset theory. Metasets enable expressing satisfaction of basic set-theoretic relations to a variety of degrees which form a Boolean algebra. Even though the theory of metasets may seem a purely abstract mathematical construction resembling its basis – the Zermelo-Fraenkel Set Theory – it is aimed at practical applications and particularly at computer implementations. It is a tool for modeling imprecise real life phenomena which are hardly representable using classical, crisp techniques. One of its advantages in this respect is the non-linear ordering of membership and equality degrees which facilitates better, more accurate representation of modeled reality and which is closer to human perception and evaluation of most vague terms.

It is worth stressing that besides the results mentioned here the notions of cardinality and equinumerosity for metasets are defined in the form allowing for straightforward algorithmization and they will be published soon. Future works on metasets focus on fast computer implementation of metasets relations and operations using the CUDA technology [4]. Another goal is defining a many-valued logic [2] for metasets based on the technique of metaset forcing [8,12]. It would allow for expressing partial membership using the language similar to the classical set theory using single relational symbol for membership.

References

1. Atanassov, K.T.: Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems 20, 87–96 (1986)
2. Bolc, L., Borowik, P.: Many-valued Logics 1: Theoretical Foundations, Springer Verlag (1992).
3. Goguen, J.: L-fuzzy Sets. Journal of Mathematical Analysis and Applications 18, 145–174 (1967)
4. Hwu, W.W.: GPU Computing Gems Emerald Edition. Applications of GPU Computing, Morgan Kaufmann (2011)
5. Kunen, K.: Set Theory, An Introduction to Independence Proofs. No. 102 in Studies in Logic and Foundations of Mathematics, North-Holland Publishing Company (1980)
6. Pawlak, Z.: Rough Sets. International Journal of Computer and Information Sciences 11, 341–356 (1982).
7. Starosta, B.: Character Recognition Java Applet, http://www.pjwstk.edu.pl/~barstar/Research/MSOCR/index.html
8. Starosta, B.: Partial Membership and Equality for Metasets. Fundamenta Informaticae, in review
9. Starosta, B.: Application of Metasets to Character Recognition. In: Proc. of 18th International Symposium, ISMIS 2009. Lecture Notes in Artificial Intelligence, vol. 5722, pp. 602–611 (2009)
10. Starosta, B.: Representing Intuitionistic Fuzzy Sets as Metasets. In: Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Volume I: Foundations. pp. 185–208 (2010)
11. Starosta, B.: Document Recognition and Understanding, chap. Character Recognition with Metasets, pp. 15–34. INTECH (2011), http://www.intechopen.com/articles/show/title/character-recognition-with-metasets
12. Starosta, B., Kosiński, W.: Forcing for Computer Representable Metasets, under preparation
13. Starosta, B., Kosiński, W.: Views on Fuzzy Sets and Systems from Different Perspectives. Philosophy and Logic, Criticisms and Applications, Studies in Fuzziness and Soft Computing, vol. 243, chap. Meta Sets. Another Approach to Fuzziness, pp. 509–522. Springer Verlag (2009)
14. Zadeh, L.A.: Fuzzy Sets. Information and Control 8, 338–353 (1965)