Representing Intuitionistic Fuzzy Sets as Metasets

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Abstract

Metaset is a new concept of set with partial membership relation. Although based on the classical set theory, the metaset theory is directed towards computer implementations and applications due to computer oriented definitions of basic relations and algebraic operations. The degrees to which membership, non-membership and uncertainty relations for metasets are satisfied, are represented by sets of nodes of the binary tree.

In this paper we focus on the representation of intuitionistic fuzzy sets by means of metasets. In particular we show how to represent an uncertainty degree by means of two metasets. Also, we define a numerical evaluation of degrees represented by sets of nodes.

As the main result we construct a family of metasets that correspond to elements of the given intuitionistic fuzzy set. Their uncertainty, non-membership and membership degrees to another dedicated metaset, evaluated as real numbers, are equal to the degrees of corresponding elements of the intuitionistic fuzzy set.

Keywords: Metasets, fuzzy sets, intuitionistic fuzzy sets.

1 Introduction

Metaset is a new concept of set with partial membership relation which is strictly based on the classical set theory. In particular, "elements" of metasets are also metasets. The language of metasets resembles the language of the Zermelo–Fraenkel set theory [2] and many properties of crisp sets are reflected in the metasets theory. However, there is a countable number of relational symbols (membership, non-membership, equality, etc.) which enable expressing various degrees to which relations may be satisfied.

On the other hand, one of the most significant characteristics of metaset theory are computer oriented definitions of basic relations and algebraic operations for metasets (we do not include the computer-oriented formulations in this paper – the interested reader is referred to [3]). The important advantage of this approach is that implementations of computer algorithms for deciding basic relations (membership, equality, etc.) and processing fuzzy data represented by means of metasets should be accurate and above all highly efficient.

Metasets – similarly to fuzzy sets – are means for representing rough, inaccurate data or collections of some entities. However – as opposed to fuzzy sets – these entities are also metasets. Thus, using metasets we may model an imprecise collection comprised of imprecise elements.

A metaset is a set, which is not completely precised, but – potentially – it might be precised in various ways. It might acquire various particular representations, which are ordinary crisp sets, depending on external circumstances. These external circumstances are formalized as interpretations of the metaset determined by branches of the binary tree. The properties of the crisp sets which are interpretations of a metaset determine the properties of the metaset itself. In particular they enable transferring of basic relations from crisp sets to metasets.

Membership degrees in metasets are expressed in terms of sets of nodes of the binary tree. These subsets may be evaluated as real numbers from the unit interval. This feature enables representation of fuzzy sets and intuitionistic fuzzy sets within metasets, what is the main topic of this paper.

2 Metasets

We introduce now fundamental concepts of the metasets theory. We focus here only on those ideas which are directly relevant to further discussion and presentation of the main result. For the detailed treatment of the metaset theory the reader is referred to [3] and [4]. We start with establishing some well known terms and notation.

2.1 The Binary Tree $\mathbb{T}$

We denote the full and infinite binary tree (see Fig. 1) with the symbol $\mathbb{T}$. Nodes of the tree are finite binary sequences denoted using square brackets, e.g.: [0], [01], [010]. The root node, which is the empty sequence, is denoted

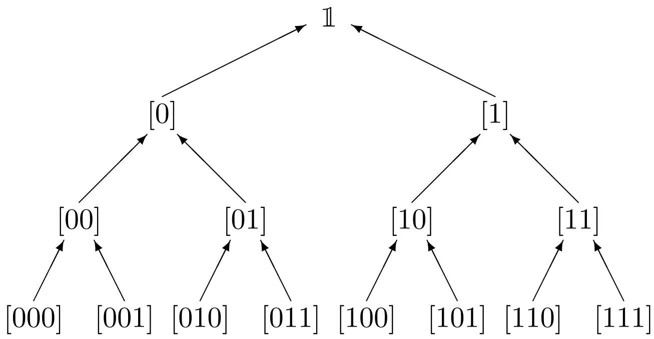

Figure 1: Ordering of nodes in the binary tree $\mathbb{T}$. Arrows point at the larger element.

with the symbol $\mathbb{1}$. The ordering of nodes in $\mathbb{T}$ is determined by inclusion of sequences: a prefix of a sequence is a larger node then the sequence itself, for instance $[0] \geq [00]$. Thus, the root $\mathbb{1}$ is the largest element. Sequences (nodes) $p \neq q \in \mathbb{T}$ of equal length are incomparable, what is denoted with the symbol $p \perp q$. If p and q are comparable we write $p \top q$. For a node $p \in \mathbb{T}$ the notation $|p|$ means the numerical value of p treated as the binary sequence representing a natural number. In the case of the empty sequence $\mathbb{1}$ we assume $|\mathbb{1}| = 0$. For instance $|[0]| = 0$, $|[1]| = 1$ and $|[11]| = 3$. We use the operation symbol $+$ as the concatenation operator for binary sequences or nodes. For instance $[00]+[10]$ is the node $[0010]$, and $p+q$ is the sequence starting with p and ending with q. Also, for $q \in \mathbb{T}$ we denote the number of occurrences of 1 in the sequence q by $\overset{1}{q}$.

The level n in the tree $\mathbb{T}$, denoted with the symbol $\mathbb{T}_n$, is the set of all binary sequences of the same length n. For instance, the level 0 contains only the root $\mathbb{1}$, the level 1 consists of two sequences: $[0]$ and $[1]$. Nodes within a level $\mathbb{T}_n$, for $n > 0$, may be ordered using another ordering than the tree ordering. If we interpret binary sequences as numbers, then the ordering of these sequences is induced by the ordering of natural numbers. For instance nodes on the level 2 are ordered as follows: $[00] \prec [01] \prec [10] \prec [11]$, since $|[00]| < |[01]| < |[10]| < |[11]|$. We will refer to this ordering as *level ordering* and denote it with the symbol $\prec$. The symbol $\#p$ denotes the level of $\mathbb{T}$ to which the node p belongs; it is the length of the binary sequence p.

A set $C \subset \mathbb{T}$ is called a *chain* in $\mathbb{T}$, if for all $p, q \in C$ holds $p \top q$. A set $A \subset \mathbb{T}$ is called *antichain* in $\mathbb{T}$, if for all $p, q \in A$ holds $p \neq q \rightarrow p \perp q$.

Thus, a chain consists of pairwise comparable nodes, whereas an antichain consists of mutually incomparable nodes. The empty set $\emptyset$ is a chain, as well as an antichain. On the Fig. 1, the elements [00], [01], [100] form a sample antichain. A *maximal antichain* is an antichain which cannot be extended by adding new elements – it is a maximal element with respect to inclusion of antichains. Examples of maximal antichains on the Fig. 1 are $\{ [0], [1] \}$ or $\{ [00], [01], [1] \}$ or even $\{ \mathbb{1} \}$. A *branch* is a maximal chain in the tree $\mathbb{T}$. For instance, the elements, [0], [01], [010] on the Fig. 1 form an initial segment of a sample branch (which always has infinite number of elements). We say that the branch $\mathcal{C}$ contains a node p whenever $p \in \mathcal{C}$. Note that $p \top q$ only, if there exists a branch containing p and q simultaneously. Similarly, $p \perp q$ whenever no branch contains both p and q. Sometimes we will treat a branch as an infinite binary sequence, in which case its elements are its prefixes.

2.2 Fundamental Definitions

Informally, a metaset might be perceived as a collection of other metasets, where each element is decorated with a label in form of a node of the binary tree $\mathbb{T}$. This collection does not necessarily have to be a set, since it might contain multiple occurrences of elements. If an element occurs more than once in the collection, then each occurrence must be labeled with different label.

More precisely, a metaset is a relation between a crisp set of other metasets and the set of nodes of the binary tree $\mathbb{T}$.

Definition 1 *A metaset is a crisp set which is either the empty set $\emptyset$, or which has the form:*

$$\tau = \{ \langle \sigma, p \rangle \mid \sigma \text{ is a metaset, } p \in \mathbb{T} \} \ .$$

Here $\langle \cdot, \cdot \rangle$ denotes an ordered pair.

The definition is recursive, however, recursion stops at the empty set, just like it is the case for crisp sets. Formally, this is a definition by induction on the well founded relation $\in$. A justification for such type of definition is presented in [2, Ch. VII, §2]. Since a metaset is a relation it is natural to consider its domain and range.

Definition 2 *The domain of a metaset τ is the following set:*

$$\operatorname{dom}(\tau) = \{ \sigma \mid \langle \sigma, p \rangle \in \tau \} \ .$$

Definition 3 *The range of the metaset* τ *is the set:*

$$\mathrm{ran}(\tau) = \{\, p \mid \langle \sigma, p \rangle \in \tau \,\} \ .$$

Thus, the domain of a metaset is the domain of the relation which the metaset is. According to this we easily see that:

$$\tau \subset \mathrm{dom}(\tau) \times \mathrm{ran}(\tau) \subset \mathrm{dom}(\tau) \times \mathbb{T} \ . \tag{1}$$

Elements of $\mathrm{dom}(\tau)$ are called *potential elements* of the metaset τ. They are called potential, since they belong to the metaset to some degree which usually is less than certainty. In many simple cases this degree is represented by the following set of nodes. The general definition of partial membership relation is presented in the sequel.

Definition 4 *Let* τ *and* σ *be arbitrary metasets. The set*

$$\tau[\sigma] = \{\, p \in \mathbb{T} \mid \langle \sigma, p \rangle \in \tau \,\}$$

is called the image of the metaset τ *at the metaset* σ*.*

Of course, if $\sigma \in \mathrm{dom}(\tau)$ then $\tau[\sigma]$ is never empty. The image $\tau[\sigma]$ is empty whenever σ is not a potential element of τ. We can easily see that:

$$\mathrm{ran}(\tau) = \bigcup_{\sigma \in \mathrm{dom}(\tau)} \tau[\sigma] \ , \tag{2}$$

$$\tau = \bigcup_{\sigma \in \mathrm{dom}(\tau)} \{\, \sigma \,\} \times \tau[\sigma] \ . \tag{3}$$

Example 1 *If* $p \in \mathbb{T}$*, then* $\tau = \{\, \langle \emptyset, p \rangle \,\}$ *is the simplest example of a non-trivial metaset. It has the single potential element which is the empty set. For the given* $q \in \mathbb{T}$ *such, that* $q \neq p$*, we may build another metaset:* $\sigma = \{\, \langle \emptyset, p \rangle , \langle \emptyset, q \rangle \,\}$*. Note, that* $\mathrm{dom}(\tau) = \mathrm{dom}(\sigma) = \{\, \emptyset \,\}$*. It is clear that* $\tau[\emptyset] = \{\, p \,\}$ *and* $\sigma[\emptyset] = \{\, p, q \,\}$*.*

We introduce now the very important class of metasets which correspond to crisp sets.

Definition 5 *A metaset* $\check{\tau}$ *is called a canonical metaset, if it is the empty set, or if it has the form:*

$$\check{\tau} = \{\, \langle \check{\sigma}, \mathbb{1} \rangle \mid \check{\sigma} \textit{ is a canonical metaset} \} \ .$$

We denote the class of canonical metasets with the symbol $\mathfrak{M}^{\mathfrak{c}}$, while the class of all metasets is denoted with $\mathfrak{M}$. Thus, a canonical metaset is a metaset whose domain includes only canonical metasets or is empty, and whose range $\operatorname{ran}(\check{\tau}) \subset \{ \mathbb{1} \}$ contains at most one element $\mathbb{1} \in \mathbb{T}$ which is the root of the tree $\mathbb{T}$.

The internal structure of a canonical metaset resembles the structure of a crisp set. If we remove second elements of each ordered pair and the pairs themselves, on each level of the membership hierarchy, leaving only the first elements, then we obtain a crisp set. We make this idea precise in the next section. Similarly, given a crisp set x, we may construct a canonical metaset corresponding to it by decorating each element on each level of membership hierarchy with the root of the tree $\mathbb{T}$. Thus, we see that there exists a natural isomorphism between canonical metasets and crisp sets. For the given crisp set X, this isomorphism labels each element $x \in X$ with the node $\mathbb{1}$ creating an ordered pair $\langle \check{x}, \mathbb{1} \rangle$ which becomes an element of the canonical metaset $\check{X}$. This process must be repeated recursively on all levels of the membership hierarchy in X to satisfy the stipulation that the potential elements of $\check{X}$ are canonical metasets themselves. Hence if $X = \{ x_i \}_{i \in I}$, then $\check{X} = \{ \langle \check{x}_i, \mathbb{1} \rangle \}_{i \in I}$. We may treat the symbol $\check{}$ as a one-argument operator which transforms the given crisp set X into the corresponding canonical metaset $\check{X}$ by labelling elements of X at all levels of the membership hierarchy with $\mathbb{1}$. If we denote the universe of all crisp sets with the letter $\mathbf{V}$, then the *canonical isomorphism* $\check{} : \mathbf{V} \mapsto \mathfrak{M}^{\mathfrak{c}}$ is defined by induction on the membership relation as follows:

$$\check{} : \quad \emptyset \quad \mapsto \quad \check{\emptyset} \,, \tag{4}$$

$$\check{} : \quad \{ x_i \}_{i \in I} \quad \mapsto \quad \{ \langle \check{x}_i, \mathbb{1} \rangle \}_{i \in I} \;. \tag{5}$$

Example 2 *In the classical set theory the natural (finite ordinal) numbers are defined with the formula $s(n) = n \cup \{ n \}$, where $s(n)$ is the successor of n. For instance:*

$$\begin{aligned} 0 &= \emptyset \,, \\ 1 &= \{ 0 \} = \{ \emptyset \} \;, \\ 2 &= \{ 0, 1 \} = \{ \emptyset, \{ \emptyset \} \} \;, \\ &\vdots \\ n &= \{ 0, 1, \ldots n-1 \} = n-1 \cup \{ n-1 \} \;. \end{aligned}$$

We may construct canonical metasets corresponding to natural numbers.

$$\begin{aligned}
\check{0} &= \emptyset \,, \\
\check{1} &= \{ \langle \check{0}, \mathbb{1} \rangle \} = \{ \langle \emptyset, \mathbb{1} \rangle \} \,, \\
\check{2} &= \{ \langle \check{0}, \mathbb{1} \rangle , \langle \check{1}, \mathbb{1} \rangle \} = \{ \langle \emptyset, \mathbb{1} \rangle , \langle \{ \langle \emptyset, \mathbb{1} \rangle \} , \mathbb{1} \rangle \} \,, \\
&\vdots \\
\check{n} &= \{ \langle \check{0}, \mathbb{1} \rangle , \ldots , \langle n \check{-} 1, \mathbb{1} \rangle \} = n \check{-} 1 \cup \{ \langle n \check{-} 1, \mathbb{1} \rangle \} \,.
\end{aligned}$$

Left hand side of each equality defines a new symbol corresponding to the canonical counterpart of a natural number.

2.3 Interpretations

An interpretation of a metaset is a crisp set. It represents some point of view on the metaset. An interpretation of a metaset is determined by a branch in the tree $\mathbb{T}$. Each metaset may have many different interpretations. Their properties imply the properties of the metaset.

Definition 6 *Let τ be a metaset and let $\mathcal{C} \subset \mathbb{T}$ be a branch. The interpretation of the metaset τ, given by the branch $\mathcal{C}$, is the crisp set:*

$$\tau_{\mathcal{C}} = \{ \sigma_{\mathcal{C}} \mid \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \} \,.$$

Informally, the process of generating the interpretation of the metaset τ involves two stages. In the first stage we remove all the ordered pairs, whose second elements do not belong to the given branch $\mathcal{C}$. The second stage replaces the remaining pairs with their first elements which are other metasets. This two-stage process is repeated recursively on all levels of the membership hierarchy: it is applied to potential elements of τ, their potential elements and so on. As the result we obtain a crisp set $\tau_{\mathcal{C}}$.

The idea behind the interpretation technique is that it allows to view a metaset as a "fuzzy" family of crisp sets. The family consists of all interpretations of the metaset: $\{ \tau_{\mathcal{C}} \mid \mathcal{C}$ is a branch in $\mathbb{T} \}$. It might be treated as "fuzzy" since some elements of the family, i.e., particular interpretations of the metaset, may occur more frequently than others, so they are members of the family to a larger degree than others. This idea becomes clear in the next section, where we define the membership relation for metasets by means of interpretations.

The interpretation technique allows to define basic set-theoretic relations and other properties for metasets so that they are consistent with similar relations and properties for crisp sets.

Example 3 *For an arbitrary branch $\mathcal{C} \subset \mathbb{T}$:*

$$\begin{aligned}
\emptyset_{\mathcal{C}} &= \emptyset = 0 \,, \\
\check{1}_{\mathcal{C}} &= \{ \langle \emptyset, \mathbb{1} \rangle \}_{\mathcal{C}} = \{ \emptyset \} = 1 \,, \\
\check{2}_{\mathcal{C}} &= \big\{ \langle \emptyset, \mathbb{1} \rangle, \langle \{ \langle \emptyset, \mathbb{1} \rangle \} , \mathbb{1} \rangle \big\}_{\mathcal{C}} = \{ \emptyset, \{ \emptyset \} \} = \{ 0, 1 \} = 2 \,. \\
&\vdots \\
\check{n}_{\mathcal{C}} &= \big\{ \langle \check{0}, \mathbb{1} \rangle , \ldots , \langle n \check{-} 1, \mathbb{1} \rangle \big\}_{\mathcal{C}} = \{ 0, 1, \ldots n - 1 \} = n \,.
\end{aligned}$$

Indeed, $\mathbb{1} \in \mathcal{C}$ for any branch $\mathcal{C}$.

Interpretations of the given canonical metaset are independent of the chosen branch $\mathcal{C}$: for all branches they are pairwise equal crisp sets.

Proposition 1 *If $\mathcal{C}'$ and $\mathcal{C}''$ are different branches and $\check{\tau}$ is a canonical metaset, then:*

$$\check{\tau}_{\mathcal{C}'} = \check{\tau}_{\mathcal{C}''} \,.$$

Moreover, any interpretation of a canonical metaset determines a reverse transformation to the canonical isomorphism.

Proposition 2 *Let x be a crisp set and let $\check{x}$ be its canonical counterpart. For any branch $\mathcal{C}$:*

$$\check{x}_{\mathcal{C}} = x \,.$$

The above propositions do not hold for metasets which are not canonical. Generally, a given metaset σ may have many different interpretations which are different crisp sets, depending on the branch. The following example illustrates this.

Example 4 *Let $p, q \in \mathbb{T}$ be incomparable, for instance: $p = [01]$, $q = [00]$. Further, let*

$$\sigma = \big\{ \langle \check{1}, p \rangle , \langle \check{2}, q \rangle \big\} \,.$$

If $\mathcal{C}$ is a branch, then we may easily see that:

$$\begin{aligned}
p \in \mathcal{C} &\quad \rightarrow \quad \sigma_{\mathcal{C}} = \{ 1 \} \,, && (\text{since } q \notin \mathcal{C}) \\
q \in \mathcal{C} &\quad \rightarrow \quad \sigma_{\mathcal{C}} = \{ 2 \} \,, && (\text{since } p \notin \mathcal{C}) \\
p \notin \mathcal{C} \wedge q \notin \mathcal{C} &\quad \rightarrow \quad \sigma_{\mathcal{C}} = 0 = \emptyset \,. && (\text{in this case } [1] \in \mathcal{C})
\end{aligned}$$

The above three cases are mutually exclusive: since $p \perp q \perp [1]$, then these nodes cannot lie on the same branch.

2.4 Basic Relations

We define the conditional membership relation for metasets so that it is possible to express partial membership of a metaset to another metaset. In fact, we define a countable number of relations for expressing different degrees of membership. This allows for using classical two-valued logic for expressing almost all subtleties concerning partial membership relation for metasets.

Definition 7 *Let $p \in \mathbb{T}$ and let τ, σ be metasets. We say that σ belongs to τ under the condition p, if for each branch $\mathcal{C}$ containing p holds $\sigma_\mathcal{C} \in \tau_\mathcal{C}$. In such case we use the notation $\sigma \epsilon_p \tau$.*

If $\sigma \epsilon_p \tau$ and $p = \mathbb{1}$, then $\sigma_\mathcal{C} \in \tau_\mathcal{C}$ for any branch $\mathcal{C}$. In such case we omit the subscript $_\mathbb{1}$ and use the notation $\sigma \epsilon \tau$.

What does the formula $\neg\, \sigma \epsilon_p \tau$ mean? By the definition it is not true, that for all branches $\mathcal{C} \ni p$ holds $\sigma_\mathcal{C} \in \tau_\mathcal{C}$. So there might exist branches such, that $\sigma_\mathcal{C} \notin \tau_\mathcal{C}$ There even may exist nodes $q \leq p$ such, that for all branches $\mathcal{C}'$ containing q holds $\sigma_{\mathcal{C}'} \notin \tau_{\mathcal{C}'}$. But also there may exist other nodes $r \leq p$, such that for all branches $\mathcal{C}''$ containing r holds $\sigma_{\mathcal{C}''} \in \tau_{\mathcal{C}''}$, i.e., $\sigma \epsilon_r \tau$. Informally speaking, if some part of σ is outside of τ, then – at the same time – other part of σ may be inside of τ. This brings the idea of partial non-membership relation of σ in τ.

Definition 8 *Let $p \in \mathbb{T}$ and let τ, σ be metasets. We say that σ does not belong to τ under the condition p, if for each branch $\mathcal{C}$ containing p holds $\sigma_\mathcal{C} \notin \tau_\mathcal{C}$. In such case we use the notation $\sigma \not\epsilon_p \tau$.*

Again, if $p = \mathbb{1}$, then we omit the subscript and simply write $\sigma \not\epsilon \tau$ to express that for any branch $\mathcal{C}$ holds $\sigma_\mathcal{C} \notin \tau_\mathcal{C}$.

Note that if $\sigma \epsilon_p \tau$, then there is an infinite number of other nodes which also specify (other degrees of) the membership: for all $q \leq p$ also holds $\sigma \epsilon_q \tau$. Indeed, since for any $\mathcal{C}$ containing p holds $\sigma_\mathcal{C} \in \tau_\mathcal{C}$, then also for any $\mathcal{C}'$ containing q holds $\sigma_{\mathcal{C}'} \in \tau_{\mathcal{C}'}$, because if a branch $\mathcal{C}$ contains q, then it also contains $p \geq q$. Similarly, if $\sigma \not\epsilon_p \tau$, then for all branches $\mathcal{C} \ni p$ holds $\sigma_\mathcal{C} \notin \tau_\mathcal{C}$. Thus, for all $q \leq p$ also holds $\sigma \not\epsilon_q \tau$. Consequently, if a relation holds under the condition p, then it also holds under the condition q, for each $q \leq p$. We say that relations are propagated down the branches.

Proposition 3 *Let σ, τ be metasets and $p, q \in \mathbb{T}$. If $q \leq p$, then*

$$\begin{aligned} \sigma \epsilon_p \tau &\rightarrow \sigma \epsilon_q \tau , \\ \sigma \not\epsilon_p \tau &\rightarrow \sigma \not\epsilon_q \tau . \end{aligned}$$

The converse implications generally do not hold. For $q \leq p$ it is not true, that $\sigma \epsilon_q \tau \rightarrow \sigma \epsilon_p \tau$. The following example demonstrates this.

Example 5 *Let $\check{\sigma}$ be a canonical metaset and let $\tau = \{ \langle \check{\sigma}, [0] \rangle \}$. Of course, $\check{\sigma} \epsilon_{[0]} \tau$. However, it is not true that $\check{\sigma} \epsilon \tau$, since if $\mathcal{C}$ is a branch containing $[1]$, then $\tau_{\mathcal{C}} = \emptyset$, so it contains no elements at all. Note also, that since $\check{\sigma}_{\mathcal{C}} \notin \emptyset = \tau_{\mathcal{C}}$, then $\check{\sigma} \not\epsilon_{[1]} \tau$.*

The above example also shows that for incomparable p and q it is possible that $\mu \not\epsilon_p \tau$ and at the same time $\mu \epsilon_q \tau$. Clearly, it is not true that $\mu \not\epsilon_p \tau \wedge \mu \epsilon_p \tau$ for any p.

One of the consequences of the proposition 3 is, that if $\sigma \epsilon_p \tau$ and $\tau[\sigma]$ contains two different $q \leq p$, then the less node q does not supply any additional membership information to the membership degree specified by p, since $\sigma \epsilon_q \tau$ is implied by $\sigma \epsilon_p \tau$. However, when $p, q \in \tau[\sigma]$ are incomparable, then they contribute independently to the overall membership degree of σ in τ. Similarly for the non-membership relation.

2.5 Evaluating Membership and Non-membership

In the language of metasets we express the degrees of membership or non-membership in terms of sets of nodes of the binary tree. On the other hand, the language of fuzzy sets usually requires numerical values to evaluate these degrees. Therefore, in this section we show how to translate degrees expressed in terms of subsets of $\mathbb{T}$ into real numbers.

The root node $\mathbb{1}$ specifies the full, absolutely certain membership. If $\sigma \epsilon \tau$, then for each branch $\mathcal{C}$ we have $\sigma_{\mathcal{C}} \in \tau_{\mathcal{C}}$. Therefore, the membership value in this case should be 1. If $\tau = \{ \langle \sigma, [0] \rangle \}$, then $\sigma \epsilon_{[0]} \tau$ but $\sigma \not\epsilon_{[1]} \tau$. In this case we assign the value of $1/2$ to the membership of σ in τ. Generally, if $p \in \mathbb{T}_k$ is a node from the k-th level of $\mathbb{T}$ and $\tau = \{ \langle \sigma, p \rangle \}$, then the membership value of σ in τ is equal to $1/2^k$. Thus, we come to the rule that each node from the k-th level supplies the factor of $1/2^k$ to the membership value.

Proposition 4 *If $A \subset \mathbb{T}$ is a maximal antichain in $\mathbb{T}$, then*

$$\sum_{p \in A} \frac{1}{2^{\# p}} = 1 \,.$$

Obviously, a maximal antichain cannot be empty. To each $p \in \mathbb{T}$ there corresponds the closed interval of the length $1/2^{\# p}$ included in the unit interval:

$$i_p = \left[\frac{|p|}{2^{\# p}}, \; \frac{|p|+1}{2^{\# p}} \right] \,. \tag{6}$$

If the measure of $U = \bigcup_{p \in A} i_p$ is less than 1, then there exist an $x \in [0 \ldots 1]$ and $\delta > 0$ such that the open interval $(x - \delta, x + \delta)$ is not covered by U. So there exists $q \in \mathbb{T}$ such, that $i_q \subset (x - \delta, x + \delta)$ is also not covered by U. Such q is incomparable to any element of A contradicting maximality of A. Thus, the measure of U is 1 and since the measure of each $i_p \cap i_q$ is 0, then $\sum_{p \in A} 1/2^{\# p} = 1$.

The above proposition suggests that if the set P is a maximal antichain, and for each $p \in P$ a relation holds under the condition p, then this relation also holds under the condition $\mathbb{1}$ – to the greatest possible degree. Moreover, adding new nodes to the P should not affect this degree.

Example 6 *Let $\tau = \left\{ \left\langle \check{0}, \mathbb{1} \right\rangle, \left\langle \check{1}, [0] \right\rangle \right\}$. Clearly* $\mathrm{dom}(\tau) = \left\{ \check{0}, \check{1} \right\}$. *It is easy to verify that $\check{0} \mathrel{\epsilon} \tau$ and $\check{1} \mathrel{\epsilon_{[0]}} \tau \wedge \check{1} \mathrel{\not\epsilon_{[1]}} \tau$. Indeed, if $\mathcal{C}^0$ is a branch containing $[0]$, then $\tau_{\mathcal{C}^0} = \{ 0, 1 \}$. If $\mathcal{C}^1$ is a branch containing $[1]$, then $\tau_{\mathcal{C}^1} = \{ 0 \}$. Now, let $\sigma = \left\{ \left\langle \check{0}, [0] \right\rangle \right\}$. Of course, $\sigma \not\in \mathrm{dom}(\tau)$. However, $\sigma \mathrel{\epsilon} \tau$, since*

$$\begin{aligned} \sigma_{\mathcal{C}^0} &= \{ 0 \} = 1 \in \{ 0, 1 \} = \tau_{\mathcal{C}^0} \,, \\ \sigma_{\mathcal{C}^1} &= \emptyset \quad\;\; = 0 \in \{ 0 \} \quad = \tau_{\mathcal{C}^1} \,. \end{aligned}$$

We conclude, that there exist metasets outside of $\mathrm{dom}(\tau)$ *which are still members of τ.*

The example shows, that members of the domain of a metaset are not the only metasets that are in the membership or non-membership relation with the given metaset, unless we consider as members only canonical metasets. Having this observation in mind we construct the numerical evaluation of membership and non-membership degrees expressed by subsets of $\mathbb{T}$. We must generalize the rule presented above to metasets not included in the domain of the given metaset.

Let τ and σ be arbitrary metasets and $S \subset \mathbb{T}$. Recall that

$$\max(S) \ = \begin{cases} \{\, s \in S \,|\, q \geq s \rightarrow q = s \,\} & \text{when } S \neq \emptyset \ , \\ \emptyset & \text{when } S = \emptyset \ . \end{cases} \tag{7}$$

denotes the set of maximal elements in S. Let

$$M^{\tau}_{\sigma} = \{\, p \in \mathbb{T} \,|\, \sigma \in_p \tau \,\} \tag{8}$$

be the set of nodes for which the membership holds. We define the numerical value of the membership degree of σ in τ as follows

$$\mathrm{val}_M(\sigma, \tau) \ = \sum_{p \in \max(M^{\tau}_{\sigma})} \frac{1}{2^{\#p}} \ , \tag{9}$$

where $\#p$ denotes the level of p in $\mathbb{T}$. Similarly, let

$$N^{\tau}_{\sigma} = \left\{\, p \in \mathbb{T} \,|\, \sigma \notin_p \tau \,\right\} \tag{10}$$

be the subset of $\mathbb{T}$ for which non-membership holds. The formula for evaluating the non-membership is

$$\mathrm{val}_N(\sigma, \tau) \ = \sum_{p \in \max(N^{\tau}_{\sigma})} \frac{1}{2^{\#p}} \ . \tag{11}$$

The reason for taking $\max(M^{\tau}_{\sigma})$ and $\max(N^{\tau}_{\sigma})$ instead of just M^{τ}_{σ} and N^{τ}_{σ} is given by the proposition 3 and is illustrated by the following example. Note, that $\max(S)$ is an antichain for any set S.

Example 7 *Let* $\tau = \{\, \langle \sigma, [00] \rangle \,, \langle \sigma, [0] \rangle \,, \langle \sigma, [10] \rangle \,\}$. *We can easily see that whenever* $p \leq [0]$ *or* $p \leq [10]$, *then* $\sigma \in_p \tau$ *holds. Also, for each* $q \leq [11]$ *holds* $\sigma \notin_q \tau$. *Thus,* $\max(M^{\tau}_{\sigma}) = \{\, [0], [10] \,\}$ *and* $\max(N^{\tau}_{\sigma}) = \{\, [11] \,\}$. *Therefore,* $\mathrm{val}_M(\sigma, \tau) = 3/4$ *and* $\mathrm{val}_N(\sigma, \tau) = 1/4$. *Removing the pair* $\langle \sigma, [00] \rangle$ *from* τ *does not affect neither membership nor its evaluation since* $[00] \leq [0]$.

We also define the evaluation for the uncertainty degree. Let

$$U^{\tau}_{\sigma} = \left\{\, p \in \mathbb{T} \,|\, \forall_{q \leq p} \neg(\sigma \in_q \tau) \ \wedge \ \forall_{q \leq p} \neg(\sigma \notin_q \tau) \,\right\} \tag{12}$$

be the set of nodes for which neither membership nor non-membership holds. Whether and when such nodes exist at all we convince in the section 3. The numerical value of uncertainty – similarly to evaluations for membership (9) and non-membership (11) – is given by the formula:

$$\mathrm{val}_U(\sigma, \tau) = \sum_{p \in \max(U^{\tau}_{\sigma})} \frac{1}{2^{\#p}} \ . \tag{13}$$

3 Representing Uncertainty

There exists metasets τ, σ and $p \in \mathbb{T}$ such, that for any $q \leq p$ neither $\sigma \;\epsilon_q\; \tau$ nor $\sigma \;\not\epsilon_q\; \tau$ holds. Since σ neither is a member nor it is not a member of τ under the condition p, and this property is maintained by all descendants of p, then the p might be considered as an uncertainty degree. We show how to construct metasets τ, σ with the above property. We start with the construction of metasets for which the membership is totally uncertain: for any $p \in \mathbb{T}$ neither $\sigma \;\epsilon_p\; \tau$ nor $\sigma \;\not\epsilon_p\; \tau$ holds. Then we modify the construction, so that it allows for expressing uncertainty degrees other than 1.

Let ω denote the set of finite ordinal numbers, i.e., the set of natural numbers. We define the metaset $\hat{\omega}$ as follows:

$$\hat{\omega} = \left\{ \langle \check{n}, p_n \rangle \mid n \in \omega \text{ and } \overset{1}{p_n} = n \right\} , \tag{14}$$

where $\check{n}$ is the canonical metaset corresponding to the ordinal n (see Ex. 2) and the symbol $\overset{1}{p_n}$ denotes the number of occurrences of 1 in the sequence p_n. Thus, the nodes p_n are such, that their representations contain exactly n occurrences of 1 and infinite number of zeros. For instance the pairs $\langle \check{0}, \mathbb{1} \rangle$, $\langle \check{0}, [00] \rangle$, $\langle \check{1}, [1000] \rangle$, $\langle \check{1}, [001] \rangle$, $\langle \check{3}, [01010100] \rangle$ belong to $\hat{\omega}$.

If $\mathcal{C}^{\mathcal{I}}$ is a branch containing infinitely many occurrences of 1, then $\hat{\omega}_{\mathcal{C}^{\mathcal{I}}} = \omega$. Indeed, for each $n \in \omega$ there exist corresponding nodes $p_n \in \mathcal{C}^{\mathcal{I}}$ whose binary representations consist of exactly n occurrences of 1 and the infinite number of 0, so that $\langle \check{n}, p_n \rangle \in \hat{\omega}$. The binary sequence p_n is the prefix of the binary sequence $\mathcal{C}^{\mathcal{I}}$, containing exactly n occurrences of 1. An obvious example of such $\mathcal{C}^{\mathcal{I}}$ is the rightmost branch in $\mathbb{T}$, i.e., the one consisting exclusively of 1 and containing no 0: $\mathbb{1}$, [1], [11], [111], ... Then, for instance, the node p_2 corresponding to $2 \in \omega$ might be $[11] \in \mathcal{C}^{\mathcal{I}}$, thus $\langle \check{2}, [11] \rangle \in \hat{\omega}$ and therefore $2 \in \hat{\omega}_{\mathcal{C}^{\mathcal{I}}}$. Similarly for other $n \in \omega$ and other branches containing infinitely many 1.

On the other hand, if $\mathcal{C}^{\mathcal{F}}$ is a branch whose representation as the binary sequence consists of finitely many, say n occurrences of 1 and infinitely many 0, then $\hat{\omega}_{\mathcal{C}^{\mathcal{F}}} = s(n)$, where $s(n)$ is the ordinal successor of n. Indeed, recall that $s(n) = \{ 0, 1, \ldots, n \}$, and let $k \in s(n)$ (or $k \leq n$ in another notation). Take a prefix of the binary representation of $\mathcal{C}^{\mathcal{F}}$ containing exactly k occurrences of 1. Such a prefix exists, since there are $n \geq k$ occurrences of 1 in the representation of $\mathcal{C}^{\mathcal{F}}$. This prefix represents the node $p_k \in \mathcal{C}^{\mathcal{F}}$ such, that $\langle \check{k}, p_k \rangle \in \hat{\omega}$. Therefore $k \in \hat{\omega}_{\mathcal{C}^{\mathcal{F}}}$. But if we pick up an $m \notin s(n)$ (i.e., $m > n$), then any $p_m \in \hat{\omega}[\check{m}]$ contains exactly m

occurrences of 1, and therefore cannot lie on the branch $\mathcal{C}^{\mathcal{F}}$, what prevents it to appear in $\hat{\omega}_{\mathcal{C}^{\mathcal{F}}}$. Thus, $m \notin \hat{\omega}_{\mathcal{C}^{\mathcal{F}}}$.

To finish the construction consider the set $\Omega = \{ \omega \}$, whose only element is ω, and its canonical counterpart $\check{\Omega} = \{ \langle \check{\omega}, \mathbb{1} \rangle \}$. We have shown above that for any $p \in \mathbb{T}$ neither $\hat{\omega} \epsilon_p \check{\Omega}$ nor $\hat{\omega} \not\epsilon_p \check{\Omega}$ holds, since for the branches $\mathcal{C}^{\mathcal{I}}$ containing infinitely many 1 the membership holds in interpretations: $\hat{\omega}_{\mathcal{C}^{\mathcal{I}}} = \omega \in \Omega = \check{\Omega}_{\mathcal{C}^{\mathcal{I}}}$, whereas for other branches the membership does not hold: $\hat{\omega}_{\mathcal{C}^{\mathcal{F}}} \notin \check{\Omega}_{\mathcal{C}^{\mathcal{F}}}$, because $\hat{\omega}_{\mathcal{C}^{\mathcal{F}}} = s(n)$ for some $n \in \omega$, depending on the branch $\mathcal{C}^{\mathcal{F}}$. In other words, we have constructed the metasets $\sigma = \hat{\omega}$ and $\tau = \check{\Omega}$ such, that for any $p \in \mathbb{T}$ neither $\sigma \epsilon_p \tau$ nor $\sigma \not\epsilon_p \tau$ holds. In this case the membersip as well as the non-membership degrees are represented by the empty subset of $\mathbb{T}$. The uncertainty degree is represented by the full tree $\mathbb{T}$.

Now we modify the construction, so that the uncertainty degree is equal to some given $p \in \mathbb{T}$. Recall that $+$ is the concatenation operator for binary sequences. For the given $p \in \mathbb{T}$, let

$$\hat{\omega}^p = \left\{ \langle \check{n}, p + p_n \rangle \mid n \in \omega \wedge \overset{1}{p_n} = n \right\} , \tag{15}$$

where the nodes p_n are – as in (14) – such, that their representations contain exactly n occurrences of 1 and infinite number of 0. For instance, if $p = [010]$, then the pairs $\langle \check{0}, [010] \rangle$, $\langle \check{0}, [01000] \rangle$, $\langle \check{1}, [0101000] \rangle$, $\langle \check{1}, [010001] \rangle$, $\langle \check{3}, [01001010100] \rangle$ belong to $\hat{\omega}^p$.

If $\mathcal{C}^{\mathcal{I}}$ is a branch containing the node p, which has infinitely many 1 in its representation, then $\hat{\omega}^p_{\mathcal{C}^{\mathcal{I}}} = \omega$, since for each $n \in \omega$ there exists $p_n \in \mathbb{T}$ such, that $\overset{1}{p_n} = n$ and $\langle \check{n}, p + p_n \rangle \in \hat{\omega}^p$, as well as $p + p_n \in \mathcal{C}^{\mathcal{I}}$.

Let $\mathcal{C}^{\mathcal{F}}$ be a branch containing p, and whose representation consists of infinitely many 0 and exactly n occurrences of 1 after the prefix p. The total number of 1 in $\mathcal{C}^{\mathcal{F}}$ is $n + \overset{1}{p}$. As before, $\hat{\omega}^p_{\mathcal{C}^{\mathcal{F}}} = s(n)$, since for $k \in s(n)$ we may find the prefix q of $\mathcal{C}^{\mathcal{F}}$, starting with p and containing $k + \overset{1}{p}$ occurrences of 1, for which $\langle \check{k}, q \rangle \in \hat{\omega}^p$. Therefore $k \in \hat{\omega}^p_{\mathcal{C}^{\mathcal{F}}}$. Again, if we chose an $m \notin s(n)$, then any $p_m \in \hat{\omega}^p[\check{m}]$ contains exactly $m + \overset{1}{p} > n + \overset{1}{p}$ occurrences of 1, so $m \notin \hat{\omega}^p_{\mathcal{C}^{\mathcal{F}}}$.

Similarly as in the previous case, we conclude that for any $q \leq p$ neither $\hat{\omega}^p \epsilon_q \check{\Omega}$ nor $\hat{\omega}^p \not\epsilon_q \check{\Omega}$ hold. Thus, we have constructed two metasets with the uncertainty degree of the membership relation equal to the given p.

What about the membership and the non-membership degrees of $\hat{\omega}^p$ in $\check{\Omega}$? If $q \in \mathbb{T}$ is incomparable to p and $\mathcal{C}$ is a branch containing q, then $\hat{\omega}^p_{\mathcal{C}} = \emptyset$ and $\check{\Omega}_{\mathcal{C}} = \{ \omega \}$, so $\hat{\omega}^p \not\epsilon_q \check{\Omega}$. If $q > p$, then neither $\hat{\omega}^p \epsilon_q \check{\Omega}$ nor

$\hat{\omega}^p \not\epsilon_q \check{\Omega}$, since the non-membership and uncertainty are established by the nodes less than q. If $q \leq p$, then also neither $\hat{\omega}^p \epsilon_q \check{\Omega}$ nor $\hat{\omega}^p \not\epsilon_q \check{\Omega}$, because of the reasons explained above.

Let us evaluate the uncertainty degrees for the defined metasets. In the first construction of this section the set $U^{\check{\Omega}}_{\hat{\omega}^p}$ (cf. (13)) contained all the nodes of $\mathbb{T}$, however $\max(U^{\check{\Omega}}_{\hat{\omega}^p}) = \{ \mathbb{1} \}$. Therefore, in the case of total uncertainty $\mathrm{val}_U(\hat{\omega}^p, \check{\Omega}) = 1/2^0 = 1$. In the second, generalized construction the set $U^{\check{\Omega}}_{\hat{\omega}^p}$ contained all $q \leq p$, for the given $p \in \mathbb{T}$. Thus, in this case $\max(U^{\check{\Omega}}_{\hat{\omega}^p}) = \{ p \}$ and $\mathrm{val}_U(\hat{\omega}^p, \check{\Omega}) = 1/2^{\# p}$.

4 Representing Intuitionistic Fuzzy Sets

We construct a sequence of metasets representing the given intuitionistic fuzzy set and another metaset necessary to model the membership, the non-membership and the uncertainty degrees. The sequence contains elements corresponding to elements of the domain of the IFS.

For simplicity, initially we assume that the domain of the IFS includes only one element, then we proceed to the general case. As a gentle introduction to the general idea we start with a simple example of an IFS with particular values for the membership and non-memberhip functions.

Let $A = \langle X, \mu_A, \nu_A \rangle$ be an IFS, where $X = \{ x \}$ and $\mu_A(x) = 1/2$, $\nu_A(x) = 1/4$. The uncertainty degree is thus equal to 1/4. Let

$$\rho^A = \{ \langle \check{n}, [0] \rangle \mid n \in \omega \} \cup \{ \langle \check{n}, [11] + p_n \rangle \mid n \in \omega \} \ , \tag{16}$$

where p_n contain exactly n occurrences of 1. We show, that

$$\begin{aligned} \rho^A \quad & \epsilon_{[0]} \ \check{\Omega} \ , & (17) \\ \rho^A \quad & \not\epsilon_{[10]} \ \check{\Omega} \ , & (18) \\ \neg \rho^A \quad & \epsilon_{[11]} \ \check{\Omega} \ , & (19) \\ \neg \rho^A \quad & \not\epsilon_{[11]} \ \check{\Omega} \ . & (20) \end{aligned}$$

Recall that $\check{\Omega} = \{ \langle \breve{\omega}, \mathbb{1} \rangle \}$ and $\check{\Omega}_{\mathcal{C}} = \{ \omega \}$ independently of the branch $\mathcal{C}$. It is clear, that the formula (17) implies $\mathrm{val}_M(\rho^A, \check{\Omega}) = 1/2$, (18) implies $\mathrm{val}_N(\rho^A, \check{\Omega}) = 1/4$, and both (19), (20) together imply $\mathrm{val}_U(\rho^A, \check{\Omega}) = 1/4$.

First, note that elements [0], [10] and [11] form the maximal antichain. Let $\mathcal{C}$ be a branch in $\mathbb{T}$. If $\mathcal{C}$ contains [0], then it does not contain either [10] or [11]. It is easy to see that $\rho^A_{\mathcal{C}} = \omega$ in this case and therefore $\rho^A \epsilon_{[0]} \check{\Omega}$.

If $\mathcal{C}$ contains [10], then $\rho^A_{\mathcal{C}} = \emptyset$, so $\rho^A \not\in_{[10]} \check{\Omega}$. Finally, let $\mathcal{C}$ contain [11]. If $\mathcal{C}$ contains infinitely many occurrences of 1, then $\rho^A_{\mathcal{C}} = \omega \in \check{\Omega}_{\mathcal{C}}$. But if $\mathcal{C}$ contains at most one 1, then $\rho^A_{\mathcal{C}} = \emptyset \notin \check{\Omega}_{\mathcal{C}}$. Also, if $\mathcal{C}$ contains $k+2$ occurrences of 1, then $\rho^A_{\mathcal{C}} = s(k) \notin \check{\Omega}_{\mathcal{C}}$, for any $k \in \mathbb{N}$. Because the membership of interpretations of $\rho^A_{\mathcal{C}}$ in $\check{\Omega}_{\mathcal{C}}$ depends on the branch and it is so for all branches containing any $q \leq [11]$, then the node [11] represents uncertainty degree whose numerical value is equal to $1/2^{\#[11]} = 1/4$.

This example reveals the path we follow to attain the general case. We still assume that A has only one element: $X = \{ x \}$. The membership degree $\mu_A(x)$ will be modelled by the metasets ρ^A, $\check{\Omega}$ and the set $\mathcal{M}_x = \rho^A[\check{x}]$, where $\check{x} \in \mathrm{dom}(\rho^A)$ corresponds to $x \in X$. Then we show, that $\mathcal{M}_x = \max(M^{\check{\Omega}}_{\rho^A})$ and $\mathrm{val}_M(\rho^A, \check{\Omega}) = \mu_A(x)$ (cf. (8) and (9)). Similarly, we define the set $\mathcal{U}_x \subset \mathbb{T}$ for determining uncertainty degree and we show, that $\mathcal{U}_x = \max(U^{\check{\Omega}}_{\rho^A})$ and $\mathrm{val}_U(\rho^A, \check{\Omega}) = 1 - \mu_A(x) - \nu_A(x)$ (cf. (12) and (13)). We also show that $\mathrm{val}_N(\rho^A, \check{\Omega}) = \nu_A(x)$ (cf. (12) and (13)).

We start with the definition of the metaset ρ^A_M which is the subset of ρ^A establishing the membership only. Then we add to it another metaset ρ^A_U for establishing uncertainty; the non-membership does not require any special additions.

If $\mu_A(x) = 1$, then we take $\rho^A = \check{\omega}$ and this finishes the whole construction, since $\nu_A(x) = 0$ and uncertainty degree is also 0 in such case. If $\mu_A(x) = 0$, then let $\rho^A_M = \emptyset$. Otherwise ($0 < \mu_A(x) < 1$), let

$$\mu_A(x) \;=\; \sum_{k=1}^{\infty} \frac{m^k_x}{2^k} \;, \quad \text{where } m^k_x \in \{ 0, 1 \} \;. \tag{21}$$

There exists k for which $m^k_x = 1$, since $\mu_A(x) > 0$. We denote the infinite branch $m^1_x m^2_x \ldots$ with the symbol $\mathcal{C}^m_x$ and by the symbol $\bar{m}^k_x$ we understand the complement of m^k_x, i.e., $\bar{m}^k_x = (m^k_x + 1) \mod 2$. In case of ambiguous representation we assume the final one, where for all k large enough holds $m^x_k = 0$. The set

$$\mathcal{M}_x = \left\{ [m^1_x \ldots m^{k-1}_x \bar{m}^k_x] \,|\, m^k_x = 1 \right\} \tag{22}$$

of nodes nodes which are siblings of those prefixes of $\mathcal{C}^m_x$, which end with 1, determines the membership degree. We define the metaset which represents the membership part so:

$$\rho^A_M = \begin{cases} \breve{\omega} & \text{if } \mu_A(x) = 1 \\ \{ \langle \breve{n}, p \rangle \mid n \in \omega \wedge p \in \mathcal{M}_x \} & \text{if } 0 < \mu_A(x) < 1 \\ \emptyset & \text{if } \mu_A(x) = 0 \; . \end{cases} \tag{23}$$

Thus, when $0 < \mu_A(x) < 1$, the canonical counterpart of each finite ordinal is paired with the direct predecessor (in the level ordering) of each prefix of the binary representation of $\mu_A(x)$, which ends with 1.

Now we switch to the representation of the uncertainty part by means of the metaset ρ^A_U. Let $\xi_A(x) = 1 - \mu_A(x) - \nu_A(x)$. If $\xi_A(x) = 1$, then the construction presented at the beginning of the previous section applies ($\mu_A(x) + \nu_A(x) = 0$ in such case). If $\xi_A(x) = 0$, then let $\rho^A_U = \emptyset$. Otherwise ($0 < \xi_A(x) < 1$) let

$$\mu_A(x) + \nu_A(x) = 1 - \xi_A(x) = \sum_{k=1}^{\infty} \frac{u^k_x}{2^k} \; , \quad \text{where } u^k_x \in \{ 0, 1 \} \; . \tag{24}$$

By the assumption, there exists k for which $u^k_x = 1$, and there are always infinitely many k for which $u^k_x = 0$. Denote the infinite branch $u^1_x \ldots u^k_x \ldots$ with the symbol $\mathcal{C}^u_x$. If there are finitely many 1 in the $\mathcal{C}^u_x$, then let the ordinal l_x be the largest k for which $u^k_x = 1$, and let $l_x = \omega$ otherwise, when there are infinitely many occurrences of 1. Also, let

$$U = \left\{ [u^1_x \ldots u^{k-1}_x \bar{u}^k_x] \mid u^k_x = 0 \wedge k < l_x \right\} \; . \tag{25}$$

Each sequence in U ends with 1. Note, that when $l_x = \omega$, then the set U is infinite. We define the set of nodes for representing uncertainty so:

$$\mathcal{U}_x = \begin{cases} U & \text{when } l_x = \omega \; , \\ U \cup \{ [u^1_x \ldots u^{l_x}_x] \} & \text{when } l_x < \omega \; . \end{cases} \tag{26}$$

If l_x is finite, then we add to U the longest sequence $[u^1_x \ldots u^k_x]$ ending with 1. Otherwise, $\mathcal{U}_x$ contains infinite number of nodes ending with 1, which are siblings of those $[u^1_x \ldots u^k_x]$ for which $u^k_x = 0$. Also, it is worth noting that for $0 < \xi_A(x) < 1$:

$$\xi_A(x) = \sum_{k=1}^{\infty} \frac{\bar{u}^k_x}{2^k} \; . \tag{27}$$

We define the metaset which represents the uncertainty part so:

$$\rho^A_U = \begin{cases} \emptyset & \text{if } \xi_A(x) = 0\ , \\ \left\{ \langle \check{n}, u + p_n \rangle \mid n \in \omega \wedge u \in \mathcal{U}_x \wedge \overset{1}{p_n} = n \right\} & \text{if } 0 < \xi_A(x) < 1\ , \\ \hat{\omega} & \text{if } \xi_A(x) = 1\ . \end{cases} \tag{28}$$

where the nodes p_n are – as previously – such, that their representations contain exactly n occurrences of 1 and infinite number of 0 (i.e., $\overset{1}{p_n} = n$), and $\hat{\omega} = \left\{ \langle \check{n}, p_n \rangle \mid n \in \omega \wedge \overset{1}{p_n} = n \right\}$, as in the section 3.

Finally, let $\rho^A = \rho^A_M \cup \rho^A_U$. Let us expose some particular cases:

$$\rho^A = \begin{cases} \breve{\omega} & \text{when } \ \mu_A(x) = 1\ , \\ \emptyset & \text{when } \ \nu_A(x) = 1\ , \\ \hat{\omega} & \text{when } \ \xi_A(x) = 1\ . \end{cases} \tag{29}$$

We must show that the degree of membership of ρ^A in $\check{\Omega}$ equals to $\mu_A(x)$, the degree of non-membership of ρ^A in $\check{\Omega}$ equals to $\nu_A(x)$, and the uncertainty degree of membership of ρ^A in $\check{\Omega}$ equals to $\xi_A(x) = 1 - \mu_A(x) - \nu_A(x)$:

$$\mathrm{val}_M(\rho^A, \check{\Omega}) = \mu_A(x)\ , \tag{30}$$

$$\mathrm{val}_N(\rho^A, \check{\Omega}) = \nu_A(x)\ , \tag{31}$$

$$\mathrm{val}_U(\rho^A, \check{\Omega}) = \xi_A(x)\ . \tag{32}$$

To prove (30) note, that if $\mu_A(x) = 1$, then $\rho^A = \breve{\omega} \mathrel{\epsilon} \check{\Omega}$ and therefore $\mathrm{val}_M(\rho^A, \check{\Omega}) = 1$. Consequently, all (30–32) hold in this case. On the other hand, if $\mu_A(x) = 0$, then $\rho^A_M = \emptyset$, so $\rho^A = \rho^A_U$ and $\neg\, \rho^A \mathrel{\epsilon_p} \check{\Omega}$ for any $p \in \mathbb{T}$, therefore, $\mathrm{val}_M(\rho^A, \check{\Omega}) = 0$. So we assume $0 < \mu_A(x) < 1$ and we show that $\max(M^{\check{\Omega}}_{\rho^A}) = \mathcal{M}_x$ (recall (8) that $M^{\check{\Omega}}_{\rho^A} = \left\{ p \in \mathbb{T} \mid \rho^A \mathrel{\epsilon_p} \check{\Omega} \right\}$). Obviously, if $p \in \mathcal{M}_x$, then $\rho^A \mathrel{\epsilon_p} \check{\Omega}$, so $p \in M^{\check{\Omega}}_{\rho^A}$ and there exists $q \in \max(M^{\check{\Omega}}_{\rho^A})$ such, that $p \leq q$. Is it possible that $p < q$? If so, then $q \in \mathcal{C}^m_x$, because – by the definition of $\mathcal{M}_x$ – all parents of p are contained in $\mathcal{C}^m_x$. By the assumption, there is an infinite number of occurrences of 0 in $\mathcal{C}^m_x$, so let $r < q$ be of form $r = [m^1_x \ldots m^{n-1}_x \bar{m}^n_x]$, where $m^n_x = 0$ and $n > \#q$. Clearly, no element of $\mathcal{M}_x$ is comparable to r. If $r \top u$ for some $u \in \mathcal{U}_x$ and $\mathcal{C}_u$ is a branch containing both r and u, then the interpretation $\rho^A_{\mathcal{C}_u}$ depends on $\mathcal{C}_u$ making $\rho^A \mathrel{\epsilon_r} \check{\Omega}$ and $\rho^A \mathrel{\not\epsilon_r} \check{\Omega}$ impossible. Since $r \leq q$, then it contradicts $\rho^A \mathrel{\epsilon_q} \check{\Omega}$ by the proposition 3, consequently $q \notin M^{\check{\Omega}}_{\rho^A}$. Otherwise, when r is incomparable to all elements of $\mathcal{U}_x$ (and also $\mathcal{M}_x$)

we have $\rho^A_{\mathcal{C}_r} = \emptyset$ for any branch $\mathcal{C}_r \ni r$. Therefore, $\neg\, \rho^A \;\epsilon_r\; \check{\Omega}$ and by proposition 3 also $\neg\, \rho^A \;\epsilon_q\; \check{\Omega}$. Consequently, $q \notin M^{\check{\Omega}}_{\rho^A}$ – a contradiction. We conclude that $\mathcal{M}_x \subset \max(M^{\check{\Omega}}_{\rho^A})$, because $\mathcal{M}_x \subset M^{\check{\Omega}}_{\rho^A}$ and for $p \in \mathcal{M}_x$ there is no $q > p$ such, that $q \in M^{\check{\Omega}}_{\rho^A}$. Also $\max(M^{\check{\Omega}}_{\rho^A}) \subset \mathcal{M}_x$, since if $q \in \max(M^{\check{\Omega}}_{\rho^A})$, i.e., $\rho^A \;\epsilon_q\; \check{\Omega}$, then $q = p$ for some $p \in \mathcal{M}_x$. Indeed, if q were incomparable to all $p \in \mathcal{M}_x$, then no branch $\mathcal{C} \ni q$ would contain any $p \in \mathcal{M}_x$. Therefore, $\neg\, \rho^A \;\epsilon_q\; \check{\Omega}$, since elements of $\mathcal{M}_x$ are the only that may guarantee the membership (branches containing elements of $\mathcal{U}_x$ give different interpretations depending on the number of 1 contained in the branch). Thus, if $q \in \max(M^{\check{\Omega}}_{\rho^A})$, then $q \top p$ for some $p \in \mathcal{M}_x$. The case of $M^{\check{\Omega}}_{\rho^A} \ni q > p \in \mathcal{M}_x$ was just proved to be impossible. Since $q \in \max(M^{\check{\Omega}}_{\rho^A})$ and $p \in M^{\check{\Omega}}_{\rho^A}$, then also $q < p$ is false. Thus, $q = p$, $q \in \max(M^{\check{\Omega}}_{\rho^A}) \subset \mathcal{M}_x$ and consequently $\mathcal{M}_x = \max(M^{\check{\Omega}}_{\rho^A})$. Let us evaluate the membership degree by applying the formula (9) to prove (30):

$$\mathrm{val}_M(\rho^A, \check{\Omega}) = \sum_{p \in \max(M^{\check{\Omega}}_{\rho^A})} \frac{1}{2^{\#p}} = \sum_{p \in \mathcal{M}_x} \frac{1}{2^{\#p}} \tag{33}$$

$$= \sum_{k \in \mathbb{N} \wedge m^k_x = 1} \frac{1}{2^k} = \sum_{k=0}^{\infty} \frac{m^k_x}{2^k} \tag{34}$$

$$= \mu_A(x)\ . \tag{35}$$

We now prove (32). First, note that if $\xi_A(x) = 1$, then $\rho^A = \hat{\omega}$, so $\mathrm{val}_U(\rho^A, \check{\Omega}) = 1$. Since for any $p \in \mathbb{T}$, neither $\rho^A \;\epsilon_p\; \check{\Omega}$ nor $\rho^A \;\not\epsilon_p\; \check{\Omega}$ hold in this case, then all (30–32) are satisfied. On the other hand, if $\xi_A(x) = 0$, then $\rho^A_U = \emptyset$, so $\rho^A = \rho^A_M$ and for each $p \in \mathbb{T}$ there exist $q \leq p$ such, that either $\rho^A \;\epsilon_q\; \check{\Omega}$ or $\rho^A \;\not\epsilon_q\; \check{\Omega}$. This implies $\mathrm{val}_U(\rho^A, \check{\Omega}) = 0$. To finish, we need to show that $\max(U^{\check{\Omega}}_{\rho^A}) = \mathcal{U}_x$ (cf. 12), similarly as for the membership. By (28), if $u \in \mathcal{U}_x$, then for any $q \leq u$ we have $\neg(\rho^A \;\epsilon_q\; \check{\Omega})$ and $\neg(\rho^A \;\not\epsilon_q\; \check{\Omega})$, therefore $\mathcal{U}_x \subset U^{\check{\Omega}}_{\rho^A}$. Let $p \in \mathcal{U}_x$ and $q \in \max(U^{\check{\Omega}}_{\rho^A})$ be such, that $p \leq q$. If we assume $p < q$, then $q \in \mathcal{C}^u_x$. Since $q \in U^{\check{\Omega}}_{\rho^A}$, then for all $r \leq q$ we have $\neg\, (\rho^A \;\epsilon_r\; \check{\Omega})$ and $\neg\, (\rho^A \;\not\epsilon_r\; \check{\Omega})$. We find r which does not satisfy this conjunction. If the branch $\mathcal{C}^u_x$ contains finitely many 1, then let $r = [u^1_x \ldots u^{l_x - 1}_x \bar{u}^{l_x}_x]$ be the sibling of the node which is the largest prefix of $\mathcal{C}^u_x$ ending with 1. If $\mathcal{C}^u_x$ contains infinitely many 1, then let $r = [u^1_x \ldots u^{n-1}_x \bar{u}^n_x]$, where $n > \#q$ and $u^n_x = 1$. In both cases $r < q$ and

r seen as the binary sequence ends with 0. Of course, r is incomparable to all elements of $\mathcal{U}_x$. If $r \top s$ for some $s \in \mathcal{M}_x$, then for $t = \min(r, s)$ we have $\rho^A \,\epsilon_t\, \check{\Omega}$ – a contradiction. If r is incomparable to all elements of $\mathcal{M}_x$, then for any branch $\mathcal{C}_r \ni r$ we have $\rho^A_{\mathcal{C}_r} = \emptyset$, so $\rho^A \not\epsilon_r \check{\Omega}$ – also a contradiction. Thus, $p = q$ and $\mathcal{U}_x \subset \max(U^{\check{\Omega}}_{\rho^A})$. Now – similarly as by membership – we show, that $\max(U^{\check{\Omega}}_{\rho^A}) \subset \mathcal{U}_x$. Assume there exists $q \in \max(U^{\check{\Omega}}_{\rho^A})$ which is incomparable to all elements of $\mathcal{U}_x$. If q is comparable to some $t \in \mathcal{M}_x$, then for $v = \min(q, t)$ we have $\rho^A \,\epsilon_v\, \check{\Omega}$, what contradicts $q \in U^{\check{\Omega}}_{\rho^A}$. If q is incomparable to all elements of $\mathcal{M}_x$, then $\rho^A \not\epsilon_q \check{\Omega}$, since for any branch $\mathcal{C} \ni q$ we have $\rho^A_{\mathcal{C}} = \emptyset$, what contradicts $q \in U^{\check{\Omega}}_{\rho^A}$ too. We conclude, that each $q \in \max(U^{\check{\Omega}}_{\rho^A})$ must be comparable to some $p \in \mathcal{U}_x$. Since we just proved that $p < q$ is not possible, then $q \leq p$. But $q < p$ is also false, since $q \in \max(U^{\check{\Omega}}_{\rho^A})$ and $p \in \mathcal{U}_x \subset \max(U^{\check{\Omega}}_{\rho^A})$. Thus $q = p$. Consequently, $\max(U^{\check{\Omega}}_{\rho^A}) \subset \mathcal{U}_x$ and finally $\max(U^{\check{\Omega}}_{\rho^A}) = \mathcal{U}_x$. Let us evaluate the uncertainty degree by applying the formula (13) to prove (32). If $\mathcal{U}_x$ is infinite, then

$$
\begin{aligned}
\mathrm{val}_U(\rho^A, \check{\Omega}) &= \sum_{p \in \max(U^{\check{\Omega}}_{\rho^A})} \frac{1}{2^{\#p}} && (36)\\
&= \sum_{p \in \mathcal{U}_x} \frac{1}{2^{\#p}} && (37)\\
&= \sum_{k>0 \wedge u^k_x = 0} \frac{1}{2^k} && (38)\\
&= \sum_{k>0 \wedge u^k_x = 0} \frac{1 - u^k_x}{2^k} && (39)\\
&= \sum_{k=1}^{\infty} \frac{1 - u^k_x}{2^k} && (40)\\
&= 1 - \sum_{k=1}^{\infty} \frac{u^k_x}{2^k} && (41)\\
&= 1 - (\mu_A(x) + \nu_A(x)) && (42)\\
&= \xi_A(x)\ . && (43)
\end{aligned}
$$

If $\mathcal{U}_x$ is finite, then

$$\begin{aligned}
\mathrm{val}_U(\rho^A, \check{\Omega}) &= \sum_{p \in \mathcal{U}_x} \frac{1}{2^{\#p}} && (44)\\
&= \frac{1}{2^{l_x}} + \sum_{u_x^k = 0 \wedge 0 < k < l_x} \frac{1}{2^k} && (45)\\
&= \sum_{k > l_x} \frac{1}{2^k} + \sum_{u_x^k = 0 \wedge 0 < k < l_x} \frac{1}{2^k} && (46)\\
&= \sum_{u_x^k = 0 \wedge l_x < k} \frac{1}{2^k} + \sum_{u_x^k = 0 \wedge 0 < k < l_x} \frac{1}{2^k} && (47)\\
&= \sum_{k > 0 \wedge u_x^k = 0} \frac{1}{2^k} && (48)\\
&= \xi_A(x)\ . && (49)
\end{aligned}$$

The equality (47) is valid, since for all $k > l_x$ holds $u_x^k = 0$, and (48) is valid, since $u_x^{l_x} = 1 \neq 0$. The last one (49) comes from (39–43).

To finish the proof we show (31). If $\nu_A(x) = 1$, then we take $\rho^A = \emptyset$, so $\max(N_{\rho^A}^{\check{\Omega}}) = \{\, \mathbb{1}\, \}$, $\mathrm{val}_N(\rho^A, \check{\Omega}) = 1$ and all (30–32) are satisfied. Otherwise, when $0 \leq \nu_A(x) < 1$, let $\mathcal{N}_x = \max(N_{\rho^A}^{\check{\Omega}})$. We claim that the set

$$D_x = \mathcal{M}_x \cup \mathcal{U}_x \cup \mathcal{N}_x \qquad (50)$$

is a maximal antichain in $\mathbb{T}$. Indeed, all the sets $\mathcal{M}_x$, $\mathcal{N}_x$ and $\mathcal{U}_x$ are antichains, since they are sets of maximal elements of other sets (or empty, which is an antichain too). To see that their elements are pairwise incomparable let $p_m \in \mathcal{M}_x$, $p_n \in \mathcal{N}_x$ and $p_u \in \mathcal{U}_x$, unless any of them is empty. We have

$$\begin{aligned}
\rho^A \ \epsilon_{p_m}\ \check{\Omega}\ , && (51)\\
\rho^A \ \not\epsilon_{p_n}\ \check{\Omega}\ , && (52)\\
\forall_{p \leq p_u} \neg\, \rho^A \ \epsilon_{p}\ \check{\Omega}\ , && (53)\\
\forall_{p \leq p_u} \neg\, \rho^A \ \not\epsilon_{p}\ \check{\Omega}\ . && (54)
\end{aligned}$$

If $p_n \leq p_m$, then by the proposition 3 and (51) we have $\rho^A\ \epsilon_{p_n}\ \check{\Omega}$, what contradicts (52). Similarly, when $p_n \geq p_m$. If $p_u \leq p_m$, then let $p \leq p_u$. The proposition 3 implies $\rho^A\ \epsilon_p\ \check{\Omega}$ what contradicts (53). Similarly, when $p_u \geq p_m$. Other cases are also analogous. We conclude that D_x is an antichain. We show, that it is a maximal one. Assume there exists a $p \in \mathbb{T}$ incomparable to all elements of D_x. If there exists a crisp set y such, that for each branch $\mathcal{C}$ containing p holds $\rho^A_{\mathcal{C}} = y$, then either $y \in \Omega$ or not. In the former case $y = \omega$ and $\rho^A \ \epsilon_p \ \check{\Omega}$, so there must exist $p_m \in \mathcal{M}_x$ such, that $p \leq p_m$ (recall that $\mathcal{M}_x = \max(M^{\check{\Omega}}_{\rho^A})$). In the latter case $\rho^A \not\epsilon_p \ \check{\Omega}$ and there exists $p_n \in \mathcal{N}_x$ such, that $p \leq p_n$ (since $\mathcal{N}_x = \max(N^{\check{\Omega}}_{\rho^A})$). Both cases yield a contradiction. Analogously we conclude that also there is no $q \leq p$ for which either $\rho^A \ \epsilon_q \ \check{\Omega}$ or $\rho^A \not\epsilon_q \ \check{\Omega}$. Therefore, $p \leq p_u$ for some $p_u \in \mathcal{U}_x$ (cf. (53) and (54)) – a contradiction. Thus, D_x is a maximal antichain and so by the proposition 4 we have

$$
\begin{aligned}
1 &= \sum_{p \in D_x} \frac{1}{2^{\#p}} \,, &(55)\\
&= \sum_{p \in \mathcal{M}_x} \frac{1}{2^{\#p}} + \sum_{p \in \mathcal{U}_x} \frac{1}{2^{\#p}} + \sum_{p \in \mathcal{N}_x} \frac{1}{2^{\#p}} \,, &(56)\\
&= \mu_A(x) + \xi_A(x) + \sum_{p \in \mathcal{N}_x} \frac{1}{2^{\#p}} \,, &(57)\\
&= 1 - \nu_A(x) + \sum_{p \in \mathcal{N}_x} \frac{1}{2^{\#p}} \,. &(58)
\end{aligned}
$$

Therefore,

$$
\nu_A(x) = \sum_{p \in \mathcal{N}_x} \frac{1}{2^{\#p}} = \sum_{p \in \max(N^{\check{\Omega}}_{\rho^A})} \frac{1}{2^{\#p}} = \mathrm{val}_N(\rho^A, \check{\Omega}) \,. \qquad (59)
$$

The first equality is the result of the previous equations, the second is the definition of $\mathcal{N}_x$ and the last comes from (11). This finishes the proof of (31). Note, that it is possible to explicitly define the set $\mathcal{N}_x$ – similarly to the sets $\mathcal{M}_x$ and $\mathcal{U}_x$ – and then prove (31) directly, like in the previous cases, however this method is quite laborious. The set $\mathcal{N}_x$ consists of two parts: one analogous to the set $\mathcal{M}_x$ and another analogous to $\mathcal{U}_x$.

To complete the whole construction, we must drop the last simplifying assumption on cardinality of A. Now let X be an arbitrary set. We extend the previous construction by repeating it for each element of X. This way we obtain a family $\{ \rho^A_x \,|\, x \in X \}$ which represents the given intuitionistic fuzzy set A. Each ρ^A_x has properties described above. In particular, the degree of membership, non-membership and uncertainty of each ρ^A_x in $\check{\Omega}$ is equal to $\mu_A(x)$, $\nu_A(x)$ and $\xi_A(x)$, respectively. Note, that the family

$\{ \rho_x^A \mid x \in X \}$ may contain elements (metasets) which are equal: if there exist $x \neq y$ in X such, that $\mu_A(x) = \mu_A(y)$ and $\nu_A(x) = \nu_A(y)$, then $\rho_x^A = \rho_y^A$.

Thus, the metaset representation of the given intuitionistic fuzzy set $A = \langle X, \mu_A, \nu_A \rangle$ is the pair $\left\langle \{ \rho_x^A \}_{x \in X}, \check{\Omega} \right\rangle$, where $\{ \rho_x^A \}_{x \in X}$ is the family of metasets representing elements of X and $\check{\Omega} = \{ \langle \check{\omega}, \mathbb{1} \rangle \}$. Elements of the family $\{ \rho_x^A \}_{x \in X}$ are constructed as described above.

5 Conclusions

We introduced the method for representing intuitionistic fuzzy sets by means of metasets. For the given IFS we constructed a family of metasets which reflect the membership, non-membership and uncertainty degrees of corresponding elements of the IFS. It is possible to represent a fuzzy set (not intuitionistic) as a metaset in a slightly different way, using a single metaset. The potential elements of such metaset correspond to elements of the fuzzy set, as opposed to the method described here, where we use an indexed family of metasets. The representation involving a single metaset has many interesting properties, but it requires that $\mu_A(x) > 0$ for all elements. The metaset $\rho_{A \cup B}$ representing the union of two fuzzy sets A and B is equal to the union $\rho_A \uplus \rho_B$ of two metasets ρ_A and ρ_B representing the fuzzy sets. Similarly for the intersection: $\rho_{A \cup B} = \rho_A \Cap \rho_B$ (the symbols $\uplus$ and $\Cap$ denote the union and the intersection of metasets). On the other hand, the complement operation for metasets and fuzzy sets do not coincide. An α-cut of a fuzzy set A is the crisp set which is the interpretation of the metaset ρ^A representing the fuzzy set A, given by the branch which is the binary representation of the real number α (cf. [6]).

The theory of metasets is a new theory of sets with partial membership relation. It is directed towards computer implementations and applications. The degrees of membership, non-membership and uncertainty are represented by sets of nodes of the binary tree. Besides the basic relations introduced in this paper, the partial equality and subset relations as well as their negations are defined. The definitions are based on the interpretation technique, similarly to the membership relation. The algebraic operations for some class of metasets are defined too. As opposed to fuzzy sets they are unambiguous and they satisfy axioms of Boolean algebra (cf. [3]). The notions of cardinality and equinumerability are defined as well.

As mentioned, large parts of the theory are computer oriented. An experimental implementation of metasets operations was developed in the Java programming language. It enables representing inaccurate data and carrying out algebraic operations on them. A character recognition system based on this implementation is described in [7]. The highly efficient production implementation is to be realized soon.

References

[1] Atanassov K. (1999) Intuitionistic Fuzzy Sets: Theory and Applications. Springer-Verlag.

[2] Kunen K. (1980) Set Theory. An Introduction to Independence Proofs. Studies in Logic and Foundations of Mathematics (102). North-Holland Publishing Company.

[3] Starosta B. and Kosiński W. (2009) Meta Sets. Another Approach to Fuzziness. In: Views on Fuzzy Sets and Systems from Different Perspective. Philosophy, Logic; Criticisms and Applications. Studies in Fuzziness and Soft Computing (243), Springer-Verlag, 509–532.

[4] Starosta B. and Kosiński W. (2009) Meta Sets. A New Concept of Intuitionistic Fuzzy Sets. In: Intuitionistic Fuzzy Sets: Recent Advances. Studies in Fuzziness and Soft Computing. Springer-Verlag.

[5] Starosta B. (2008) Meta Sets. A Fresh View on Fuzziness. Proc. of X International PhD Workshop OWD 2008, Conference Archives, PTETIS 2008, 97–102.

[6] Starosta B. (2009) Fuzzy Sets as Metasets. Proc. of XI International PhD Workshop OWD 2009, Conference Archives, PTETIS 2009, 11–15.

[7] Starosta B. (2009) Application of Metasets to Character Recognition. Proc. of 18th International Symposium, ISMIS 2009, LNAI 5722, Springer 2009, 602–611.

[8] Zadeh L.A. (1965) Fuzzy sets. Information and Control, 8, 338–353.