Application of Meta Sets to Character Recognition

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Abstract. A new approach to character recognition problem, based on meta sets, is introduced and developed. For the given compound character pattern consisting of a number of character samples accompanied by their corresponding quality degrees, and for the given testing character sample, the main theorem of the paper gives means to evaluate the correlation between the testing sample and the compound pattern. It also enables calculation of similarity degrees of the testing sample to each pattern element. The quality degrees and the correlation are expressed by means of membership degrees of meta sets representing samples in the meta set representing the compound pattern. The similarity degrees are expressed as equality degrees of these meta sets.
The meta set theory is a new alternative to the fuzzy set theory. By the construction of its fundamental notions it is directed to efficient computer implementations. This paper presents an example of application of the theory to a real-life problem.

1 Motivations

The theory of meta sets is a new set theory with non-binary ("fuzzy") membership relation. It uses a language similar to the language of the classical set theory ([1]). However, there is an infinite, countable number of membership, non-membership, equality and inequality relational symbols to enable expressing various degrees of satisfaction of the relations. It is worth noting, that algebraic operations for meta sets satisfy Boolean algebra axioms ([3]). The meta set theory is meant to be an alternative to the fuzzy set theory ([2]). Although is is better fitted within the classical set theory than the fuzzy set theory – in particular, elements of meta sets are also meta sets – it was designed so as to enable efficient computer (or even hardware) implementations. For the detailed treatment of the idea of meta set the reader is referred to [3] and [4]. This paper introduces only the concepts directly relevant to character recognition.

The theory is new and under development. We demonstrate here the first example of its application to a real-life problem. We stress, that our main goal is to manifest the fact that the concept of meta set properly describes "fuzzy" relations and is applicable to real problems. To clarify the presentation we concentrate on a very particular, simplified case. Further in this section we explain the general idea of our approach. Section 2 introduces basic definitions and proves the main theorem. Section 3 reveals the idea of application of meta sets to character recognition.

The abstract concepts presented here are practically tested by means of an experimental computer program based on the implementation of meta sets operations. The program enables defining character patterns, supplying testing samples and evaluating similarity degrees. The results seem to be consistent with human intuition with respect to similarity of characters or letters.

1.1 The General Idea

Let us consider a number of different (possibly hand-written) samples of some letter, e.g. `'L'`, and let us denote them with the symbols $\pi^1, \pi^2, \ldots \pi^n$. Further, let us assign to each sample π^i a quality degree P^i which measures how close is the sample to the ideal. The quality degrees are to be supplied by an expert or a user. They represent his or her point of view on the shape of the letter. If we manage to represent each sample π^i as a meta set,[1] and each degree P^i as a set of nodes of the binary tree,[2] then the set $\pi = \bigcup_{i=1}^{i=n} \left\{ \pi^i \right\} \times P^i$ may be treated as another meta set. This meta set expresses our opinion on how the letter `'L'` should look like, based on the compound pattern comprised of a number of estimated samples. We will match new samples of the letter against this pattern to measure their quality degree or similarity to the pattern π.

Let us then supply another sample of the letter `'L'` and let us represent it as a meta set σ, similarly to the pattern elements π^i. We may ask what is the membership degree of the sample σ to the pattern π. The Theorem 1 gives the answer to this question. It also allows for evaluation of equality degrees of σ to each π^i, which express similarity of the supplied sample to each pattern element. Since these degrees are meant to express character resemblance, then the higher the membership (equality) degree – the greater resemblance of the sample and the pattern (pattern element).

2 The Theory of Meta Sets

We now establish some basic terms and notation. Then we define fundamental meta set theory concepts and prove their most important properties relevant to character recognition. This section ends with the main theorem which is applied in the next section.

[1] See Sect. 3.
[2] See Definition 1.

2.1 Fundamental Definitions

We use the symbol $\mathbb{T}$ for the infinite binary tree with the root $\mathbb{1}$ which is its largest element. The nodes of the tree $\mathbb{T}$ might be considered as finite binary sequences, the root $\mathbb{1}$ being the empty sequence. A branch in the tree $\mathbb{T}$ is a maximal chain (a maximal set of pairwise comparable nodes). It might be represented as an infinite binary sequence. The n-th level of the tree $\mathbb{T}$, denoted by $\mathbb{T}_n$, is the set of all binary sequences of the same length n, for instance $\mathbb{T}_1 = \{ 0, 1 \}$, $\mathbb{T}_2 = \{ 00, 01, 10, 11 \}$, whereas $\mathbb{T}_0 = \{ \mathbb{1} \}$.

We now define the fundamental notion of meta set. A meta set might be perceived as a crisp set whose elements are accompanied by sets of nodes of the binary tree. These sets of nodes express the membership degrees of elements to the set. Since elements are also meta sets, then their elements are also accompanied by nodes of $\mathbb{T}$, and so on, recursively. The recursion stops at the empty set $\emptyset$ by the Axiom of Foundation in the Zermelo-Fraenkel set theory ([1]).

Definition 1. *A meta set is a crisp set which is either the empty set $\emptyset$ or which has the form:*

$$\tau = \{ \langle \sigma, p \rangle : \sigma \text{ is a meta set, } p \in \mathbb{T} \} \quad .$$

Here $\langle \cdot, \cdot \rangle$ denotes an ordered pair.

Thus, from the classical set theory point of view, a meta set is a relation between a crisp set of other meta sets and a set of nodes of the tree $\mathbb{T}$. The nodes measure the membership degree of an element to the set in such way, that greater nodes (in the tree ordering) represent higher membership degrees; the root $\mathbb{1}$ designates the full, classical membership. For two comparable nodes $q > p$, the greater one q supplies more membership information and therefore, the smaller one p does not influence the overall membership degree of an element to the meta set. On the other hand, if p and q are incomparable, then they both independently contribute to the membership degree (cf. Lemmas 1 and 2 in [4]).

A meta set may also be perceived as a "fuzzy" family of crisp sets thanks to the interpretation technique introduced by the following definition. Each member of the family represents some specific, particular point of view on the meta set.

Definition 2. *Let τ be a meta set and let $\mathcal{C}$ be a branch in the binary tree $\mathbb{T}$. The interpretation of the meta set τ, given by the branch $\mathcal{C}$, is the following crisp set:*

$$\tau_{\mathcal{C}} = \{ \sigma_{\mathcal{C}} \colon \langle \sigma, p \rangle \in \tau \wedge p \in \mathcal{C} \} \quad .$$

Thus, branches in $\mathbb{T}$ allow for producing crisp sets out of the meta set. The family $\{ \tau_{\mathcal{C}} \colon \mathcal{C} \text{ is a branch in } \mathbb{T} \}$ consists of interpretations of the meta set τ. Properties of these interpretations induce properties of the meta set. The family might be considered "fuzzy" because some interpretations might occur "more frequently" than others, depending on the structure of τ. The frequency allows for introducing the membership degree of each interpretation set in the whole family. This way, a meta set perceived as a "fuzzy" family of crisp sets, resembles a fuzzy set. Of course, this informal reasoning may be made rigorous. Note, that all interpretations of the empty meta set $\emptyset$ are equal to itself: $\emptyset_{\mathcal{C}} = \emptyset$.

We now define basic set-theoretic relations for meta sets.

Definition 3. *Let $p \in \mathbb{T}$ and let τ, μ be meta sets. We say that μ belongs to τ under the condition p, if for each branch $\mathcal{C}$ containing p we have $\mu_{\mathcal{C}} \in \tau_{\mathcal{C}}$. We use the symbol ϵ_p for the relation of being a member under the condition p: $\mu \ \epsilon_p \ \tau$.*

In other words, $\mu \ \epsilon_p \ \tau$, whenever the (crisp) membership relation holds for interpretations of τ and μ determined by p. Similarly we define conditional equality relations.

Definition 4. *Let $p \in \mathbb{T}$ and let τ, μ be meta sets. We say that μ is equal to τ under the condition p, if for each branch $\mathcal{C}$ containing p we have $\mu_{\mathcal{C}} = \tau_{\mathcal{C}}$. We use the symbol $\approx_p$ for the relation of being equal under the condition p: $\mu \approx_p \tau$.*

The relations $\epsilon_{\mathbb{1}}$ and $\approx_{\mathbb{1}}$ designate full membership and equality, like the standard relations for crisp sets. If $\tau \approx_{\mathbb{1}} \mu$, then all their interpretations are pairwise equal: $\tau_{\mathcal{C}} = \mu_{\mathcal{C}}$ for any $\mathcal{C}$. Similarly for the $\epsilon_{\mathbb{1}}$ relation. On the other hand, if $p < q < \mathbb{1}$, then $\approx_p$ means "less equal" than $\approx_q$, and ϵ_p denotes smaller degree of membership than ϵ_q.

2.2 Basic Properties of Meta Sets

Let us state some basic properties of meta sets needed further. We start with the simplest example of conditional equality.

Lemma 1. *Let $p, q \in \mathbb{T}_n$ be nodes of the binary tree from the n-th level and let $\tau = \{ \langle \emptyset, p \rangle \}$ and $\sigma = \{ \langle \emptyset, q \rangle \}$ be meta sets. If $p = q$, then $\forall_{r \in \mathbb{T}_n} \tau \approx_r \sigma$. If $p \neq q$, then $\forall_{r \in \mathbb{T}_n} \ r \notin \{ p, q \} \Leftrightarrow \tau \approx_r \sigma$.*

Proof. First, assume that $p = q$ and let $\mathcal{C}_p$ be a branch in $\mathbb{T}$ containing p. By the Definition 2 we see that $\tau_{\mathcal{C}_p} = \{ \emptyset \} = \sigma_{\mathcal{C}_p}$. Since the crisp equality holds for interpretations given by p, then by the Definition 4 also $\tau \approx_p \sigma$. If $n \neq 0$, then there exist other nodes in the level n, so let $r \in \mathbb{T}_n$, $r \neq p$ and let $\mathcal{C}$ be a branch containing r (of course, it cannot contain p). Clearly $\tau_{\mathcal{C}} = \emptyset = \sigma_{\mathcal{C}}$, so the crisp equality holds for these interpretations too, and consequently $\tau \approx_r \sigma$. Therefore, $\forall_{r \in \mathbb{T}_n} \tau \approx_r \sigma$.

To prove the second part (which makes sense for $n > 0$) assume $p \neq q$ and let $r \in \mathbb{T}_n$ be such that $r \neq p$ and $r \neq q$ (such r exist for $n > 1$). If $\mathcal{C}$ is a branch containing r, then similarly as before we have $\tau_{\mathcal{C}} = \emptyset = \sigma_{\mathcal{C}}$, and therefore $\tau \approx_r \sigma$. On the other hand (for $n > 0$), if $r = q$ and $\mathcal{C}_q$ is a branch containing q, then $\tau_{\mathcal{C}_q} = \emptyset$ (since $\mathcal{C}_q$ cannot contain p), whereas $\sigma_{\mathcal{C}_q} = \{ \emptyset \}$, so $\tau_{\mathcal{C}_q} \neq \sigma_{\mathcal{C}_q}$, and by the Definition 4, we have $\neg \tau \approx_q \sigma$. Similarly for $r = p$ and a branch $\mathcal{C}_p \ni p$: $\tau_{\mathcal{C}_p} = \{ \emptyset \} \neq \emptyset = \sigma_{\mathcal{C}_p}$, so $\neg \tau \approx_p \sigma$ holds too. □

Note, that we do not define here the relation $\not\approx_p$, so $\neg \tau \approx_p \sigma$ is not equivalent to $\tau \not\approx_p \sigma$. The formula $\neg \tau \approx_p \sigma$ simply means that there exists a branch $\mathcal{C}$ containing p such, that $\tau_{\mathcal{C}} \neq \sigma_{\mathcal{C}}$. In some particular cases we can say even more; the following proposition handles one of such simple cases.

Proposition 1. *Let $S, R \subset \mathbb{T}_n$ be not empty, $q \in \mathbb{T}_n$ and let $\sigma = \{\emptyset\} \times S$ and $\rho = \{\emptyset\} \times R$ be meta sets. If $\neg\, \sigma \approx_q \rho$, then for any branch $\mathcal{C}$ containing q holds $\sigma_{\mathcal{C}} \neq \rho_{\mathcal{C}}$.*

Proof. According to the Definition 4, $\sigma \approx_q \rho$ only if for each branch $\mathcal{C}$ containing q holds $\sigma_{\mathcal{C}} = \rho_{\mathcal{C}}$. Thus, if $\neg\, \sigma \approx_q \rho$, then there must exist a branch $\bar{\mathcal{C}}$ such, that $\sigma_{\bar{\mathcal{C}}} \neq \rho_{\bar{\mathcal{C}}}$. However, if $\mathcal{C}'$ and $\mathcal{C}''$ are any branches containing q, then $\sigma_{\mathcal{C}'} = \sigma_{\mathcal{C}''}$ and also $\rho_{\mathcal{C}'} = \rho_{\mathcal{C}''}$, because $\mathcal{C}' \cap S = \mathcal{C}'' \cap S \subset \{q\}$ and similarly $\mathcal{C}' \cap R = \mathcal{C}'' \cap R \subset \{q\}$ (branches are chains in $\mathbb{T}$, whereas S and R are antichains, so their intersections may contain at most one element). Thus, because all interpretations of σ given by branches containing q are equal (similarly for τ) and for some interpretation $\mathcal{C}$ holds $\sigma_{\mathcal{C}} \neq \tau_{\mathcal{C}}$, then this must hold for all branches. In other words, if $\neg\, \sigma \approx_q \rho$, then for all branches $\mathcal{C}$ containing q holds $\sigma_{\mathcal{C}} \neq \rho_{\mathcal{C}}$. □

The next lemma generalizes the Lemma 1 to arbitrary subsets of $\mathbb{T}_n$. It enables evaluation of the equality degree, in other words – similarity degree, of two character samples.

Lemma 2. *Let $P, Q \subset \mathbb{T}_n$ be not empty and let $\tau = \{\langle\emptyset, p\rangle : p \in P\}$ and $\sigma = \{\langle\emptyset, q\rangle : q \in Q\}$. If $R = P \cap Q \cup (\mathbb{T}_n \setminus P) \cap (\mathbb{T}_n \setminus Q)$, then the following implications hold:*

$$r \in R \;\Rightarrow\; \tau \approx_r \sigma \;, \tag{1}$$

$$r \in \mathbb{T}_n \setminus R \;\Rightarrow\; \neg\, \tau \approx_r \sigma \;. \tag{2}$$

Proof. Assume that $r \in P \cap Q$. If $\mathcal{C}$ is a branch containing r, then clearly $\tau_{\mathcal{C}} = \{\emptyset\} = \sigma_{\mathcal{C}}$, and therefore $\tau \approx_r \sigma$. If $r \in (\mathbb{T}_n \setminus P) \cap (\mathbb{T}_n \setminus Q)$ and $\mathcal{C}$ is a branch containing r, then $\tau_{\mathcal{C}} = \emptyset = \sigma_{\mathcal{C}}$, so $\tau \approx_r \sigma$ holds too. This proves (1).

To prove (2) note, that

$$\begin{aligned}
\mathbb{T}_n \setminus R &= \mathbb{T}_n \setminus \big(P \cap Q \cup (\mathbb{T}_n \setminus P) \cap (\mathbb{T}_n \setminus Q)\big) \;, \\
&= (\mathbb{T}_n \setminus P \cap Q) \cap \big(\mathbb{T}_n \setminus (\mathbb{T}_n \setminus P) \cap (\mathbb{T}_n \setminus Q)\big) \;, \\
&= (\mathbb{T}_n \setminus P \cap Q) \cap (P \cup Q) \;, \\
&= \big((\mathbb{T}_n \setminus P) \cup (\mathbb{T}_n \setminus Q)\big) \cap (P \cup Q) \;, \\
&= (\mathbb{T}_n \setminus P) \cap Q \cup (\mathbb{T}_n \setminus Q) \cap P \;.
\end{aligned}$$

If $r \in (\mathbb{T}_n \setminus P) \cap Q$, and $\mathcal{C}$ is a branch containing r, then $\tau_{\mathcal{C}} = \emptyset$ and $\sigma_{\mathcal{C}} = \{\emptyset\}$, so $\neg\, \tau \approx_r \sigma$. Similarly, if $r \in (\mathbb{T}_n \setminus Q) \cap P$, then $\tau_{\mathcal{C}} = \{\emptyset\}$ and $\sigma_{\mathcal{C}} = \emptyset$, so $\neg\, \tau \approx_r \sigma$. Thus, for $r \in \mathbb{T}_n \setminus R$ we obtain $\neg\, \tau \approx_r \sigma$. □

The last lemma is the meta set version of the obvious fact known from the crisp set theory: $x = y \wedge y \in z \Rightarrow x \in z$.

Lemma 3. *If $p \in \mathbb{T}$ and τ, σ, λ are meta sets such, that $\tau \approx_p \sigma$ and $\sigma \epsilon_p \lambda$, then also $\tau \epsilon_p \lambda$.*

Proof. If $\mathcal{C}$ is an arbitrary branch containing p, then by the assumptions $\tau_{\mathcal{C}} = \sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{C}} \in \lambda_{\mathcal{C}}$. Therefore, also $\tau_{\mathcal{C}} \in \lambda_{\mathcal{C}}$, what implies $\tau \epsilon_p \lambda$. □

We now state the main theorem which allows for calculation of the membership degree of a supplied sample to the compound pattern. The membership degree measures the quality of the sample i.e., its similarity to the defined pattern. In the following theorem the meta set σ represents testing character sample, each π^i for $i = 1 \ldots n$ represents a compound pattern element and ρ is the compound character pattern built up of elements π^i.

Theorem 1. *Let $P^i, R^i, S \subset \mathbb{T}_n$ for $i = 1 \ldots k$ be not empty. Let $\sigma = \{\,\emptyset\,\} \times S$, $\pi^i = \{\,\emptyset\,\} \times P^i$ for $i = 1 \ldots k$ and $\rho = \bigcup_{i=1}^{k} \{\,\pi^i\,\} \times R^i$ be meta sets. For the sets $Q^i = S \cap P^i \cup (\mathbb{T}_n \setminus S) \cap (\mathbb{T}_n \setminus P^i)$, $i = 1 \ldots k$, and $U = \bigcup_{i=1}^{k} Q^i \cap R^i$, the following holds:*

$$q \in Q^i \;\Rightarrow\; \sigma \approx_q \pi^i \;, \tag{3}$$

$$q \in \mathbb{T}_n \setminus Q^i \;\Rightarrow\; \neg\, \sigma \approx_q \pi^i \;. \tag{4}$$

$$u \in U \;\Rightarrow\; \sigma \,\epsilon_u\, \rho \;, \tag{5}$$

$$u \in \mathbb{T}_n \setminus U \;\Rightarrow\; \neg\, \sigma \,\epsilon_u\, \rho \;, \tag{6}$$

Proof. The Lemma 2 proves (3) and (4).

To prove (5) and (6) let $R = \bigcup_{i=1}^{k} R^i$ and let $\bar{Q}^i = \mathbb{T}_n \setminus Q^i$. We may split each R^i into two parts: $R^i = \big(R^i \setminus Q^i\big) \cup \big(R^i \cap Q^i\big) = R^i \cap \bar{Q}^i \,\cup\, R^i \cap Q^i$. Therefore,

$$R \;=\; \bigcup_{i=1}^{k} R^i \cap \bar{Q}^i \;\cup\; \bigcup_{i=1}^{k} R^i \cap Q^i \;=\; \bigcup_{i=1}^{k} \big(R^i \cap \bar{Q}^i\big) \;\cup\; U \;.$$

Let $u \in \mathbb{T}_n$ and let $\mathcal{C}$ be a branch containing u.

First, let $u \in R$. If $u \in U$, then $u \in R^i \cap Q^i$ for some $i \in \{\,1 \ldots k\,\}$. By (3) this implies $\sigma \approx_u \pi^i$, since $u \in Q^i$. By the construction of ρ ($u \in R^i$, so $\langle \pi^i, u \rangle \in \rho$) and by the Definition 3 we have $\pi^i \,\epsilon_u\, \rho$. Thus, by the Lemma 3 we obtain $\sigma \,\epsilon_u\, \rho$, what proves (5). If $u \notin U$ (but still $u \in R$), then let $I \subset \{\,1 \ldots k\,\}$ be the set of all those i, that $u \in R^i \cap \bar{Q}^i$. Since $u \in C$ and for each $i \in I$ the intersection $R^i \cap \mathcal{C}$ contains at most one element which is u, then by the Definition 2: $\rho_{\mathcal{C}} = \big\{\, \pi^i_{\mathcal{C}} \colon R^i \cap \mathcal{C} \neq \emptyset \,\big\} = \big\{\, \pi^i_{\mathcal{C}} \colon u \in R^i \,\big\}$. Note, that $\big\{\, i \colon u \in R^i \,\big\} = \big\{\, i \colon u \in R^i \cap \bar{Q}^i \,\big\} \cup \big\{\, i \colon u \in R^i \cap Q^i \,\big\} = I$, since $u \notin U$. Thus, $\rho_{\mathcal{C}} = \big\{\, \pi^i_{\mathcal{C}} \colon i \in I \,\big\}$ and for $i \in I$ holds $\pi^i_{\mathcal{C}} \in \rho_{\mathcal{C}}$. However by (4) and by the Proposition 1 we get $\sigma_{\mathcal{C}} \neq \pi^i_{\mathcal{C}}$ for $i \in I$. Since $\sigma_{\mathcal{C}}$ is different than all the members of $\rho_{\mathcal{C}}$, then $\sigma_{\mathcal{C}} \notin \rho_{\mathcal{C}}$ for any $\mathcal{C} \ni u$, and consequently $\neg\, \sigma \,\epsilon_u\, \rho$.

If $u \in \mathbb{T}_n \setminus R$, then $\rho_{\mathcal{C}} = \emptyset$, therefore also $\neg\, \sigma \,\epsilon_u\, \rho$. This proves (6) and the whole theorem, since either $u \in U$ or $u \in R \setminus U$ or $u \in \mathbb{T}_n \setminus R$. □

The equality degrees as well as the membership degree are subsets of the n-th level of the tree $\mathbb{T}$. Since there are 2^n nodes on this level, then we may easily map these degrees to rational numbers from the unit interval, dividing the cardinality of the subset representing a degree by 2^n. This mapping allows for evaluating obtained results by means of numbers, what is more human friendly.

3 Character Recognition

Characters might be depicted using rectangular matrices of $width \times height$ cells. For simplicity we consider here a very particular case when $width \cdot height = 2^n$ for some n. The general case involves additional techniques which are beyond the scope of this paper.

We will explain the method for encoding character samples as meta sets and the method for encoding quality degrees of samples as membership degrees. Then, applying the Theorem 1, we will be able to calculate the quality degree of a new testing sample and its similarity to pattern elements.

3.1 Characters as Meta Sets

To represent a character sample as a meta set, first we must establish a mapping of cells of the matrix to nodes of the n-th level of the tree $\mathbb{T}$. We focus here on 4×4 matrices in order to simplify formulas and figures; in practical applications we would rather use matrices comprised of 32 or 64 cells. We map cells of the matrix to nodes from $\mathbb{T}_4$ as on the Fig. 1 (other mappings are acceptable too).

0000	0001	0010	0011
0100	0101	0110	0111
1000	1001	1010	1011
1100	1101	1110	1111

Fig. 1. Mapping of cells of the 4×4 matrix to nodes in $\mathbb{T}_4$

By marking appropriate cells we may draw a character on the matrix. For instance the Fig. 2 represents two versions of the letter `'L'`: marked cells contain nodes from $\mathbb{T}_4$ (depicted as binary sequences), whereas unmarked ones are empty.

0000			
0100			
1000			
1100	1101	1110	

0000			
0100			
1000			
1100	1101	1110	1111

Fig. 2. Two versions of the letter `'L'` represented on the matrix: π^1 and π^2.

Once the mapping is established we define the meta set representing the given character sample to be the crisp set of ordered pairs whose first element is the empty meta set $\emptyset$ and the second element is a node from $\mathbb{T}_4$ corresponding to a marked cell. For instance, the character on the left matrix from the Fig. 2 is represented by the following meta set π^1:

$$\pi^1 = \{ \langle \emptyset, 0000 \rangle , \langle \emptyset, 0100 \rangle , \langle \emptyset, 1000 \rangle , \langle \emptyset, 1100 \rangle , \langle \emptyset, 1101 \rangle , \langle \emptyset, 1110 \rangle \} \quad .$$

Similarly we define π^2 representing `'L'` from the right matrix of the Fig. 2:

$$\pi^2 = \{ \langle\emptyset, 0000\rangle , \langle\emptyset, 0100\rangle , \langle\emptyset, 1000\rangle , \langle\emptyset, 1100\rangle , \langle\emptyset, 1101\rangle , \langle\emptyset, 1110\rangle , \langle\emptyset, 1111\rangle \} \ .$$

The meta sets π^1 and π^2 are different views on the letter `'L'`. Probably one of them is better and another worse. Therefore, we assign them quality degrees in form of sets of nodes or – in other words – in form of membership degrees in some meta set ρ which will represent the compound pattern, and whose domain is comprised of π^1 and π^2. In this paper we assume that the sets of nodes representing quality degrees are subsets of the same level of $\mathbb{T}$ that is mapped to cells of the matrix (here it is $\mathbb{T}_4$). This implies that membership degrees are directly proportional to cardinalities of sets of nodes (it is not true in general, cf. [3], [4]). Thus, the larger cardinality of the set, the greater membership degree, and consequently, the better quality. Assuming that π^1 represents the letter `'L'` better than π^2 we construct the meta set ρ as follows:

$$\rho = \left\{ \pi^1 \right\} \times \{ 0000, 0001, 0010, 0011 \} \ \cup \ \left\{ \pi^2 \right\} \times \{ 1110, 1111 \} \ .$$

We have chosen here some arbitrary sets of nodes, small enough to make formulas simple, yet they express the fact, that the π_1 is better than π_2.

The sets of nodes representing quality degrees should reflect our perception of the quality of the letters. The better one is accompanied by a larger subset of $\mathbb{T}_4$, representing greater membership degree. The best one, if it existed, should have the whole $\mathbb{T}_4$ as the representant of its quality. This rule leaves some indeterminacy. It is possible to express equal degrees of membership by different sets, which differently influence the result. The precise selection of one of the equivalent subsets is subject to experimentation, similarly to the internal structure of a neural network, which usually cannot be determined a priori by some rule, but has to be tuned experimentally to achieve the best result. On the other hand, it is possible to formulate and prove some laws which simplify selection of the subsets best suited for the given task.

The ratio of the number of different nodes paired with π^1 to the number of all nodes in $\mathbb{T}_4$ is the numerical value of the quality degree of π^1, similarly for π^2. For π^1 this ratio is 1/4 and for π^2 it is 1/8, so the fact that the former resembles the letter `'L'` better than the latter is properly reflected. Note, that we have given π^1 and π^2 small ratings in order to simplify formulas; in practice, these ratings should be close to 1, since these samples resemble the letter `'L'` quite well.

3.2 Evaluating Quality of a Testing Sample

Let σ be a meta set representing a testing sample supplied by a user, like the one from the Fig. 3: $\sigma = \{ \langle\emptyset, 0100\rangle , \langle\emptyset, 1000\rangle , \langle\emptyset, 1100\rangle , \langle\emptyset, 1101\rangle , \langle\emptyset, 1110\rangle \}$.

To what measure does this sample match our view of the letter `'L'`, or – in other words – what is the degree of membership of σ to ρ? And what are the equality degrees of σ and each π^i (they express the resemblance of the supplied sample to each pattern element)? By the Theorem 1 we may calculate the answer

0100			
1000			
1100	1101	1110	

Fig. 3. A testing sample σ of the letter `'L'`, supplied by a user

as follows. The constructions of σ, ρ and each π^i imply that the sets S, P^i and R^i from the Theorem 1 have the following contents:

$$\begin{aligned} S &= \{\, 0100, 1000, 1100, 1101, 1110 \,\} \quad , \\ P^1 &= \{\, 0000, 0100, 1000, 1100, 1101, 1110 \,\} \quad , \\ P^2 &= \{\, 0000, 0100, 1000, 1100, 1101, 1110, 1111 \,\} \quad , \\ R^1 &= \{\, 0000, 0001, 0010, 0011 \,\} \quad , \\ R^2 &= \{\, 1110, 1111 \,\} \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} Q^1 &= S \cap P^1 \cup (\mathbb{T}_4 \setminus S) \cap (\mathbb{T}_4 \setminus P^1) = S \cup (\mathbb{T}_4 \setminus P^1) = \mathbb{T}_4 \setminus \{\, 0000 \,\} \quad , \\ Q^2 &= S \cap P^2 \cup (\mathbb{T}_4 \setminus S) \cap (\mathbb{T}_4 \setminus P^2) = S \cup (\mathbb{T}_4 \setminus P^2) = \mathbb{T}_4 \setminus \{\, 0000, 1111 \,\} \quad , \\ U &= Q^1 \cap R^1 \cup Q^2 \cap R^2 = \{\, 0001, 0010, 0011 \,\} \cup \{\, 1110 \,\} \quad , \end{aligned}$$

and finally, for any $p \in \mathbb{T}_4$ we obtain $\sigma \ \epsilon_p \ \rho \Leftrightarrow p \in U$ and $\sigma \approx_p \pi^1 \Leftrightarrow p \in Q^1$ and $\sigma \approx_p \pi^2 \Leftrightarrow p \in Q^2$.

As we see, σ represents the letter `'L'` equally well as π^1 and better than π^2, since U has the same number of elements as R^1 and more than R^2. The numerical ratio for the membership degree of σ in ρ equals $1/4$, like it is the case for π^1. The sets Q^1 and Q^2 measure similarity of σ to π^1 and π^2 respectively. Since Q^1 includes Q^2, it follows that σ resembles π^1 better than π^2.

In general, the obtained results strongly depend not only on the cardinality of R^1 and R^2, but rather on the nodes they contain. If we change the contents of the sets R^i, but preserve their cardinalities, the results might be completely different. This variability enables supposing additional semantics on the quality degrees R^i, besides the linear ordering of their cardinalities. An example of such semantics is stressing some important areas of the matrix diminishing at the same time the relevance of other cells, i.e., pixels in characters which define the compound pattern. It is done simply by choosing elements of R^i from some crucial area of the matrix, like, for instance, the cells that might contain the dot over the letter `'i'`.

The core problem in grading the compound pattern elements, i.e. defining the sets R^i, is to achieve a result (the set U for a supplied sample) consistent with human intuition – a human perception of similar characters. This problem is partially open and is subject to investigations.

4 Conclusions and Further Work

We have described the method for using meta sets to grade similarity of characters for the particular case of characters represented on matrices comprised of 2^n cells. The general case requires additional facts from the meta set theory and it is to be published soon. Note that the presented mechanism seems to be applicable not only to characters (like letters), but to any type of data expressible by means of sets of finite binary sequences, when the problem involves evaluating a degree to which some predefined compound data pattern is matched by a supplied sample.

The existing computer program already handles the general case of arbitrary rectangular matrices. Initial tests confirm that the presented mechanism is able to properly reflect human notion of similarity of character patterns. Particularly – when supplied with accurate pattern data – the program reasonably grades samples which are not member of the compound pattern set (ρ), i.e., it interpolates human view on similarity of characters.

We stress the fact that the theory of meta sets, especially in the form introduced in [3], is directed towards computer implementations and applications. It is because of possible to achieve efficiency of algorithms realizing fundamental relations and operations. The current implementation has a testing character and is to be replaced by a final product in the future. We expect interesting results and computer applications once the final implementation is ready. The theory is under development, new papers on this subject are under preparation; this one includes only the facts that are substantial for the discussed problem.

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