Theory of Computing Systems manuscript No.
(will be inserted by the editor)

Model Checking Abilities of Agents: A Closer Look

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The date of receipt and acceptance will be inserted by the editor

Abstract Alternating-time temporal logic (ATL) is a logic for reasoning about open computational systems and multi-agent systems. It is well known that ATL model checking is *linear in the size of the model*. We point out, however, that the size of an ATL model is usually *exponential in the number of agents*. When the size of models is defined in terms of *states and agents* rather than *transitions*, it turns out that the problem is (1) $\mathbf{\Delta_3^P}$-complete for concurrent game structures, and (2) $\mathbf{\Delta_2^P}$-complete for alternating transition systems. Moreover, for "Positive ATL" that allows for negation only on the level of propositions, model checking is (1) $\mathbf{\Sigma_2^P}$-complete for concurrent game structures, and (2) **NP**-complete for alternating transition systems. We show a nondeterministic polynomial reduction from checking arbitrary alternating transition systems to checking turn-based transition systems, We also discuss the determinism assumption in alternating transition systems, and show that it can be easily removed.

In the second part of the paper, we study the model checking complexity for formulae of ATL *with imperfect information* (ATL$_{ir}$). We show that the problem is $\mathbf{\Delta_2^P}$-complete in the number of transitions and the length of the formula (thereby closing a gap in previous work of Schobbens [37]). Then, we take a closer look and use the same fine structure complexity measure as we did for ATL with perfect information. We get the surprising result that checking formulae of ATL$_{ir}$ is also $\mathbf{\Delta_3^P}$-complete in the general case, and $\mathbf{\Sigma_2^P}$-complete for "Positive ATL$_{ir}$". Thus, model checking agents' abilities for both perfect and imperfect information systems belongs to the same complexity class when a finer-grained analysis is used.

`Keywords:` multi-agent systems, model checking, computational complexity.

1 Introduction

Alternating-time temporal logic [3–5] is one of the most interesting frameworks that emerged recently for reasoning about computational systems. One of the most appreciated features of ATL is its model checking complexity: *linear in the size of the model* (more precisely, in the *number of transitions* in the model) *and the size of the formula.* Thus, the complexity is the same as for computation tree logic CTL (despite ATL being strictly more expressive than CTL). While the result is certainly attractive, it guarantees less than one could expect. We point out that, unlike in CTL, the number of transitions outgoing from a single global state in an ATL model is usually *exponential* in the number of agents. While it is well-known that the number of states in a model can be exponential in the size of a higher-level description of the system, it also turns out that the size of an *explicit* ATL model is usually exponential in the number of agents, *even when no higher level description is considered.*

Following this observation, we study the precise ATL model checking complexity for *explicit models* when *the size of models is defined in terms of states* rather than transitions, and *the number of agents is considered a parameter of the problem.* In fact, we show that the model checking problem is intractable in such a setting of input parameters. Firstly, it turns out that the problem is $\mathbf{\Sigma_2^P}$-complete for the language of "Positive ATL" (where negation is allowed only on the level of propositions) and the semantics based on concurrent game structures (CGS). Secondly, model checking "Positive ATL" is "only" **NP**-complete when an earlier semantics, based on alternating transition systems (ATS), is used. Using our ideas, Laroussinie et al. [29] have proved that model checking formulae of the full ATL is $\mathbf{\Delta_3^P}$-complete for CGS, and $\mathbf{\Delta_2^P}$-complete for ATS; we cite their results, too.[1]

Additionally, we point out that ATL model checking over the broader class of nondeterministic alternating transition systems is still $\mathbf{\Delta_2^P}$-complete for the full language, and **NP**-complete for the "positive" sublanguage—which suggests that using the more general class of ATS may be a good choice in practice.

The results mentioned above apply to general, arbitrary models (be it CGS or ATS). On the other hand, model checking ATL for turn-based models (i.e., those in which only one agent/process at a time is executing an action) can still be done in deterministic polynomial time. We show how, for an arbitrary alternating transition system M, a turn-based system M'

[1] A note of explanation. $\mathbf{\Delta_2^P} = \mathbf{P^{NP}}$ is the class of problems that can be solved by a *deterministic* Turing machine in polynomial time with adaptive calls to an **NP** oracle. $\mathbf{\Sigma_2^P} = \mathbf{NP^{NP}}$ is the class of problems that can be solved by a *non-deterministic* Turing machine in polynomial time with calls to an **NP** oracle. $\mathbf{\Delta_3^P} = \mathbf{P^{\Sigma_2^P}}$ is the class of problems that can be solved by a *deterministic* Turing machine in polynomial time with adaptive calls to a $\mathbf{\Sigma_2^P}$ oracle [35, 7].
Contrary to what the index suggests, $\mathbf{\Delta_{i+1}^P}$ belongs still to the i-th level of the polynomial hierarchy.

can be constructed, so that a combination of *choices* in M corresponds to a combination of *strategies* in a fragment of M'. We then propose a translation of ATL formulae into ATL$^+$ formulae, such that the original formula holds in M, q iff the translated formula holds in M', q. Finally, we point out that the latter can be model-checked in time $\mathbf{\Delta_2^P}$ (respectively **NP** for "Positive ATL"), and thus provide another (slightly more general) proof that the original problem is in $\mathbf{\Delta_3^P}$ (resp. $\mathbf{\Sigma_2^P}$ for "Positive ATL"). The translation of models is independent from the translation of formulae in our construction, which allows for "pre-compiling" models when one wants to check various properties of a particular multi-agent system.

The last part of the paper is concerned with ATL *with imperfect information* (ATL$_{ir}$), introduced by Schobbens in [37]. Since no satisfying semantics based on alternating transition systems for strategic abilities under imperfect information has been proposed so far, we present our results for an epistemic extension of concurrent game structures only. First, we close a gap in Schobbens's results, and show that model checking ATL$_{ir}$ formulae is $\mathbf{\Delta_2^P}$-complete with respect to the number of transitions in the model and the length of the formula (thus confirming the initial intuition of Schobbens [37]). We also show that the problem is **NP**-complete for "Positive ATL$_{ir}$". Next, we demonstrate that model checking ATL$_{ir}$ is $\mathbf{\Delta_3^P}$-complete when the size of models is defined in terms of states and agents rather than transitions (and $\mathbf{\Sigma_2^P}$-complete for "Positive ATL$_{ir}$" in the same setting). We point out that the result is somewhat surprising: *checking abilities of agents acting under imperfect information falls into the same complexity class as checking abilities of agents in perfect information scenarios* when a finer-grained analysis is used.

This article is organised as follows. In Section 2 we introduce ATL and its semantics. Several variants of this logic are considered and the notions of *perfect* and *imperfect* information in these systems are precisely defined. Section 3 presents known and new results on the complexity of model checking ATL with concurrent game structures. In Section 4 we consider model checking ATL with alternating transition systems. We also show that the usual *singleton* requirement in ATS can be relaxed without affecting the complexity. In Section 5 we relate general ATS and turn-based systems. Section 6 contains our main results with respect to agents with imperfect information. They suggest, rather surprisingly, that there is no major difference in the model checking complexity between games of perfect and imperfect information. We conclude with Section 7.

1.1 History of the Results

This article is based on preliminary results presented in a series of conference papers [20–22]. In [20] we considered model checking of ATL, observing (correctly) that the "perceived" size of ATL models is very sensitive to the measure one applies (much more so than CTL models). We concluded that the model checking problem was $\mathbf{\Sigma_2^P}$-complete for CGS, and **NP**-complete for ATS. Unfortunately, *these* claims were incorrect, as Laroussinie, Markey and Oreiby pointed out in [29]. The error was related to the way we handled negation in our model checking algorithms. Laroussinie and colleagues used our ideas to obtain the *correct* results, namely to prove that ATL model checking is $\mathbf{\Delta_3^P}$-complete for concurrent game structures, and $\mathbf{\Delta_2^P}$-complete for alternating transition systems [29]. Still, they observed that our algorithms *were* correct for "Positive ATL" – i.e., ATL without negated cooperation modalities. We summarize all the relevant results in Section 3 of this paper, to get the complete picture.

Another paper [21], where we reported complexity results on model checking ATL$_{ir}$, suffered from the error mentioned above. Again, our claims were correct for "Positive ATL$_{ir}$", but incorrect for model checking of the full logic. In Section 6.2, we present an entirely new result, proving that model checking of full ATL$_{ir}$ is $\mathbf{\Delta_2^P}$-hard (and hence, by Schobbens's result [37], also $\mathbf{\Delta_2^P}$-complete) with respect to the number of transitions in the model. Then we use the results, obtained by Laroussinie et al. for ATL [29], to establish the precise model checking complexity of ATL$_{ir}$ with respect to the number of states and agents.

2 ATL: A Logic of Strategic Ability

The logic of ATL [3–5] was originally invented to capture properties of *open computer systems* (such as computer networks), where different components can act autonomously, and computations in such systems result from their combined actions. Alternatively, ATL can be seen as a logic for systems involving multiple agents, that allows one to reason about what agents can achieve in game-like scenarios. ATL can be also understood as a generalisation of the well-known branching time logic CTL [14,13], in which path quantifiers E ("there is a path") and A ("for each path") are replaced by *cooperation modalities* $\langle\!\langle A\rangle\!\rangle$ that express strategic abilities of agents and their teams.

Formula $\langle\!\langle A\rangle\!\rangle\varphi$ expresses that agents A have a collective strategy to enforce φ. ATL formulae include temporal operators: "$\bigcirc$" ("in the next state"), $\Box$ ("always from now on") and $\mathcal{U}$ ("until"). An additional operator $\Diamond$ ("sometime from now on") can be defined as $\Diamond\varphi \equiv \top\,\mathcal{U}\,\varphi$. Like in CTL, every occurrence of a temporal operator is preceded by exactly one cooperation modality (this variant of the language is sometimes called "vanilla" ATL). The broader language of ATL*, in which no such restriction is imposed, is discussed briefly in Section 2.3.

Formally, the recursive definition of ATL formulae is:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\!\langle A\rangle\!\rangle \bigcirc\varphi \mid \langle\!\langle A\rangle\!\rangle\Box\varphi \mid \langle\!\langle A\rangle\!\rangle\varphi\,\mathcal{U}\,\varphi.$$

"Positive ATL*"* is the subset of ATL in which the use of negation is limited to propositional formulae. Formally, it can be defined by the following

grammar:

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle\!\langle A \rangle\!\rangle \bigcirc \varphi \mid \langle\!\langle A \rangle\!\rangle \Box \varphi \mid \langle\!\langle A \rangle\!\rangle \varphi \mathcal{U} \varphi.$$

A number of different semantics and model classes have been defined for ATL, most of them equivalent (cf. [16,17]). Among these, *concurrent game structures* [5] are probably the most natural and easiest to come up with when modeling concrete problem domains. Moreover, they are the easiest to extend to the imperfect information case, because actions have global identity in concurrent game structures (cf. [18]). However, it seems that *alternating transition systems*, introduced in the more preliminary papers [3, 4] may offer some advantage in terms of model checking complexity (see the results in Sections 3 and 4).

In what follows, we begin with a brief presentation of the two most prominent semantics, based on concurrent game structures and alternating transition systems. In Section 2.4, we are extending the scope of ATL with the possibility that some agents have imperfect information about the current state of the world. Research on this subject is far from being complete, yet a number of ATL extensions have already been proposed to cope with such systems: from the logics of ATEL [40,41] and "ATL with incomplete information" [5] to more sophisticated approaches like ATOL and ATEL-R* [23], ATL$_{ir}$ and ATL$_{iR}$ [37], ETSL [42], and CSL [24]. Among these, ATL$_{ir}$ seems to stand out for its simplicity and conceptual clarity; also (unlike for logics of agents with perfect recall, e.g. ATEL-R* and ATL$_{iR}$), its model checking procedure is decidable. We believe that ATL$_{ir}$, while probably *not* the definitive ATL extension for games with imperfect information,[2] includes constructs that are indispensable when addressing such games. Thus, we treat ATL$_{ir}$ as a kind of "core" ATL-based language for strategic ability under imperfect information, and present its syntax and semantics in Section 2.4.

2.1 Strategic Abilities with Concurrent Game Structures

Concurrent game structures (CGS) [5], can be defined as tuples

$$M = \langle \mathbb{A}\mathrm{gt}, St, \mathit{\Pi}, \pi, Act, d, o \rangle,$$

where:

- $\mathbb{A}\mathrm{gt} = \{a_1, ..., a_k\}$ is a finite nonempty set of all agents,
- St is a nonempty set of states,
- $\mathit{\Pi}$ is a set of atomic propositions,
- $\pi : \mathit{\Pi} \to \mathcal{P}(St)$ is a valuation of propositions,
- Act is a finite nonempty set of (atomic) actions;
- function $d : \mathbb{A}\mathrm{gt} \times St \to \mathcal{P}(Act)$ defines actions available to an agent in a state, and

[2] ATOL, for example, is strictly more expressive with the same model checking complexity.

– o is a (deterministic) transition function that assigns outcome states $q' = o(q, \alpha_1, \ldots, \alpha_k)$ to states and tuples of actions.

Remark 1 Firstly, this variant of concurrent game structures differs slightly from the original CGS [5]: we represent agents and their actions with symbolic labels, whereas in [5] they are represented with natural numbers.

Secondly, determinism is not a crucial issue here, as systems with nondeterministic outcome of agents' actions can be modeled easily by introducing a new, fictional agent, "nature", which settles all nondeterministic transitions.

A *strategy* of agent a is a conditional plan that specifies what a is going to do in each possible state. Thus, a strategy can be represented with a function $s_a : St \to Act$, such that $s_a(q) \in d_a(q)$. A *collective strategy* for a group of agents $A = \{a_1, ..., a_r\}$ is simply a tuple of strategies $S_A = \langle s_{a_1}, ..., s_{a_r} \rangle$, one per agent from A.

Remark 2 This is an important deviation from the original semantics of ATL [3–5], where strategies assign agents' choices to *sequences* of states, which suggests that agents can recall the whole history of each game. In this article, however, we employ "memoryless" strategies. While the choice of one or another notion of strategy affects the semantics of the full ATL*, and most ATL extensions (e.g. for games with imperfect information), it should be pointed out that both types of strategies yield equivalent semantics for "pure" ATL [37].

A *path* in M is an infinite sequence of states that can result from subsequent transitions, and refers to a possible course of action (or a possible computation). Function $out(q, S_A)$ returns the set of all paths that may occur when agents A execute strategy S_A from state q onward:[3]

$out(q, S_A) = \{\lambda = q_0 q_1 q_2 ... \mid q_0 = q$ and for each $i = 1, 2, ...$ there exists a tuple of agents' decisions $\langle \alpha^{i-1}_{a_1}, ..., \alpha^{i-1}_{a_k} \rangle$ such that $\alpha^{i-1}_a \in d_a(q_{i-1})$ for every $a \in \mathbb{A}\mathrm{gt}$, and $\alpha^{i-1}_a \in S_A(a)(q_{i-1})$ for every $a \in A$, and $o(q_{i-1}, \alpha^{i-1}_{a_1}, ..., \alpha^{i-1}_{a_k}) = q_i\}$.

Let $\lambda[i]$ denote the ith position in computation λ (starting from $i = 0$). The semantics of ATL is defined via the clauses below. Informally speaking, $M, q \models \langle\langle A \rangle\rangle \Phi$ iff there exists a collective strategy S_A such that Φ holds for all computations from $out(q, S_A)$.

$M, q \models p$ iff $q \in \pi(p)$ (where $p \in \Pi$);
$M, q \models \neg\varphi$ iff $M, q \not\models \varphi$;
$M, q \models \varphi \vee \psi$ iff $M, q \models \varphi$ or $M, q \models \psi$;
$M, q \models \langle\langle A \rangle\rangle \bigcirc \varphi$ iff there is a collective strategy S_A such that, for each path $\lambda \in out(S_A, q)$, we have $M, \lambda[1] \models \varphi$;

[3] The notation $S_A(a)$ stands for the strategy s_a of agent a in the tuple $S_A = \langle s_{a_1}, ..., s_{a_r} \rangle$.

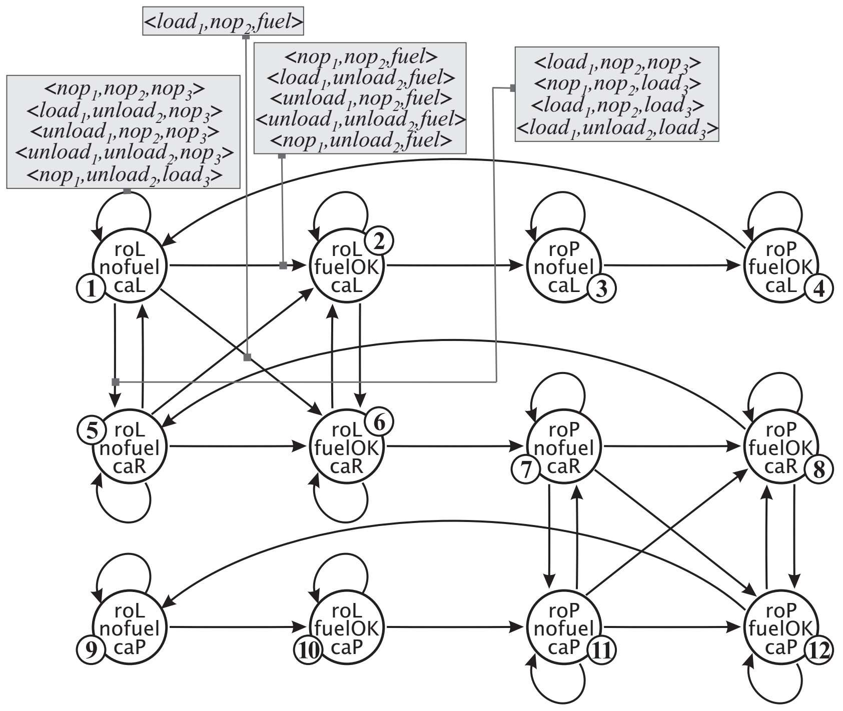


Fig. 1 A CGS for Simple Rocket Domain

$M, q \models \langle\langle A \rangle\rangle \Box \varphi$ iff there exists S_A such that, for each $\lambda \in out(S_A, q)$, we have $M, \lambda[i] \models \varphi$ for every $i \geq 0$;

$M, q \models \langle\langle A \rangle\rangle \varphi \mathcal{U} \psi$ iff there exists S_A such that, for each $\lambda \in out(S_A, q)$, there is $i \geq 0$ for which $M, \lambda[i] \models \psi$, and $M, \lambda[j] \models \varphi$ for each $0 \leq j < i$.

Example 1 Consider a modified version of the Simple Rocket Domain from [8]. There is a rocket that can be moved between London (roL) and Paris (roP), and piece of cargo that can lie in London (caL), Paris (caP), or inside the rocket (caR). Three agents are involved: 1 who can load the cargo, unload it, or move the rocket; 2 who can unload the cargo or move the rocket, and 3 who can load the cargo or supply the rocket with fuel. Each agent can also stay idle at a particular moment (the *nop* – "no-operation" actions). The "moving" action has the highest priority. "Loading" is executed when the rocket does not move and more agents try to load than to unload; "unloading" works in a similar way (in a sense, the agents "vote" whether the cargo should be loaded or unloaded). Finally, "fueling" can be accomplished only when the rocket tank is empty (alone or in parallel with loading or unloading). The rocket can move only if it has some fuel (fuelOK), and the fuel must be refilled after each flight. The concurrent game structure for the domain is shown in Figure 1 (we will refer to this model as M_1). All the transitions for state 1 (the cargo and the rocket are in London, no fuel in the rocket) are labeled; output of agents' choices for other states is analogous.

Example ATL formulae that hold in $M_1, 1$ are: $\neg\langle\langle 1\rangle\rangle\Diamond \mathsf{caP}$ (agent 1 cannot deliver the cargo to Paris on his own), $\langle\langle 1,3\rangle\rangle\Diamond \mathsf{caP}$ (1 and 3 can deliver the cargo if they cooperate), and $\langle\langle 2,3\rangle\rangle\Box(\mathsf{roL} \wedge \langle\langle 2,3\rangle\rangle\Diamond \mathsf{roP})$ (2 and 3 can keep the rocket in London forever, and at the same time retain the ability to change their strategy and move the rocket to Paris).

It is worth pointing out that the CTL path quantifiers A and E can be embedded in ATL in the following way: $\mathsf{A}\varphi \equiv \langle\langle\varnothing\rangle\rangle\varphi$ and $\mathsf{E}\varphi \equiv \langle\langle\mathbb{A}\mathrm{gt}\rangle\rangle\varphi$. Note that the determinism of the transition function is essential for the latter property. In a deterministic system, a collective strategy for the "grand coalition" of agents $\mathbb{A}\mathrm{gt}$ determines a *single* path in the model. In contrast, this is usually not the case in non-deterministic systems. Thus, it may be the case that there is a single path for which property φ holds (i.e., we have $\mathsf{E}\varphi$), and yet the agents are not able to enforce it, so $\langle\langle\mathbb{A}\mathrm{gt}\rangle\rangle\varphi$ does not hold.

On the other hand, $\mathsf{A}\varphi$ *is* equivalent to $\langle\langle\varnothing\rangle\rangle\varphi$ even when we abandon the determinism assumption (to see this, it is sufficient to check what the semantic clauses for $\langle\langle\varnothing\rangle\rangle\bigcirc\varphi$, $\langle\langle\varnothing\rangle\rangle\Box\varphi$ and $\langle\langle\varnothing\rangle\rangle\varphi\,\mathcal{U}\,\psi$ look like). This allows us to define $\mathsf{E}\bigcirc\varphi$ as $\neg\langle\langle\varnothing\rangle\rangle\bigcirc\neg\varphi$, $\mathsf{E}\Box\varphi$ as $\neg\langle\langle\varnothing\rangle\rangle\Diamond\neg\varphi$, and $\mathsf{E}\Diamond\varphi$ as $\neg\langle\langle\varnothing\rangle\rangle\Box\neg\varphi$. Still, as demonstrated in [28], $\mathsf{E}\varphi\,\mathcal{U}\,\psi$ cannot be expressed with any combination of the above operators.[4]

2.2 Semantics of ATL Based on ATS

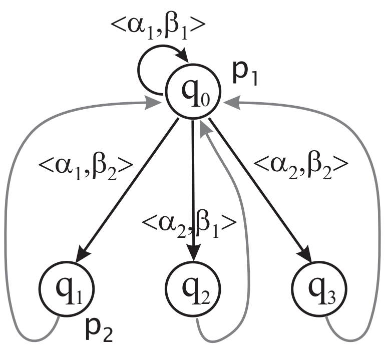
Previous versions of ATL were defined over alternating transition systems [3, 4]. An *alternating transition system* (ATS) is a tuple

$$M = \langle \mathbb{A}\mathrm{gt}, St, \Pi, \pi, \delta\rangle,$$

where:

- $\mathbb{A}\mathrm{gt}$ is a non-empty finite set of *agents*, St is a non-empty set of *states*, Π is a set of (atomic) *propositions*, and $\pi : St \to \mathcal{P}(\Pi)$ is a *valuation* of propositions;
- $\delta : St \times \mathbb{A}\mathrm{gt} \to \mathcal{P}(\mathcal{P}(St))$ is a function that maps pairs $\langle state, agent\rangle$ to non-empty families of choices with respect to possible next states. The idea is that, at state q, agent a chooses a set $Q_a \in \delta(q,a)$ thus forcing the outcome state to be from Q_a. The resulting transition leads to a state which is in the intersection of all Q_a for $a \in \mathbb{A}\mathrm{gt}$ and so it reflects the will of all agents. Since the system is required to be deterministic (given the state and the agents' decisions), $Q_{a_1} \cap ... \cap Q_{a_k}$ must always be a singleton.

[4] See also Remark 4, and the expressivity results from [29].



$\mathbb{A}\mathrm{gt} = \{a, b\}$
$St = \{q_0, q_1, q_2, q_3\}$
$\Pi = \{p_1, p_2\}$
$\pi(p_1) = \{q_0\}$, $\pi(p_2) = \{q_1\}$

$\delta(q_0, a) = \{\overset{\alpha_1}{\{q_0, q_1\}}, \overset{\alpha_2}{\{q_2, q_3\}}\}$
$\delta(q_0, b) = \{\overset{\beta_1}{\{q_0, q_2\}}, \overset{\beta_2}{\{q_1, q_3\}}\}$

$\delta(q_1, b) = \delta(q_2, a) = \delta(q_3, a) = \{\{q_0\}\}$
$\delta(q_1, a) = \delta(q_2, b) = \delta(q_3, b) = \{\{q_0, q_1\}\}$

Fig. 2 Alternating transition system M_2: two agents, each has two choices at state q_0

In an ATS, the type of a strategy function is slightly different since choices are sets of states now, and a strategy is represented as a mapping $s_a : St \to \mathcal{P}(St)$, such that $s_a(q) \in \delta(q, a)$. The rest of the semantics looks exactly the same as for concurrent game structures. In particular, the semantic clauses are exactly the same as the ones in Section 2.1.

Example 2 Consider ATS M_2 from Figure 2. We use symbols α_1, α_2 and β_1, β_2 as shorthands for the choices, to make the example easier to read. The following ATL formulae hold in M_2, q_0: $\neg\langle\!\langle a\rangle\!\rangle\Diamond \mathsf{p}_2$ (a cannot enforce that p_2 is eventually true), $\langle\!\langle a, b\rangle\!\rangle\Box \mathsf{p}_1$ (a and b can cooperate to guarantee that p_1 always holds), and $\langle\!\langle a\rangle\!\rangle \bigcirc(\mathsf{p}_1 \vee \mathsf{p}_2)$ (a can achieve $\mathsf{p}_1 \vee \mathsf{p}_2$ in the next step).

Note that M_2 is not “tight” in the sense that some choices include states that cannot be reached via these choices. It can be tightened by removing q_1 from the choices in $\delta(q_1, a)$, $\delta(q_2, b)$ and $\delta(q_3, b)$, which yields an equivalent *tight* ATS. We discuss the notion of tightness in Section 4.1 more formally.

It is worth pointing out that alternating transition systems are usually less natural and more difficult to come up with than concurrent game structures; they are also larger in most cases (cf. [19], Section 2.7.4). More precisely: for each ATS there exists an isomorphic CGS, but the reverse does not hold. Moreover, alternating transition systems do not allow easily for extensions (e.g. with the possibility that agents may have imperfect information). This subject is discussed in more detail in [18,17,19].

Remark 3 Note that the determinism assumption *is* significant in the case of ATS. Unlike for CGS, adding an auxiliary player (“nature”) to an existing alternating transition system is neither easy nor straightforward. The problem is to extend the existing choice function δ so that it still satisfies the rigid formal requirement that *all* the intersections of choices are singletons. Designing a completely new ATS from scratch is probably an easier solution.

We note here that model checking of ATL formulae has been proved to be linear in the size of the model and the length of the formula for both concurrent game structures [5] and alternating transition systems [4], which coincides with the model checking complexity for CTL [9]. We will discuss this issue in more detail in Section 3.1.

2.3 Beyond ATL: ATL$^+$ and ATL*

The full language of ATL* [4,5] is usually presented as consisting of

1. *state formulae* $\langle\!\langle A\rangle\!\rangle\varphi$, expressing strategic abilities of agents to enforce specific paths of computation, and
2. *path formulae* $\bigcirc\varphi$ and $\varphi\,\mathcal{U}\,\psi$, expressing temporal properties of paths.

Both state and path formulae can be combined using Boolean operators. State formulae are interpreted in states, with $M, q \models \langle\!\langle A\rangle\!\rangle\varphi$ meaning "there is S_A such that, for each path $\lambda \in out(q, S_A)$, we have $M, \lambda \models \varphi$". Path formulae are interpreted in paths, with $M, \lambda \models \bigcirc\varphi$ and $M, \lambda \models \varphi\,\mathcal{U}\,\psi$ defined in the obvious way.

ATL* is more costly in computational terms. Model checking ATL* with memoryless strategies (i.e., the variant that we are interested in here) is **PSPACE**-complete [37]. Model checking ATL* with perfect recall is even more expensive: it is 2EXPTIME-complete in the number of transitions in the model and the length of the formula [5].

In this article, we are only interested in its subset ATL$^+$ [37], in which each temporal operator is preceded by a single cooperation modality, modulo Boolean operators. That is, $\langle\!\langle A\rangle\!\rangle$ is followed by a Boolean combination of path formulae $\bigcirc\varphi$, $\varphi\,\mathcal{U}\,\psi$, in which φ, ψ are state formulae again. As an example, the following is an ATL$^+$ formula: $\langle\!\langle a\rangle\!\rangle(\Box(\mathsf{p}_1 \vee \mathsf{p}_2) \wedge \Diamond\mathsf{p}_1)$. It states that a has a strategy to visit state q_0 at least once, while staying in states q_0, q_1 all the time (note, by the way, that the formula holds in M_2, q_0 from Example 2).

ATL$^+$ can be seen as a generalisation of CTL$^+$ [15]. Model checking of ATL$^+$ has been proved Δ_3-complete in the number of transitions and the length of the formula (for both memoryless and perfect recall strategies) [37], while CTL$^+$ model checking is Δ_2-complete [30]. However, the ATL$^+$ and CTL$^+$ formulae that we use in this article can be model checked in nondeterministic polynomial time (cf. Section 5.2).

2.4 Strategic Abilities under Imperfect Information

ATL and its models include no way of addressing uncertainty that an agent or a process may have about the current situation; moreover, strategies in ATL can define different choices for any pair of different states, hence implying that an agent can recognise each (global) state of the system, and act accordingly. Thus, it can be argued that the logic is tailored for describing and analyzing systems in which every agent/process has *complete and accurate knowledge* about the current state of the system. This is usually not the case for most application domains, where a process can access its *local* state, but the state of the environment and the (local) states of other agents can be observed only partially.

One of the main challenges, when a logic of strategic abilities under imperfect information is addressed, is the question of how agents' knowledge should interfere with the agents' available strategies. When reasoning about what an agent can *enforce*, it seems more appropriate to require the agent to know his winning strategy rather than to know only that such a strategy exists [18,23,25]. This problem is closely related to the distinction between knowledge *de re* and knowledge *de dicto*, well known in the philosophy of language [36], as well as in research on the interaction between knowledge and action [33,34,43]. Several variations on "ATL with imperfect information" have been proposed [23,37,25,42,24], yet none of them has been commonly accepted. In this article, we treat Schobbens' ATL$_{ir}$ and ATL$_{iR}$ [37] as "core", minimal ATL-based languages for strategic ability under imperfect information. The first logic enables reasoning about agents that have no implicit memory of the game (i.e., they use "memoryless" strategies), while the latter is guided by the assumption that agents can always memorise the whole game. As agents seldom have unlimited memory, and logics of strategic ability with imperfect information and perfect recall are believed to have undecidable model checking, we use ATL$_{ir}$ as *the* logic of strategic ability under uncertainty here.

ATL$_{ir}$ includes the same formulae as ATL, only the cooperation modalities are presented with a subscript: $\langle\!\langle A\rangle\!\rangle_{ir}$ to indicate that they address agents with imperfect *information* and imperfect *recall*. Like for ATL, *"Positive* ATL$_{ir}$*"* is the subset of ATL$_{ir}$ in which the use of negation is limited to propositional formulae. Models of ATL$_{ir}$, *imperfect information concurrent game structures* (i-CGS), can be presented as concurrent game structures augmented with a family of indistinguishability relations $\sim_a \subseteq St \times St$, one per agent $a \in \mathbb{A}\mathrm{gt}$. The relations describe agents' uncertainty: $q \sim_a q'$ means that, while the system is in state q, agent a considers it possible that it is in q' now. Each $\sim_a$ is assumed to be an equivalence. It is required that agents have the same choices in indistinguishable states: if $q \sim_a q'$ then $d(a, q) = d(a, q')$.

Again, a *strategy* of an agent a is a conditional plan that specifies what a is going to do in each possible state. An executable (deterministic) plan must prescribe the same choices for indistinguishable states. Therefore ATL$_{ir}$ restricts the strategies that can be used by agents to the set of so called uniform strategies. A *uniform strategy* of an agent a is defined as a function $s_a : St \to Act$, such that: (1) $s_a(q) \in d(a, q)$, and (2) if $q \sim_a q'$ then $s_a(q) = s_a(q')$. A *collective strategy* for a group of agents $A = \{a_1, ..., a_r\}$ is a tuple of strategies $S_A = \langle s_{a_1}, ..., s_{a_r}\rangle$, one per each agent from A. A collective strategy is uniform if it contains only uniform individual strategies. Again, function $out(q, S_A)$ returns the set of all paths that may result from agents

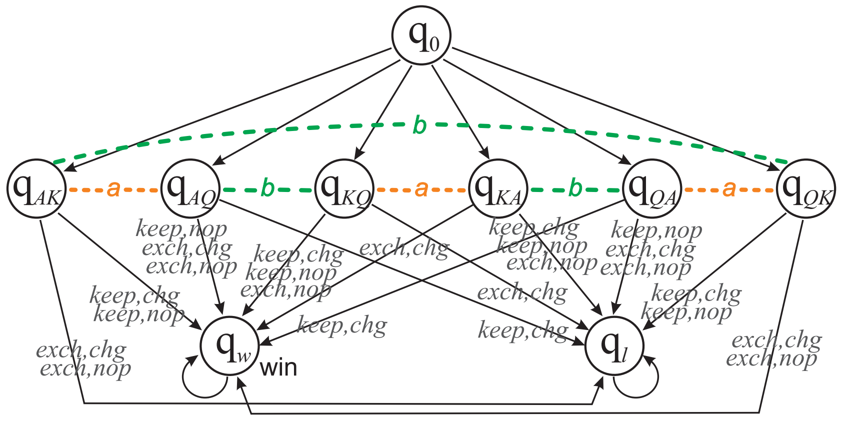


Fig. 3 Gambling Robots game

A executing strategy S_A from state q onward. The semantics of cooperation modalities in ATL_{ir} is defined as follows:

$M, q \models \langle\langle A\rangle\rangle_{ir} \bigcirc \varphi$ iff there is a uniform collective strategy S_A such that, for each $a \in A$, q' such that $q \sim_a q'$, and path $\lambda \in out(S_A, q')$, we have $M, \lambda[1] \models \varphi$;

$M, q \models \langle\langle A\rangle\rangle_{ir} \Box \varphi$ iff there exists a uniform S_A such that, for each $a \in A$, q' such that $q \sim_a q'$, and $\lambda \in out(S_A, q')$, we have $M, \lambda[i] \models \varphi$ for each $i \geq 0$;

$M, q \models \langle\langle A\rangle\rangle_{ir} \varphi \mathcal{U} \psi$ iff there exist a uniform strategy S_A such that, for each $a \in A$, q' such that $q \sim_a q'$, and $\lambda \in out(S_A, q')$, there is $i \geq 0$ for which $M, \lambda[i] \models \psi$, and $M, \lambda[j] \models \varphi$ for each $0 \leq j < i$.

That is, $\langle\langle A\rangle\rangle_{ir}\varphi$ if agents A have a uniform strategy, such that for each path *that can possibly result from execution of the strategy according to at least one agent from* A, φ is the case.

Example 3 (Gambling robots) Two robots (a and b) play a simple card game. The deck consists of Ace, King and Queen (A, K, Q). Normally, it is assumed that A is the best card, K the second best, and Q the worst. Therefore A beats K and Q, K beats Q, and Q beats no card. At the beginning of the game, the "environment" agent deals a random card to both robots (face down), so that each player can see his own hand, but he does not know the card of the other player. Then robot a can exchange his card for the one remaining in the deck (action *exch*), or he can keep the current one (*keep*). At the same time, robot b can change the priorities of the cards, so that Q becomes better than A (action *chg*) or he can do nothing (*nop*), i.e. leave the priorities unchanged. If a has a better card than b after that, then a win is scored, otherwise the game ends in a "losing" state. A CGS M_1 for the game is shown in Figure 3.

It is easy to see that $M_1, q_0 \models \neg\langle\langle a\rangle\rangle_{ir}\Diamond \mathsf{win}$, because, for each a's (uniform) strategy, if it guarantees a win in e.g. state q_{AK} then it fails in q_{AQ}

(and similarly for other pairs of indistinguishable states). Let us also observe that $M_1, q_0 \models \neg\langle\langle a, b\rangle\rangle_{ir}\Diamond\mathsf{win}$: in order to win, a must *exchange* his card in state q_{QK}, so he must *exchange* his card in q_{QA} too (by uniformity), and playing *exch* in q_{QA} leads to the losing state. On the other hand, $M_1, q_{AQ} \models \langle\langle a, b\rangle\rangle_{ir}\bigcirc\mathsf{win}$ (a winning strategy: $s_a(q_{AK}) = s_a(q_{AQ}) = s_a(q_{KQ}) = \mathit{keep}$, $s_b(q_{AQ}) = s_b(q_{KQ}) = s_b(q_{AK}) = \mathit{nop}$; q_{AK}, q_{AQ}, q_{KQ} are the states that must be considered by a and b in q_{AQ}). Still, $M_1, q_{AK} \models \neg\langle\langle a, b\rangle\rangle_{ir}\bigcirc\mathsf{win}$.

Schobbens [37] proved that ATL$_{ir}$ model checking is **NP**-hard and $\mathbf{\Delta_2^P}$-easy. He also conjectured that the problem might be $\mathbf{\Delta_2^P}$-complete. We discuss the issue in more detail, and formally confirm his intuition in Section 6.

Remark 4 The CTL universal path quantifier A can be expressed in ATL$_{ir}$ in the following way: $\mathsf{A}\varphi \equiv \langle\langle\varnothing\rangle\rangle_{ir}\varphi$. The existential path quantifier E, however, is not fully expressible when cooperation modalities quantify over uniform strategies only. Like for non-deterministic models, it may be the case that there is a single path for which property φ holds (i.e., we have $\mathsf{E}\varphi$), and yet even the "grand coalition" of agents $\mathbb{A}\mathrm{gt}$ is not able to enforce it (because $\mathbb{A}\mathrm{gt}$ can now use only uniform strategies), so $\langle\langle\mathbb{A}\mathrm{gt}\rangle\rangle_{ir}\varphi$ does not hold. Moreover, $\mathsf{E}\varphi\,\mathcal{U}\,\psi$ cannot be expressed as a combination of $\mathsf{A}\varphi\,\mathcal{U}\,\psi$, $\mathsf{E}\Diamond\varphi$, $\mathsf{E}\Box\varphi$, $\mathsf{A}\Box\varphi$, $\mathsf{E}\bigcirc\varphi$, and $\mathsf{A}\bigcirc\varphi$ (cf. [28], and the remark at the end of Section 2.1).

3 Complexity of ATL Model Checking Revisited

The model checking problem asks, given model M, state q in M, and formula φ, whether φ holds in M, q. Model checking of temporal logics is usually computationally cheaper than satisfiability checking or theorem proving, while often being at least as useful because the designer or user of a system can come up with a precise model of the system behaviour (e.g. a graph with all the actions that may be executed) in many cases. For ATL, model checking has been proved *linear in the size of the models and formulae.* This seems to be a very good property, but unfortunately it guarantees less than one could expect.

3.1 Model Checking ATL: Easy or Hard?

It has been known for a long time that formulae of CTL can be checked in time linear with respect to the size of the model and the length of the formula [9]. One of the main results concerning ATL states that its formulae can also be model-checked in deterministic linear time.

Proposition 1 ([4, 5]) *The* ATL *model checking problem is* PTIME*-complete, and can be done in time* $\mathbf{O}(ml)$*, where* m *is the number of transitions in the model and* l *is the length of the formula.*

The ATL *model checking algorithm from [4, 5] is presented in Figure 4.*

```
function mcheck_1(M, φ).
Returns the set of states in model $M = \langle \mathbb{A}\mathrm{gt}, St, \Pi, \pi, o\rangle$ for which formula $\varphi$ holds.
case $\varphi \in \Pi$ : return $\pi(p)$
case $\varphi = \neg\psi$ : return $St \setminus mcheck_1(M, \psi)$
case $\varphi = \psi_1 \vee \psi_2$ : return $mcheck_1(M, \psi_1) \cup mcheck_1(M, \psi_2)$
case $\varphi = \langle\langle A\rangle\rangle \bigcirc \psi$ : return $pre_1(M, A, mcheck_1(M, \psi))$
case $\varphi = \langle\langle A\rangle\rangle \Box \psi$ :
  $Q_1 := St$;   $Q_2 := mcheck_1(M, \psi)$;   $Q_3 := Q_2$;
  while $Q_1 \not\subseteq Q_2$
  do  $Q_1 := Q_2$;   $Q_2 := pre_1(M, A, Q_1) \cap Q_3$ od;
  return $Q_1$
case $\varphi = \langle\langle A\rangle\rangle \psi_1 \mathcal{U} \psi_2$ :
  $Q_1 := \emptyset$;   $Q_2 := mcheck_1(M, \psi_1)$;
  $Q_3 := mcheck_1(M, \psi_2)$;
  while $Q_3 \not\subseteq Q_1$
  do  $Q_1 := Q_1 \cup Q_3$;   $Q_3 := pre_1(M, A, Q_1) \cap Q_2$ od;
  return $Q_1$
end case
```

```
function pre_1(M, A, Q).
Auxiliary function; returns the exact set of states $Q'$ such that, when the system is in a state $q \in Q'$, agents $A$ can cooperate and enforce the next state to be in $Q$.
return $\{q \mid \exists\alpha_A \forall\alpha_{\mathbb{A}\mathrm{gt}\setminus A}\ o(q, \alpha_A, \alpha_{\mathbb{A}\mathrm{gt}\setminus A}) \in Q\}$
```

Fig. 4 The ATL model checking algorithm from [5]

While the result is certainly attractive, it should be kept in mind that it is only relative to the size of models and formulae, and these can be very large for most application domains. Indeed, it is well known that the number of states in a model is usually exponential in the size of a higher-level description of the problem domain for both CTL and ATL models. Consider, for example, a system whose state space is defined by r Boolean variables (binary attributes). Obviously, the number of global states in the system is $n = 2^r$. A more general approach is presented in [27], where the "higher level description" is defined in terms of so called *concurrent programs*, that can be used for simulating Boolean variables, but also processes or agents acting in parallel. Each concurrent program $C = \langle C_1, ..., C_k\rangle$ implicitly generates a system of global states which is defined as the product automaton of C. The main result concerning model checking is that checking CTL formulae is **PSPACE**-complete in the size of the concurrent program (and the length of the formula) [27].[5]

[5] We also note in passing that, for *some* high-level system descriptions, even the computation of $\langle\langle A\rangle\rangle \bigcirc$ may require **PSPACE**, or even **NEXPTIME** [10, 11], but these results are not relevant for our discussion here.

Thus, there are basically two kinds of results regarding model checking CTL and ATL. On the one hand, the problem is computationally easy with respect to CTL/ATL models one uses when defining semantics (sometimes called *global state graphs* [9] or *explicit models* [32]). On the other hand, the problem is very hard with respect to more compact representations (e.g. concurrent programs), mainly because these representations unravel into exponentially large explicit models. As a concurrent program may be seen as a system involving k agents, this already shows that having multiple agents can make models (and model checking) explode *with respect to a high level description*. What we point out in this article is that the complexity of $\mathbf{O}(ml)$ includes potential intractability *even on the level of explicit models* if the size of models is defined in terms of states rather than transitions, and the number of agents is a parameter of the problem rather than a fixed value. We state the observation formally as follows.

Remark 5 Let n be the number of states in an explicit ATL model M. It was already observed in [5] that the number of transitions in M is *not* bounded by n^2, because transitions are labeled with tuples of agents' choices. Here, we make the observation more precise.

Let k denote the number of agents, and d the maximal number of available decisions per agent per state. Then, $m = O(nd^k)$. In consequence, the ATL model checking algorithms from [4,5] run in time $O(nd^k l)$, and hence their complexity is exponential if the number of agents is a parameter of the problem.

Example 1 is quite illustrative in this respect. The state space refers to valuations of only three attributes (two binary, and one ternary), which yields 12 states. And the number of transitions is already 216, despite the fact that the system includes only three agents, and each agent has only two or three actions available at each state.

Remark 6 Note that, for turn-based models, only one agent is playing at a time, so the number of transitions is $O(nd)$, and hence model checking can be done in time $O(ndl)$.

Throughout the rest of Section 3, we report results on the complexity of model checking ATL formulae over concurrent game structures, with n, k, d, l as input parameters. We show that the problem is $\mathbf{\Sigma_2^P}$-complete for "Positive ATL", and we cite [29], where model checking of full ATL is proved to be $\mathbf{\Delta_3^P}$-complete. The results seem natural as soon as we re-formulate $M, q \models \langle\langle a_1, ..., a_r \rangle\rangle \bigcirc \varphi$ as $\exists(\alpha_1, ..., \alpha_r)\forall(\alpha_{r+1}, ..., \alpha_k)\ M, o(q, \alpha_1, ..., \alpha_k) \models \varphi$, which bears close resemblance to the problem of QSAT_2. Before we discuss them formally, however, we must make one more important remark.

Remark 7 Note that the transition function o must be kept external to the model checking algorithm, or represented in a somehow "compressed" way. Otherwise the algorithm requires exponential amount of memory to store the function, and in consequence the problem is not even in PSPACE.

One way to achieve this is to assume that the transition function is implemented as an external procedure (more precisely: deterministic Turing machine) that, given state q and actions $\alpha_1, ..., \alpha_k$, returns the value of $o(q, \alpha_1, ..., \alpha_k)$ in polynomial time.

Another way is to represent transitions in a more compact way. Laroussinie and colleagues (after an idea that we developed in [20]) propose the notion of an *implicit concurrent game structure.* An implicit CGS [29] is a CGS where, in each state q, the transition is defined by a finite sequence $((\varphi_1, q_1), ..., (\varphi_n, q_n))$. In the sequence, each q_i is a state, and each φ_i is a boolean combination of propositions $\hat{\alpha}^a$, where $\alpha \in d(a, q)$; $\hat{\alpha}^a$ stands for "agent a chooses action α". The transition function is now defined as follows:

$$o(q, \alpha_1, ..., \alpha_k) = q_i \textit{ iff } i \textit{ is the lowest index such that } \{\hat{\alpha_1}^1, ..., \hat{\alpha_k}^k\} \models \varphi_i.$$

It is required that $\varphi_n \equiv \top$, so that no agent can enforce a deadlock. Every CGS can be encoded as an implicit CGS, with each φ_i being of polynomial size with respect to the number of states and actions [29].

3.2 "Positive ATL" Model Checking for CGS is $\mathbf{\Sigma_2^P}$-hard

Firstly, we show that model checking of "positive" ATL formulae over concurrent game structures is $\mathbf{\Sigma_2^P}$-hard. We show this through a polynomial reduction of QSAT$_2$ to the model checking problem.

Definition 1 (QSAT$_i$: satisfiability for quantified Boolean formulae with i alternations of quantifiers)

Input: k propositional variables $p_1, ..., p_k$ (partitioned into i sets $P_1, ..., P_i$), and a Boolean formula θ that includes no other variables.
Problem: QSAT$_i$ asks if $\exists P_1 \forall P_2 \exists P_3 \ldots \Delta P_i\ \theta$ (where $\Delta = \forall$ if i is even, and $\exists$ if i is odd), i.e. whether there is a valuation of propositions in P_1 such that, for all valuations of propositions in P_2, there exists a valuation of propositions in P_3 etc., such that θ is satisfied.

We use θ as a symbol for the Boolean formula that appears in QSAT$_i$, to distinguish it from the ATL formula in the model checking problem which we usually denote by φ. QSAT$_i$ is known to be $\mathbf{\Sigma_i^P}$-complete [35]. Obviously, in QSAT$_2$, we have only two sets of variables: P_1 and P_2.

To obtain the reduction, we construct a concurrent game structure M with 3 states: $St = \{q_0, q_\top, q_\bot\}$, and k agents: $\mathbb{A}\mathrm{gt} = \{a_1, ..., a_k\}$ that "decide" at q_0 upon valuations of propositions $p_1, ..., p_k \in P_1 \cup P_2$, respectively. Thus, agent a_i can "declare" proposition p_i true (action $\top$) or false (action $\bot$). Every tuple of actions from $\mathbb{A}\mathrm{gt}$ corresponds to a valuation $v_1, ..., v_k$ of the propositions, and vice versa. Now, the transitions from q_0 are defined in the following way:

$$o(q_0, v_1, ..., v_k) = \begin{cases} q_\top \text{ if } v_1, ..., v_k \models \theta \\ q_\bot \text{ else} \end{cases}$$

Transitions from $q_\top$ and $q_\bot$ do not matter. Note that $v_1, ..., v_k \models \theta$ can be verified in time and space linear in $|\theta|$, so o has a polynomial representation with respect to the size of the original problem.[6] Finally, we define proposition sat to hold only in state $q_\top$. Note that the agents "controlling" propositions from P_1 can enforce the next state to be $q_\top$ if, and only if, they can declare such a valuation of "their" propositions that θ is satisfied regardless of the opponents' choices:

Lemma 1 *Let A be the group of agents "responsible" for propositions P_1, i.e. $a_i \in A$ iff $p_i \in P_1$. Then, $\exists P_1 \forall P_2\ \theta$ iff $M, q_0 \models \langle\langle A \rangle\rangle \bigcirc \mathsf{sat}$.*

Proposition 2 *Model checking formulae of "Positive* ATL*" over concurrent game structures is $\mathbf{\Sigma_2^P}$-hard.*

3.3 "Positive ATL" Model Checking for CGS is in $\mathbf{\Sigma_2^P}$

In order to demonstrate membership in $\mathbf{\Sigma_2^P}$ of the model checking problem, we show an algorithm that computes the set of states in which formula φ holds, and lies in $\mathbf{NP^{NP}}$. A careful analysis of the algorithms proposed in [4, 5] reveals that the intractability is due to the pre-image operator *pre*, which is called at most n times for each subformula of φ.[7] Indeed, as we saw in the previous section, checking what a coalition can enforce in a single step (e.g., $M, q \models \langle\langle A \rangle\rangle \bigcirc \mathsf{sat}$) lies very close to the standard $\mathbf{\Sigma_2^P}$-complete problem of QSAT$_2$.

We show that checking a more sophisticated "positive" formula of ATL is no more complex than this. The main idea of the algorithm is as follows.

1. We guess nondeterministically *all* the choices that will be needed for any call to function *Pre* (that is, for each coalition A that occurs in φ, and for each state $q \in St$). The choices are then stored in the array *choice*.
2. We employ the standard model checking algorithm from Figure 4 with one important modification: every time function $pre_2(M, A, Q_1)$ is called, it assumes the subsequent A's choices from the array, and checks whether $q \in pre_2(M, A, Q_1)$ by calling an **NP** oracle (*is there a response from the opposition in q that leads to a state outside Q_1?*) and inverting its answer.

The detailed algorithm is shown in Figures 5 and 6. As the number of iterations, as well as the number of calls to *pre*, in the algorithm from Figure 4 is $O(nl)$, we get a nondeterministic polynomial algorithm that makes calls to an **NP** oracle.

Lemma 2 *Function $mcheck_2$ defines a nondeterministic Turing machine that runs in time $O(nkl)$, making calls to an* **NP** *oracle. The size of the witness is $O(nkl)$. The oracle is a nondeterministic Turing machine that runs in time $O(n + k)$.*

[6] Note that, in fact, this is a simple example of an implicit CGS.

[7] We recall that n is the number of states in M.

```
function mcheck_2(M, φ);
Returns the set of states in M, in which formula φ holds.

■ assign cooperation modalities in $\varphi$ with subsequent numbers $1, ..., c$;
    // note that $c \leq l$;  by $c(\varphi)$, we denote the number of coop. modalities in $\varphi$
    // we will denote the coalition from the $i$th cooperation modality in $\varphi$ as $\varphi[i]$
■ for each $i = 1, ..., c$, assign the agents in $\varphi[i]$ with numbers $1, ..., k_c$;
    // note that $k_c \leq k$ and $k_c \leq l$
    // we will denote the $j$th agent in $A$ with $A[j]$
■ guess an array $choice$ such that, for each $i = 1, ..., c$, $q \in St$, and $j = 1, ..., k_c$, we have that $choice[i][q][j] \in d_{\varphi[i][j]}(q)$;
    // at this point, the optimal choices for all coalitions in $\varphi$ are guessed
    // note that the size of $choice$ is $O(nkl)$
    // by $choice|_i$, we will denote the array $choice$ with rows $1, ..., i-1$ removed
■ return $eval_2(M, \varphi, choice)$;
```

```
function eval_2(M, φ, choice);
Returns the states in which φ holds, given choices for all the coalitions from φ.

case $\varphi \in \Pi$ : return $\{q \mid \varphi \in \pi(q)\}$;
case $\varphi = \neg\psi$ : return $Q \setminus eval_2(M, \psi, choice)$;
case $\varphi = \psi_1 \vee \psi_2$ : return $eval_2(M, \psi_1, choice) \cup eval_2(M, \psi_2, choice|_{c(\psi_1)+1})$;
case $\varphi = \langle\langle A\rangle\rangle \bigcirc \psi$ : return $pre_2(M, A, eval_2(M, \psi, choice|_2), choice[1])$;
case $\varphi = \langle\langle A\rangle\rangle \Box \psi$ : $Q_1 := St$;   $Q_2 := Q_3 := eval_2(M, \psi, choice|_2)$;
  while $Q_1 \not\subseteq Q_2$ do $Q_1 := Q_1 \cap Q_2$; $Q_2 := pre_2(M, A, Q_1, choice[1]) \cap Q_3$
od;
  return $Q_1$;
case $\varphi = \langle\langle A\rangle\rangle \psi_1 \mathcal{U} \psi_2$ : $Q_1 := \emptyset$;   $Q_2 := eval_2(M, \psi_1, choice|_2)$;
  $Q_3 := eval_2(M, \psi_2, choice|_{c(\psi_1)+2})$;
  while $Q_3 \not\subseteq Q_1$ do $Q_1 := Q_1 \cup Q_3$; $Q_3 := pre_2(M, A, Q_1, choice[1]) \cap Q_2$
od;
  return $Q_1$;
end case
```

Fig. 5 Nondeterministic algorithm for model checking formulae of "Positive ATL"; part I.

Proposition 3 *Model checking formulae of "Positive* ATL*" over concurrent game structures is in* $\mathbf{\Sigma_2^P}$.

The following theorem is an immediate corollary:

Theorem 1 *Model checking "Positive* ATL*" formulae over* CGS *is* $\mathbf{\Sigma_2^P}$*-complete with respect to the number of (global) states, actions and agents, and the length of the formula. It is* $\mathbf{\Sigma_2^P}$*-complete even for formulae that include only one cooperation modality, and only the "nexttime" temporal operator* $\bigcirc$.

```
function pre_2(M, A, Q_1, thischoice);
Returns the set of states, for which the A's choices from thischoice enforce
that the next state is in Q_1, regardless of what agents from Agt \ A do.
■ $Q_2 := \varnothing$;
■ for each $q \in St$: if $oracle_2(M, A, Q_1, thischoice, q) = yes$ then $Q_2 := Q_2 \cup \{q\}$ fi;
■ return $Q_2$;
```

```
function oracle_2(M, A, Q_1, thischoice, q);
Returns yes if, and only if, the A's choices from thischoice in q enforce that
the next state is in Q_1, regardless of what agents from Agt \ A do.
■ guess an array resp such that, for each $a \in \mathbb{A}gt \setminus A$, we have $resp[a] \in d_a(q)$;
    // at this point, the most dangerous response from the opposition is guessed
    // note that the size of resp is $O(k)$
■ if $o(q, thischoice[q], resp) \in Q_1$ then return yes else return no fi;
```

Fig. 6 Nondeterministic algorithm for model checking formulae of "Positive ATL": part II.

3.4 Full ATL

As pointed out by Laroussinie, Markey and Oreiby, the algorithm in Figures 5 and 6 can be easily adapted to handle arbitrary ATL formulae in time $\mathbf{\Delta_3^P}$. In that case, strategies are guessed for each cooperation modality separately (and it is not necessary to guess them in advance).

Proposition 4 *Model checking formulae of* ATL *over concurrent game structures is in* $\mathbf{\Delta_3^P}$.

Laroussinie et al. have also proved that the bound is tight.

Theorem 2 ([29]) *Model checking* ATL *over* CGS *is* $\mathbf{\Delta_3^P}$*-complete in the number of states, actions and agents in the model, and the length of the formula.*

4 Model Checking with Alternating Transition Systems

Model checking ATL over CGS is $\mathbf{\Delta_3^P}$-complete (and $\mathbf{\Sigma_2^P}$-complete for "positive" formulae) when the size of models is defined in terms of the number of states, and the number of agents is a parameter of the problem. However, the transition function in a CGS refers to choices that are abstract, while in alternating transition systems the function already encodes some information about possible outcomes of actions. One obvious advantage is that, in an ATS, the transition function is already represented in a compact way: for n states, k agents and at most d decisions per agent and state, the size of function δ is $O(n^2kd)$, while an explicit transition table in a CGS may require $O(nd^k)$ memory cells in general. In this section, we show that using ATS results in some advantages in terms of model checking complexity: it still sits in the nondeterministic polynomial hierarchy, but one level lower. Firstly, we demonstrate that the model checking of "Positive ATL" is in **NP** in Section 4.1. Secondly, in Sections 4.2 and 4.3, we define a variant of the Boolean satisfiability problem that we call "single false clause SAT" (*Sfc-SAT*), prove that it is **NP**-complete, and present a reduction of *Sfc-SAT* to the model checking problem. In Section 4.4, we cite [29] again, where correct complexity results for full ATL were presented.

Modeling systems via ATS is usually troublesome in practice, mostly due to the "singleton" requirement. In Section 4.5 we point out that, if we relax the requirement and allow for nondeterministic ATS's, we obtain the same model checking complexity for a strictly larger class of models.

4.1 Model Checking "Positive ATL" over ATS is in **NP**

Unlike in concurrent game structures, choices in alternating transition systems already contain some information about which states can possibly be achieved through them. More precisely, α includes *all* the states that can be achieved through α. Had it contained *only* such states, checking if it enforces φ would have been easy (it would have been sufficient to check whether φ holds in all $q' \in \alpha$). However, the latter condition is not true in general. [16] introduces the notion of a *tight* ATS: all states q' to which no transition exists from q are removed from agents' choices at q (i.e. from the elements of $\delta(q, a)$ for all $a \in \mathbb{A}\mathrm{gt}$). Still, this is not enough for our purposes, because $\alpha \in \delta(q, a)$ may include states that are reachable from q in general, but not by executing α. In this section, we propose a stronger notion of tightness, and show a nondeterministic algorithm to model check "Positive ATL" formulae over such ATS's. We also present a nondeterministic algorithm to "tighten" an ATS, and point out how these algorithms can be combined to obtain a procedure that model checks "Positive ATL" formulae over arbitrary ATS's in nondeterministic polynomial time. In the following, we assume without loss of generality that $A = \{a_1, ..., a_r\}$ for some $r \le k$.

Definition 2 *Let $\alpha_A = \langle \alpha_1, ..., \alpha_r \rangle$ be a collective choice of A at q, i.e. $\alpha_i \in \delta(q, a_i)$. State q' is α_A-reachable from q if there is a combination of responses from the rest of agents: $\alpha_{r+1}, ..., \alpha_k$, $\alpha_i \in \delta(q, a_i)$ such that $q' \in \alpha_1 \cap ... \cap \alpha_k$.*

Definition 3 ATS M *is* strongly tight *if, for each $q \in St, a \in \mathbb{A}\mathrm{gt}$, we have that for each $q' \in \alpha_a \in \delta(q, a)$, q' is α_a-reachable from q.*

Lemma 3 *Let M be strongly tight, $\alpha_1, ..., \alpha_r$ be choices of $a_1, ..., a_r$ at q, and $q' \in \alpha_1 \cap ... \cap \alpha_r$. Then q' is $\langle \alpha_1, ..., \alpha_r \rangle$-reachable from q.*

Every ATS can be made strongly tight via the procedure in Figure 7. Moreover, "Positive ATL" formulae can be model-checked over strongly tight

```
function $tighten(M)$;
For each $a_i \in \mathbb{A}\mathrm{gt}$, $q \in St$, $\alpha_i \in \delta(q, a_i)$, and $q' \in \alpha_i$:
  ■ guess the "opposition" responses $\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_k$;
  ■ if $q' \notin \alpha_1 \cap ... \cap \alpha_k$ then remove $q'$ from $\alpha_i$;
```

Fig. 7 Algorithm for "tightening" alternating transition systems

```
function $pre_3(M, A, Q1)$;
  ■ $pre := \emptyset$;
  ■ for each $q \in St$:
      – guess $\alpha_a \in \delta(q, a)$ for each $a \in A$;
      – if $\bigcap_{a \in A} \alpha_a \subseteq Q1$ then $pre := pre \cup \{q\}$;
  ■ return $pre$;
```

Fig. 8 New pre-image function for model checking ATL over alternating transition systems

ATS's via the original ATL model checking algorithm from Figure 4, with function $pre(A, Q)$ implemented as in Figure 8. We observe that – if we assign numbers $1, ..., |\delta(q, a)|$ to choices from $\delta(q, a)$ for all q, a at the beginning, so that the choices are further identified by abstract labels rather than their content – all the "guessing" operations are independent from each other when we evaluate a "positive" formula. Thus, we can apply the same trick as in Section 3.3, and guess *all* the necessary information beforehand. The size of the witness is $O(n^2k^2d+nkl)$, hence we obtain an **NP** algorithm for the model checking.

Proposition 5 *Model checking formulae of "Positive* ATL*" over alternating transition systems is in* **NP**.

4.2 Single False Clause SAT

To prove **NP**-hardness of model checking "Positive ATL" over alternating transition systems, we define the following variant of the SAT problem.

Definition 4 (Sfc-SAT: single false clause SAT)

Input: (1) n clauses: $C_1, ..., C_n$, in k propositions: $p_1, ..., p_k$ such that for each valuation of $p_1, ..., p_k$, exactly one clause is false*;*
(2) numbers $m \le n$, $r \le k$.

Problem: Is there a valuation of $p_1, ..., p_r$ such that all clauses $C_1|_r, ..., C_m|_r$ are satisfied? *Clause $C|_r$ is obtained from clause C by deleting all literals that refer to propositions $p_{r+1}, ..., p_k$ (we keep only the literals up to r).*

Remark 8 Obviously, *Sfc-SAT* is in **NP** (it is sufficient to guess a valuation and check whether it is a good one).

In order to show that *Sfc-SAT* is **NP**-hard, we show that 3-SAT can be reduced to it. In 3-SAT, we are given m clauses $C_1, ..., C_m$ over r propositions $p_1, ..., p_r$ such that each clause C_i contains at most three literals: $C_i = l_{i,1} \lor l_{i,2} \lor l_{i,3}$ ($l_{i,j}$ are p_l or $\neg p_l$, $1 \le i \le m$). This special instance of the satisfiability problem is also **NP**-complete [35]. Note that the m and the r are the respective numbers occurring as inputs in Definition 4. To show that 3-SAT can be reduced to *Sfc-SAT*, we demonstrate that there are propositions $p_{r+1}, \dots, p_k$, and clauses $C'_1, ..., C'_n$, with $m \le n$, $C_i \subseteq C'_i$ and $C'_i|_r = C_i$ for $i \le m$, such that for each valuation of $p_1, \dots, p_k$, exactly one of C'_i is false.

What does the last condition mean for a set of clauses $C'_1, ..., C'_n$? Basically, it means that these clauses represent all 2^k possibilities of choosing truth values for $p_1, \dots, p_k$. So, the problem in the reduction is to *extend the given clauses by new variables* and to *add new clauses*. This has to be done so that the length of the new problem is still polynomial in the length of the given 3-SAT instance.

We assume without loss of generality that none of $C'_1, ..., C'_m$ contains a complementary pair of literals (otherwise the clause would be satisfiable under all valuations and could be safely discarded as it does not matter for the overall satisfiability problem). In order to extend clauses $C_1, ..., C_m$ in an appropriate way, we use auxiliary formulae α_i and β, defined in the following way:

α_i: We construct formulae α_i stating that *a selected clause number is $i \le m$*. To be more precise, we introduce $t := \lceil \log m \rceil$ new variables $y_1, \dots, y_t$ and define conjunctions α_i ($i = 1, \dots, m$) over these variables as follows (this idea is due to Thomas Eiter [12]). We write each number $1, \dots, m$ in binary and represent each (of the t) digits by the new variables (a 1 is represented by the variable itself, a 0 by the negation of the variable). The i'th digit is then represented by y_i if it is 1 and by $\neg y_i$ if it is 0. Thus, for each valuation of the new variables, only one conjunction α_i can be true, namely the one representing the number coded in the binary representation.
Note that we can also represent numbers greater than m (up to the next power of 2, namely 2^t). These conjunctions do not correspond to the m original clauses from the 3-SAT problem. In our reduction, we have to distinguish between them. Therefore we introduce a formula β in the next step.

β: We construct a formula β stating that the selected clause number is less than or equal to m. Thus, β satisfies the following equivalences: $\beta \Leftrightarrow \bigvee_{i=1}^{m} \alpha_i \Leftrightarrow \bigwedge_{i=m+1}^{2^{\lceil \log m \rceil}} \neg\alpha_i$.
We therefore define a set of clauses β, which describe all valuations corresponding to numbers strictly greater than m. Thus we have:

$$\beta \Leftrightarrow \bigwedge_{i=1}^{m} \neg\alpha_i.$$

Realising β as a set of clauses is simple: we just take α_m and check that they coincide on an initial segment and then a negated variable occurs (where in α_m a positive variable is located).
β can also be written as a set of clauses

$$\beta \Leftrightarrow \bigwedge_{i=m+1}^{2^{\lceil \log m \rceil}} \neg\alpha_i .$$

Note that the last formula is a set clauses (because all $\neg\alpha_i$ are clauses), and hence we need at most $2^{\lceil \log m \rceil} - m$ many clauses to represent β (which is never more than m). We denote these clauses by $C_1^\beta, \ldots, C_m^\beta$. Each clause C_j^β states, that *the selected clause has not the number* $m+j$. In the following we sometimes use β to represent the (at most) m clauses $C_1^\beta, \ldots, C_m^\beta$.

Extending the clauses: For each $C_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$ we construct the *remaining* 7 clauses (all parities of the 3 variables) and add $\neg\alpha_i$. So, for each C_i we get 8 clauses $C'_{i,0}, \ldots C'_{i,7}$, where $C'_{i,0} = C_i \vee \neg\alpha_i$ and $(C'_{i,0} \wedge \ldots \wedge C'_{i,7}) \Leftrightarrow \neg\alpha_i$. Note, again, that $\neg\alpha_i$ is always a clause. We observe also that the m clauses $C_1, ..., C_m$, which we originally started with (as an instance of 3-SAT), are, by construction, exactly $C'_{1,0}|_r, C'_{2,0}|_r, \ldots, C'_{m,0}|_r$.

Reduction: The (at most) $s := m + m \times 8$ clauses:

$$C_1^\beta, \ldots, C_m^\beta, \qquad \text{and} \qquad C'_{i,j} \;\; (1 \le i \le m,\, 0 \le j \le 7),$$

over $k = r + \lceil \log m \rceil$ variables, represent an instance of *Sfc-SAT*, such that if we choose $m \le n$ and $r \le k$, then we get the 3-SAT problem we started with.

Why are the clauses above an instance of *Sfc-SAT*? The fact that we get back the 3-SAT problem has already been shown. It is also obvious that the constructed instance is polynomial in the size of the instance we started with. So it remains to show that for each valuation of all the variables, *exactly one clause is false*. Let a valuation be given. We must consider two cases:

1. Exactly one of the $\alpha_1, \ldots, \alpha_m$ is true, say α_{i_0} (this is decided by the newly introduced variables). Then all clauses $C'_{i,j}$ with $i \neq i_0$ are true (because $\neg\alpha_i$ is true and it occurs as a disjunct in all these clauses). Of the 8 clauses $C'_{i_0,j}$ $(0 \le j \le 7)$, exactly one is false, namely the one contradicting the valuation of the three old variables occurring in the original C_i (note that all possibilities are covered with the 8 cases). Clearly, β (i.e all clauses C_j^β) is true as well.
2. None of the $\alpha_1, \ldots, \alpha_m$ is true. But then all clauses $C'_{i,j}$ are true and only β is false, i.e. exactly one of the clauses C_j^β.

These are all the cases, because α_i (resp. C_j^β) are pairwise inconsistent by construction: any two different conjunctions α_i, α_j (resp. C_i^β, C_j^β) with

$i \neq j$ contain at least one pair of complementary literals. This gives us the following result:

Proposition 6 *Sfc-SAT is* **NP**-*complete.*

4.3 Reduction of Sfc-SAT to "Positive ATL" Model Checking over ATS

To obtain the reduction, we construct an ATS M with $St = \{q_0, C_1, ..., C_n\}$, i.e. one state per clause plus an initial state. Next, we "simulate" propositions $p_1, ..., p_k$ with agents $a_1, ..., a_k$. Each agent "declares" his proposition true or false in the initial state q_0. Thus, agent a_i has two available choices at q_0: to declare p_i true or to declare p_i false; a choice of a_i is represented with the set of clauses that are *not* made true by setting the value of p_i in this particular way. For example, for clauses $C_1 = p_1 \vee \neg p_2, C_2 = p_2$, a_1's choices are represented as $\{C_2\}, \{C_1, C_2\}$: if p_1 is set to true, only C_2 can be false, but if p_1 is set to false, both C_1, C_2 can remain unsatisfied. Choices and transitions at states $C_1, ..., C_m$ do not really matter. There is only one atomic proposition, $\mathsf{therest}$, with $\pi(\mathsf{therest}) = \{C_{m+1}, ..., C_n\}$.

Note that each combination of choices from $a_1, ..., a_k$ at q_0 corresponds to a single valuation of $p_1, ..., p_k$, and vice versa. Moreover, a clause is not satisfied by a valuation iff no proposition "makes" it true. Thus, the set of clauses, unsatisfied by a valuation, is equal to the intersection of sets of clauses that are not "made" true by each single proposition. By definition of *Sfc-SAT*, such an intersection is always a singleton, which proves that M is indeed an ATS.

Lemma 4 *There is a valuation of $p_1, ..., p_r$ such that all clauses $C_1|r, ..., C_m|r$ are satisfied iff $M, q_0 \models \langle\!\langle a_1, ..., a_r \rangle\!\rangle \bigcirc \mathsf{therest}$.*

Proof **[⇒]** Suppose that there is such a valuation of $p_1, ..., p_r$. Thus, regardless of the actual valuation of $p_{r+1}, ..., p_k$, clauses $C_1, ..., C_t$ must be true, and hence for each valuation of $p_{r+1}, ..., p_k$, the single unsatisfied clause must be among $C_{t+1}, ..., C_n$. Re-writing it in terms of the ATS M: there is a collective choice of $a_1, ..., a_r$ such that, for each tuple of choices from the other agents, the resulting next state must be among $C_{t+1}, ..., C_n$. In consequence, $M, q_0 \models \langle\!\langle a_1, ..., a_r \rangle\!\rangle \bigcirc \mathsf{P}$.

[⇐] Let $M, q_0 \models \langle\!\langle a_1, ..., a_r \rangle\!\rangle \bigcirc \mathsf{P}$. Thus, there is a collective choice $S_{\{a_1,...,a_r\}}$ such that, for each tuple of choices from the other agents, the resulting next state is always among $C_{t+1}, ..., C_n$. We take the valuation of $p_1, ..., p_r$ that corresponds to this $S_{\{a_1,...,a_r\}}$. By the shape of the construction, each $C_i \in \{C_1, ..., C_t\}$ must be true for each valuation of $p_{r+1}, ..., p_k$. In particular, C_i is true for the valuation $p_{r+1} = \bot, ..., p_k = \bot$. Thus, $C_i|r$ is also true.

Note that the reduction can be done in time polynomial in n, k. Computing the agents' choice sets is the hardest point here, and it can be done in time $O(k^2 n)$. The resulting model includes $n + 1$ states, k agents, and

$d = 2$ choices per agent per state – and the length of the resulting formula is $l = r + 2 \leq k + 2$, which concludes the reduction, and proves that the model checking problem is **NP**-hard. Thus, we have the following.

Theorem 3 *Model checking "Positive* ATL*" formulae over* ATS *is* **NP***-complete with respect to the number of (global) states, actions and agents, and the length of the formula. It is* **NP***-complete even for formulae that include only one cooperation modality, and only the "nexttime" temporal operator* $\bigcirc$.

4.4 Full ATL

Like in the case of CGS, the algorithm presented in Section 4.1 can be easily adapted to model-check arbitrary ATL formulae in time $\mathbf{\Delta_2^P}$. Again, strategies are guessed for each cooperation modality separately (and it is not necessary to guess them in advance).

Proposition 7 *Model checking formulae of* ATL *over alternating transition systems is in* $\mathbf{\Delta_2^P}$.

Laroussinie, Markey and Oreiby have proved that the bound is tight.

Theorem 4 ([29]) *Model checking* ATL *over* ATS *is* $\mathbf{\Delta_2^P}$*-complete in the number of states, actions and agents in the model, and the length of the formula.*

4.5 Model Checking with Nondeterministic Transition Systems

Alternating transition systems were proposed as models for open computational systems, and the way in which the transition function is constructed reflects this intention. The problem with ATS's is that they are *not* modular, partly due to the "singleton intersection" requirement: legality of a choice cannot be defined in isolation from the rest of the choices in a given state. Adding another process to the system usually requires thorough reconstruction of the model: in particular, new states must be added, and agents' choices extended so that *each* intersection is again a singleton. We suggest that the requirement can be relaxed, yielding a more general (and more flexible) class of models with the same ATL model checking complexity. To show this, we define *non-deterministic alternating transition systems* (NATS) in the same way as ATS, except that no requirement on function δ is imposed.[8] Obviously, model checking ATL formulae over NATS is $\mathbf{\Delta_2^P}$-hard (and **NP**-hard for "positive" formulae), because ATS are special cases of NATS. Moreover, the model checking algorithm, depicted in Section 4.1, can be applied to NATS as well.

[8] Traditionally, the transition relation is required to be serial in models of temporal logic, in order to make sure that the time "flows forever". This can be ensured by requiring that, for each tuple of choices $\alpha_i \in \delta(q, a_i)$, the intersection $\alpha_1 \cap ... \cap \alpha_k$ is non-empty. Our argument in this section is valid for such a variant of NATS, too.

```
case $\varphi = \langle\langle A\rangle\rangle \psi_1 \mathcal{W} \psi_2$ :
    $Q_1 := mcheck(M, \psi_1) \cup mcheck(M, \psi_2)$;
    $Q_2 := St$;
    $Q_3 := mcheck(M, \psi_2)$;
    while $Q_2 \neq Q_1$
    do  $Q_2 := Q_1$;    $Q_1 := Q_2 \setminus \Big((St \setminus pre(M, A, Q_2)) \setminus Q_3\Big)$ od;
    return $Q_1$
```

Fig. 9 Subroutine for model checking "weak until"

Theorem 5 *Model checking* ATL *formulae over* NATS *is* $\mathbf{\Delta_2^P}$*-complete for the full language, and* **NP***-complete for the "positive" sublanguage.*

This suggests that using the more general class of NATS may be beneficial for most purposes: we can get rid of the rigid and highly inconvenient "singleton" requirement without any computational cost! But, as we already pointed out in Sections 2.1 and 2.2, the existential path quantifier E from CTL cannot be fully embedded in ATL, when the latter has its semantics defined over NATS. This looks as a serious shortcoming in terms of expressivity at first glance. However, we observe that we can deal with this problem by adding another temporal operator to the language of ATL. The operator we propose to add is the "weak until" operator $\mathcal{W}$, known in temporal logic for a long time [31], although not as popular as the "strong until" $\mathcal{U}$. Formula $\varphi \mathcal{W} \psi$ means that, if ψ becomes eventually true, then φ holds until the first occurrence of ψ, otherwise φ holds for ever.[9] The formal semantics of "weak until" can be defined as follows:

$M, q \models \langle\langle A\rangle\rangle \varphi \mathcal{W} \psi$ iff there exists S_A such that, for each $\lambda \in out(S_A, q)$: (1) there is $i \geq 0$ for which $M, \lambda[i] \models \psi$, and $M, \lambda[j] \models \varphi$ for each $0 \leq j < i$, or (2) $M, \lambda[j] \models \varphi$ for each $j \geq 0$.

In Figure 9, we present a simple extension of the model checking algorithm from Figure 4 that deals with "weak until" formulae in a way analogous to model checking $\langle\langle A\rangle\rangle \Box\varphi$ and $\langle\langle A\rangle\rangle \varphi \mathcal{U} \psi$. The model checking algorithm from Section 4.1 can be augmented in the same way. It is easy to see that the complexity of the algorithms stays the same as before. Moreover, we show that adding $\langle\langle A\rangle\rangle \varphi \mathcal{W} \psi$ to ATL allows to express the full power of CTL (and more), through the following translation:

$$\begin{aligned}
\mathsf{A}\bigcirc\varphi &\equiv \langle\langle \varnothing\rangle\rangle \bigcirc\varphi \\
\mathsf{A}\varphi \mathcal{U} \psi &\equiv \langle\langle \varnothing\rangle\rangle \varphi \mathcal{U} \psi \\
\mathsf{A}\varphi \mathcal{W} \psi &\equiv \langle\langle \varnothing\rangle\rangle \varphi \mathcal{W} \psi
\end{aligned}$$

[9] We note that "weak until" is not expressible even in ATL with deterministic transitions, cf. [29].

$$\begin{aligned}
\mathsf{E}\bigcirc\varphi &\equiv \neg\mathsf{A}\bigcirc\neg\varphi\\
\mathsf{E}\varphi\,\mathcal{U}\,\psi &\equiv \neg\mathsf{A}(\neg\psi)\,\mathcal{W}\,(\neg\varphi\wedge\neg\psi)\\
\mathsf{E}\varphi\,\mathcal{W}\,\psi &\equiv \neg\mathsf{A}(\neg\psi)\,\mathcal{U}\,(\neg\varphi\wedge\neg\psi)\\
\Diamond\varphi &\equiv \top\,\mathcal{U}\,\varphi\\
\Box\varphi &\equiv \varphi\,\mathcal{W}\,\bot
\end{aligned}$$

Note that formulae $\langle\!\langle A\rangle\!\rangle\Box\varphi$ do not have to be included in the definition of ATL explicitly any more, since the $\Box$ operator can be derived from $\mathcal{W}$.

Theorem 6 ATL *with "weak until" covers the full expressive power of* CTL *even when nondeterministic alternating transition systems are used as models. Moreover, model checking* ATL *with "weak until" over nondeterministic* ATS *is:*

1. **P***-complete (linear time) with respect to the number of transitions in the model and the length of the formula;*
2. **NP***-complete for the "positive" sublanguage with respect to the number of states, agents and decisions in the model and the length of the formula;*
3. $\mathbf{\Delta_2^P}$*-complete for the full language with respect to the number of states, agents and decisions in the model and the length of the formula.*

Finally, we observe that ATS have already been used in the work on implementing symbolic model checking for ATL [26], probably because of the compact representation of the transition function.[10] We proved in this section that using ATS offers also some computational advantage over CGS. Theorem 6 suggests that *designing* ATS does not have to be such a painstaking process.

5 Turning Game Models Turn-Based

In this section, we demonstrate how strategic ability in arbitrary ATS's can be translated into strategic ability in turn-based systems. More precisely, we show how, for an arbitrary alternating transition system M, a turn-based system M' can be constructed, so that a combination of *choices* in M corresponds to a combination of *strategies* in a fragment of M'. We then propose a translation of ATL formulae into ATL$^+$ formulae, such that the original formula holds in M, q if, and only if, the translated formula holds in M', q. Finally, we point out that the latter can be model-checked in

[10] The authors of [26] define the semantics of ATL in terms of concurrent game structures, but the model checking algorithm they present uses *postconditions* to specify the possible outcomes of a choice. A postcondition is taken to be simply a set of states, and choices by different agents executed in parallel lead to the state from the intersection of the postconditions (it is even assumed that the intersection must be a singleton).

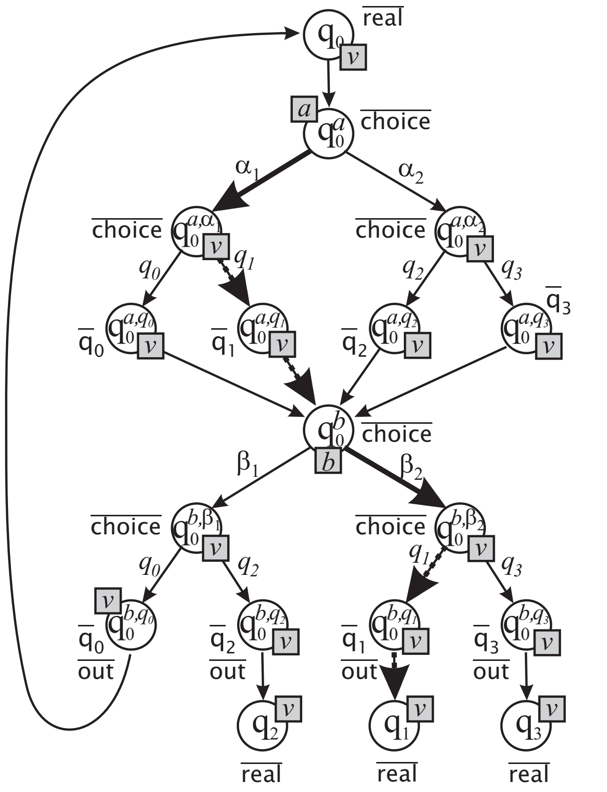


Fig. 10 A fragment of M'_1: simulation of outgoing transitions from q_0

nondeterministic polynomial time ($\mathbf{\Delta_2^P}$ for full ATL, and **NP** for "Positive ATL"), and provide another (slightly more general) proof of the upper bounds for both variants of the language.

The translation of models is independent from the translation of formulae in our construction, which allows for "pre-compiling" models when one wants to check various properties of a particular multi-agent system.

5.1 Translation of Models

Let $M = \langle \mathbb{A}\mathrm{gt}, St, \Pi, \pi, \delta \rangle$ be an ATS. We construct a turn-based ATS $M' = \langle \mathbb{A}\mathrm{gt}', St', \Pi', \pi', \delta' \rangle$ as follows:

- $\mathbb{A}\mathrm{gt}' = \mathbb{A}\mathrm{gt} \cup \{\mathbf{v}\}$: we add an additional agent $\mathbf{v}$ ("verifier") to the original set of players. Verifier helps to find out the right outcome state, given the choices from all agents (i.e. the sole state which belongs to the intersection of their choices);
- $St' = St \cup \bigcup_{a \in \mathbb{A}\mathrm{gt}} (dec(a) \cup exec(a) \cup outcome(a))$, where:

  - $dec(a) = \{q^a \mid q \in St\}$ are the "dummy states" from which agent a's decisions are simulated; by x^ρ, we will denote a copy of item x, labeled with superscript ρ.
  - $exec(a) = \{q^{a,S} \mid q \in St, S \in \delta(q,a)\}$ simulate the situations between a's decision making and the execution of a decision.
  - $outcome(a) = \{q^{a,q'} \mid q \in St, q' \in \bigcup_{S \in \delta(q,a)} S\}$ are the dummy states that simulate possible outcomes of a's decisions.
- $\Pi' = \{\overline{\mathsf{q}} \mid q \in St\} \cup \{\overline{\mathsf{real}}, \overline{\mathsf{choice}}, \overline{\mathsf{out}}\}$. Proposition $\overline{\mathsf{real}}$ marks the original, "real" states from M; $\overline{\mathsf{choice}}$ labels the dummy states that simulate situations before and after a choice, $\overline{\mathsf{out}}$ marks the *final* outcome states before the next "real" state is reached, and $\overline{\mathsf{q}}_\mathsf{i}$ mark "outcome" dummy states that refer to a transition ending up in state q_i. Thus:
  - $\pi'(\overline{\mathsf{real}}) = St, \qquad \pi'(\overline{\mathsf{choice}}) = \bigcup_{a \in \mathbb{A}\mathrm{gt}}(dec(a) \cup exec(a))$,
  - $\pi'(\overline{\mathsf{out}}) = outcome(a_k)$,
  - $\pi'(\overline{\mathsf{q}}_\mathsf{i}) = \{q^{a,q_i} \mid q^{a,q_i} \in outcome(a), a \in \mathbb{A}\mathrm{gt}\}$.
- The "decision" states are "owned" by the decision making players; the rest of the states is owned by verifier:
  - $\delta(q, \mathbf{v}) = \{St'\}$ for $q \in dec(a), a \in \mathbb{A}\mathrm{gt}$,
  - $\delta(q, a) = \{St'\}$ for $q \notin dec(a)$.
- Choices of the original agents remain the same as in M, but they are split between "choice" states. Verifier makes substantial choice only at the "execution" dummy states. Transitions from the "outcome" dummy states are automatic, and lead to the decision node of the next player. Choices executed by agents at decision nodes lead to their corresponding execution states, and verifier's actions at execution nodes lead to their corresponding outcome nodes.
  - $\delta(q^a, a) = \{\{q^{a,Q_1}\}, ..., \{q^{a,Q_i}\}\}$ for $q^a \in dec(a)$, $\delta(q, a) = \{Q_1, ..., Q_i\}$;
  - $\delta(q^{a,S}, \mathbf{v}) = \{\{q^{a,q_1}\}, ..., \{q^{a,q_i}\}\}$ for $q^{a,S} \in exec(a)$ and $S = \{q_1, ..., q_i\}$.
  - $\delta(q^{a_i,q_j}, \mathbf{v}) = \{\{q^{a_{i+1}}\}\}$ for $i < k$, and $\delta(q^{a_k,q_j}, \mathbf{v}) = \{\{q^{a_k}\}\}$.

Example 4 Consider a fragment of the alternating transition system M_2, depicted in Figure 2. The fragment of the resulting ATS M_2', that refers to the transitions starting from q_0, is shown in Figure 10. (Remember, we use symbols α_1, α_2 and β_1, β_2 as shorthand for the choices to make the example easier to read, but in fact these are sets of states and not abstract labels.) The collective strategy of $\{a, b\}$, that corresponds to the combination of choices $\langle \alpha_1, \beta_2 \rangle$ in the original ATS, is marked with bold arrows. The only verifier's response, that yields a path with exactly one $\overline{\mathsf{q}}_\mathsf{i}$ proposition holding along it, is also indicated.

Note that, for each state q in M, the transformation of the outgoing transitions requires that we process all the choices from $\delta(q, a)$ once; we must also process the "contents" of each choice (i.e. all the states included in the choice) – but only once, too. Moreover, the resulting substructure includes at most $O(kdn)$ outgoing transitions per node.

Proposition 8 *The translation of M can be done in time $O(n^2kd)$, and M' includes $m' = O(n^2kd)$ states.*

5.2 Translation of Formulae

Let φ, ψ be ATL formulae, whose interpretations in M are $[\![\varphi]\!], [\![\psi]\!]$ respectively.[11] We define the translation of complex formulae in the following way:

$$
\begin{aligned}
tr_M(\neg\varphi) &= \neg[\![\varphi]\!] \\
tr_M(\varphi \wedge \psi) &= [\![\varphi]\!] \wedge [\![\psi]\!] \\
next_M(\varphi) &= \neg \bigvee_{q_i \notin [\![\varphi]\!]} (\overline{\mathsf{q_i}} \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}}) \mathcal{U} \; \overline{\mathsf{out}} \\
tr_M(\langle\!\langle A \rangle\!\rangle \bigcirc \varphi) &= \langle\!\langle A \rangle\!\rangle next_M(\varphi) \\
tr_M(\langle\!\langle A \rangle\!\rangle \Box \varphi) &= [\![\varphi]\!] \wedge \langle\!\langle A \rangle\!\rangle \Box (\overline{\mathsf{real}} \rightarrow \langle\!\langle \varnothing \rangle\!\rangle next_M(\varphi)) \\
tr_M(\langle\!\langle A \rangle\!\rangle \varphi \mathcal{U} \psi) &= [\![\psi]\!] \vee ([\![\varphi]\!] \wedge \langle\!\langle A \rangle\!\rangle (\overline{\mathsf{real}} \rightarrow \langle\!\langle \varnothing \rangle\!\rangle next_M(\varphi)) \mathcal{U} \; [\![\psi]\!]).
\end{aligned}
$$

The idea is as follows: the paths that matter are the ones where only a single proposition $\overline{\mathsf{q_i}}$ occurs in each subpath between two subsequent "real" states – they correspond to intersections of the agents' choices that can be found along the subpath. For $\langle\!\langle A \rangle\!\rangle \bigcirc \varphi$, we want to make sure that A have a strategy to enforce that all such subpaths until the next "real" node refer to states from $[\![\varphi]\!]$. In other words, A must have a strategy such that *no* initial subpath occurs that refers to some $q_i \notin [\![\varphi]\!]$. For $\langle\!\langle A \rangle\!\rangle \Box \varphi$, the same must hold for subpaths after *each* "real" node etc. Note that $tr_M(\Phi)$ is a formula of ATL$^+$, since it includes Boolean combinations of temporal formulae.[12] The following proposition states that the translation is correct.

Proposition 9 *Let φ be an* ATL *formula that does not include special propositions* $\overline{\mathsf{real}}$, $\overline{\mathsf{out}}$, $\overline{\mathsf{choice}}$ *and* $\overline{\mathsf{q_i}}$. *Let M be an* ATS, *and q a state in M. Then:*

$$M, q \models \Phi \textit{ iff } M', q \models tr_M(\Phi).$$

Proof We prove the proposition for the case $\Phi \equiv \langle\!\langle A \rangle\!\rangle \bigcirc \varphi$. The other cases follow from respective fixpoint characterisations of $\langle\!\langle A \rangle\!\rangle \Box \varphi$ and $\langle\!\langle A \rangle\!\rangle \varphi \mathcal{U} \psi$.

[⇒] Let $M, q \models \langle\!\langle A \rangle\!\rangle \bigcirc \varphi$, $A = \{a_1, ..., a_r\}$. Suppose that $M', q \not\models \langle\!\langle A \rangle\!\rangle \neg \bigvee_{q_i \notin [\![\varphi]\!]} (\overline{\mathsf{q_i}} \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}}) \mathcal{U} \; \overline{\mathsf{out}}$. Note that $\overline{\mathsf{out}}$ holds for the first time exactly after $3k$ transitions from q in M'. Thus, for each strategy S'_A in M' there is a path $\lambda \in out(q, S'_A)$, and a state $q_i \notin [\![\varphi]\!]$, such that $M, \lambda[j] \models (\overline{\mathsf{q_i}} \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}})$ for all $j = 0, ..., 3k-1$. We take any strategy S_A in M, find the corresponding S'_A with $s'_a(q^a) = s_a(q)$ for $a \in A$, and then we take the above λ and q_i. We set the choices of the opponents in M, q to $s_{a_j}(q) = \sigma$ such that $\lambda[3j-2] = q^{a_j, \sigma}$, $a_j \notin A$. By construction, $q_i \in s_{a_1}(q) \cap ... \cap s_{a_k}(q)$, which gives a contradiction.

[11] We will abuse the notation slightly by using $[\![\varphi]\!]$ to denote also $\bigvee_{q_i \in [\![\varphi]\!]} \overline{\mathsf{q_i}}$, a formula that holds exactly in the states from $[\![\varphi]\!]$.

[12] We assume that this is the "memoryless" version of ATL$^+$, with strategies represented as functions from *states* to choices.

[$\Leftarrow$] Similarly: we take the "winning" strategy in M', construct the corresponding strategy in M (or rather its relevant part for state q), and show that no combination of responses from $a_{r+1}, ..., a_k$ can lead to a state $q' \notin [\![\varphi]\!]$.

Example 5 Consider models M_2 and M_2' again. Formula $\langle\!\langle a \rangle\!\rangle \bigcirc (\mathsf{p}_1 \vee \mathsf{p}_2)$ holds in M_2, q_0, and indeed $M_2', q_0^a \models \langle\!\langle a \rangle\!\rangle \neg\big((\overline{\mathsf{q}}_2 \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}})\,\mathcal{U}\ \overline{\mathsf{out}} \vee (\overline{\mathsf{q}}_3 \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}})\,\mathcal{U}\ \overline{\mathsf{out}}\big)$. On the other hand, $M_2', q_0 \not\models \langle\!\langle a \rangle\!\rangle \bigcirc \mathsf{p}_1$, and also $M_2', q_0^a \not\models \langle\!\langle a \rangle\!\rangle \neg\big((\overline{\mathsf{q}}_1 \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}})\,\mathcal{U}\ \overline{\mathsf{out}} \vee (\overline{\mathsf{q}}_2 \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}})\,\mathcal{U}\ \overline{\mathsf{out}} \vee (\overline{\mathsf{q}}_3 \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}})\,\mathcal{U}\ \overline{\mathsf{out}}\big)$.

Proposition 10 *The length of $tr_M(\varphi)$ is $l' = O(n+l)$, where l is the length of φ, and n is the number of states in M.*

The following nondeterministic algorithm can be used to model check formula $\langle\!\langle A \rangle\!\rangle \varphi$ of the "memoryless" ATL* in model M':

1. Recursively compute the interpretations of the state subformulae of φ in M' (e.g., for $\varphi \equiv \bigcirc\psi$, compute the set of states that satisfy ψ);
2. Guess the collective strategy S_A. Note that the size of S_A is $O(nkd)$;
3. "Trim" model M', removing all A's choices that do not appear in S_A. As M' is turn-based, the operation requires only $O(nkd)$ steps, and yields a turn-based ATS M'' with no more states and transitions than M';
4. Model-check CTL* formula $\mathsf{A}\varphi$ in M''.

Note that $\mathsf{A}\ next_M(\varphi) \Leftrightarrow \neg\mathsf{E} \bigvee_{q_i \notin [\![\varphi]\!]} (\overline{\mathsf{q}}_\mathsf{i} \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}})\,\mathcal{U}\ \overline{\mathsf{out}} \Leftrightarrow \neg \bigvee_{q_i \notin [\![\varphi]\!]} \mathsf{E}(\overline{\mathsf{q}}_\mathsf{i} \vee \overline{\mathsf{real}} \vee \overline{\mathsf{choice}})\,\mathcal{U}\ \overline{\mathsf{out}}$, which is a formula of "vanilla" CTL, and can be model-checked in deterministic polynomial time.[13] Note also that an array of strategies for all the cooperation modalities occurring in a complex "positive" formula can be guessed *before* the translation of the formula (as strategy $s_a(q) = \alpha$ in M transformed to an equivalent strategy $s_a'(q^a) = \{\{q^{a,\alpha}\}\}$ in M'). The size of the witness is still $O(nkdl)$, which gives us the following.

Corollary 1 *Model checking of an* ATL *formula φ in an* ATS *M is in $\mathbf{\Delta_2^P}$ with respect to n, k, d, l. Model checking "Positive* ATL*" is even in* **NP**.

Thus, we obtained a proof of membership in $\mathbf{\Delta_2^P}$ (resp. **NP**), alternative to the one in Section 4. We want to emphasize that the above algorithm is somewhat more general than the one in Section 4.1, because it does *not* employ "tightening" of the model. In principle, an equivalent tight model exists for each ATS if we consider alternating transition systems in isolation. However, the same does not have to hold when we extend ATL and ATS with additional modalities. For instance, for an ATL extension that handles imperfect information, we may want to require that a single strategy specifies identical choices in indistinguishable states (cf. [41]), which means that a choice must include all the states that are considered as possible outcomes

[13] We thank an anonymous reviewer of MFCS'05 for pointing this out.

by an agent in a given situation, and not only the ones that can *physically* occur [38]. In consequence, such a kind of alternating *epistemic* transition systems cannot be tight in most cases. The above algorithm is valid for all ATS, even for those which cannot be tightened in a given context.

6 Model Checking Strategic Abilities of Agents under Incomplete Information

In this section, we consider model checking of ATL *with imperfect* (or *incomplete*) *information*. Since no satisfying semantics based on alternating transition systems has been proposed so far for strategic abilities under imperfect information, we present our results for an extension of concurrent game structures only.

Schobbens [37] proved that ATL$_{ir}$ model checking is intractable: more precisely, it is **NP**-hard and $\mathbf{\Delta_2^P}$-easy (i.e., can be solved through a polynomial number of calls to an oracle for some problem in **NP**) when the size of the model is defined in terms of the number of transitions. He also conjectured that the problem might be $\mathbf{\Delta_2^P}$-complete.

This section contains several new results. Firstly, we close the gap and prove that model checking ATL$_{ir}$ is $\mathbf{\Delta_2^P}$-hard, and hence indeed $\mathbf{\Delta_2^P}$-complete with respect to the number of transitions in the model and the length of the formula. The proof proceeds by a reduction of the **SNSAT** problem to ATL$_{ir}$ model checking, presented in Section 6.3. **NP** and $\mathbf{\Delta_2^P}$ are quite close, both belonging to the first level of the polynomial hierarchy, so our result might seem a minor one – although, technically, it was not that trivial to prove it. On the other hand, its importance goes well beyond model checking of ATL$_{ir}$. In fact, Theorem 10 yields immediate corollaries with $\mathbf{\Delta_2^P}$-completeness of other logics like ATOL [23], "Feasible ATEL" [25], CSL [24] etc., and $\mathbf{\Delta_2^P}$-hardness of ETSL [42].

We also point out that the problem is **NP**-complete for the "positive" sublanguage of ATL$_{ir}$.

Secondly, we show that the problem is $\mathbf{\Delta_3^P}$-complete in the number of states, agents and decisions in the model, and the length of the formula (and it is "only" $\mathbf{\Sigma_2^P}$-complete for "Positive ATL$_{ir}$"). Therefore, the problem sits in the same complexity class as model checking strategic abilities for *perfect* information games with respect to these parameters. We believe this is good news, as far as complexity is concerned, for agent logics dealing with imperfect information.

Finally, we point out that the difference between the perfect and imperfect information case lies in the modularity of strategies with respect to the property that the agents may want to enforce. For perfect information games, potential successfulness of sub-strategies is more independent and they can be computed (or guessed) incrementally, while imperfect information strategies refuse incremental analysis.

6.1 Existing Results

Model checking ATL$_{ir}$ has been proved to be **NP**-hard and $\mathbf{\Delta_2^P}$-easy in the number of transitions and the length of the formula [37]. Membership in $\mathbf{\Delta_2^P}$ was demonstrated through the following observation. If the formula to be model checked is of the form $\langle\!\langle A\rangle\!\rangle_{ir}\varphi$ (φ being $\bigcirc\psi$, $\Box\psi$ or $\psi_1\,\mathcal{U}\,\psi_2$), where φ contains no more cooperation modalities, then it is sufficient to guess a strategy for A, "trim" the model by removing all transitions that will never be executed (according to this strategy), and model check CTL formula $\mathsf{A}\varphi$ in the resulting model. Thus, model checking an arbitrary ATL$_{ir}$ formula can be done by checking the subformulae iteratively, which requires a polynomial number of calls to an **NP** algorithm.[14]

NP-hardness follows from a reduction of the well known **SAT** problem. Here, we present a reduction which is somewhat different from the one in [37]. We will adapt it in Section 6.3 to prove $\mathbf{\Delta_2^P}$-hardness. In **SAT**, we are given a CNF formula $\varphi \equiv C_1 \wedge \ldots \wedge C_n$ involving k propositional variables from set $X = \{x_1, ..., x_k\}$. Each clause C_i can be written as $C_i \equiv x_1^{s_{i,1}} \vee \ldots \vee x_k^{s_{i,k}}$, where $s_{i,j} \in \{+,-,0\}$; x_j^+ denotes a positive occurrence of x_j in C_i, x_j^- denotes an occurrence of $\neg x_j$ in C_i, and x_j^0 indicates that x_j does not occur in C_i. The problem asks if $\exists X.\varphi$, that is, if there is a valuation of $x_1, ..., x_k$ such that φ holds.

We construct the corresponding i-CGS M_φ as follows. There are two players: verifier $\mathbf{v}$ and refuter $\mathbf{r}$. The refuter decides at the beginning of the game which clause C_i will have to be satisfied: it is done by proceeding from the initial state q_0 to a "clause" state q_i. At q_i, verifier decides (by proceeding to a "proposition" state $q_{i,j}$) which of the literals $x_j^{s_{i,j}}$ from C_i will be attempted. Finally, at $q_{i,j}$, verifier attempts to prove C_i by declaring the underlying propositional variable x_j true (action $\top$) or false (action $\bot$). If she succeeds (i.e., if she executes $\top$ for x_j^+, or executes $\bot$ for x_j^-), then the system proceeds to the "winning" state $q_\top$. Otherwise, the system stays in $q_{i,j}$. Additionally, "proposition" states referring to the same variable are indistinguishable for verifier, so that she has to declare the same value of x_j in all of them within a uniform strategy. A sole ATL$_{ir}$ proposition yes holds only in the "winning" state $q_\top$. Obviously, states corresponding to literals x_j^0 can be omitted from the model.

Speaking more formally, $M_\varphi = \langle \mathbb{A}\mathrm{gt}, St, \Pi, \pi, Act, d, o, \sim_1, ..., \sim_k\rangle$, where:

- $\mathbb{A}\mathrm{gt} = \{\mathbf{v}, \mathbf{r}\}$,
- $St = \{q_0\} \cup St_{cl} \cup St_{prop} \cup \{q_\top\}$, where $St_{cl} = \{q_1, \ldots, q_n\}$, and $St_{prop} = \{q_{1,1}, \ldots, q_{1,k}, \ldots, q_{n,1}, \ldots, q_{n,k}\}$;
- $\Pi = \{\mathsf{yes}\}$, $\quad \pi(\mathsf{yes}) = \{q_\top\}$,
- $Act = \{1, ..., \max(k,n), \top, \bot\}$,

[14] The algorithm from [21] can be also used to demonstrate the upper bounds for the complexity of this problem.

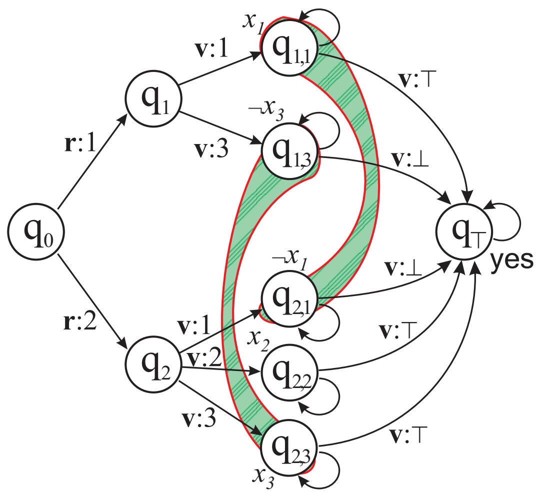


Fig. 11 An *i*-CGS for checking satisfiability of $\varphi \equiv (x_1 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$

- $d(\mathbf{v}, q_0) = d(\mathbf{v}, q_\top) = \{1\}$, $\quad d(\mathbf{v}, q_i) = \{1, ..., k\}$,
  $d(\mathbf{v}, q_{i,j}) = \{\top, \bot\}$,
  $d(\mathbf{r}, q) = \{1, ..., n\}$ for $q = q_0$, and $d(\mathbf{r}, q) = \{1\}$ otherwise;
- $o(q_0, 1, i) = q_i$, $\quad o(q_i, j, 1) = q_{i,j}$,
  $o(q_{i,j}, \top, 1) = q_\top$ if $s_{i,j} = +$, and $q_{i,j}$ otherwise,
  $o(q_{i,j}, \bot, 1) = q_\top$ if $s_{i,j} = -$, and $q_{i,j}$ otherwise;
- $q_0 \sim_{\mathbf{v}} q$ iff $q = q_0$, $\quad q_i \sim_{\mathbf{v}} q$ iff $q = q_i$, $\quad q_{i,j} \sim_{\mathbf{v}} q$ iff $q = q_{i',j}$.

As an example, model M_φ for $\varphi \equiv (x_1 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$ is presented in Figure 11.

Theorem 7 *φ is satisfiable iff* $M_\varphi, q_0 \models \langle\langle \mathbf{v} \rangle\rangle_{ir} \Diamond \mathsf{yes}$.

Proof **(⇒)** Firstly, if there is a valuation that makes φ true, then for each clause C_i one can choose a literal out of C_i that is made true by the valuation. The choice, together with the valuation, corresponds to a uniform strategy for $\mathbf{v}$ such that, for all possible executions, $q_\top$ is achieved at the end.

(⇐) Conversely, if $M_\varphi, q_0 \models \langle\langle \mathbf{v} \rangle\rangle_{ir} \Diamond \mathsf{yes}$, then there is a strategy $s_{\mathbf{v}}$ such that $q_\top$ is achieved for all paths from $out(q_0, s_{\mathbf{v}})$. But then the valuation, which assigns propositions $x_1, ..., x_k$ with the same values as $s_{\mathbf{v}}$, satisfies φ.

Both the number of states and transitions in M_φ are linear in the length of φ, and the construction of M requires linear time too. Thus, the model checking problem for ATL$_{ir}$ is **NP**-hard. Note that it is **NP**-hard even for formulae with a single cooperation modality, and turn-based models with at most two agents.[15]

[15] In fact, it is **NP**-hard even for models with a single agent, although the construction must be a little different to demonstrate this.

```
function $mcheck_4(M, \varphi)$;
Returns the set of states in $M$, in which formula $\varphi$ holds.

■ assign cooperation modalities in $\varphi$ with subsequent numbers $1, ..., c$;
    // note that $c \leq l$
    // we will denote the coalition from the $i$th cooperation modality in $\varphi$ as $\varphi[i]$
■ for each $i = 1, ..., c$, assign the agents in $\varphi[i]$ with numbers $1, ..., k_c$;
    // note that $k_c \leq k$ and $k_c \leq l$
    // we will denote the $j$th agent in $A$ with $A[j]$
■ guess an array $choice$ such that, for each $i = 1, ..., c$, $q \in St$, and $j = 1, ..., k_c$, we have that $choice[i][q][j] \in d_{\varphi[i][j]}(q)$, and for each $q' \in St$ such that $q \sim_{\varphi[i][j]} q'$ we have $choice[i][q][j] = choice[i][q'][j]$;
    // at this point, the optimal choices for all coalitions in $\varphi$ are guessed
    // note that the size of $choice$ is $\mathbf{O}(ml)$
    // by $choice|_i$, we will denote the array $choice$ with rows $1, ..., i-1$ removed
■ return $eval_4(M, \varphi, choice)$;
```

```
function $eval_4(M, \varphi, choice)$;
Returns the states in which $\varphi$ holds, given choices for all the coalitions from $\varphi$.

case $\varphi \in \Pi$ : return $\{q \mid \varphi \in \pi(q)\}$;
case $\varphi = \neg\psi$ : return $Q \setminus eval_4(M, \psi, choice)$;
case $\varphi = \psi_1 \vee \psi_2$ : return $eval_4(M, \psi_1, choice) \cup eval_4(M, \psi_2, choice)$;
case $\varphi = \langle\langle A\rangle\rangle T\psi$, where $T = \bigcirc$ or $\Box$ :
  $Q_1 := eval_4(M, \psi, choice|_2)$;  $M' := trim(M, choice[1])$;
  add to $M'$ new proposition $\overline{\mathsf{p}}$ with $\pi(\overline{\mathsf{p}}) = Q_1$;
  $Q_2 := mcheck_{CTL}(M', \mathsf{A}T\,\overline{\mathsf{p}})$;
  return $\{q \in St \mid \forall a, q' \,.\, a \in A \wedge q \sim_a q' \Rightarrow q' \in Q_2\}$;
case $\varphi = \langle\langle A\rangle\rangle \psi_1 \,\mathcal{U}\, \psi_2$ :
  $c'$ := the number of cooperation modalities in $\psi_1$;
  $Q_1 := eval_4(M, \psi_1, choice|_2)$;  $Q_2 := eval_4(M, \psi_2, choice|_{c'+2})$;
  $M' := trim(M, choice[1])$;
  add to $M'$ new propositions $\overline{\mathsf{p}}_1, \overline{\mathsf{p}}_2$ with $\pi(\overline{\mathsf{p}}_1) = Q_1$, $\pi(\overline{\mathsf{p}}_2) = Q_2$;
  $Q_3 := mcheck_{CTL}(M', \mathsf{A}\overline{\mathsf{p}}_1 \,\mathcal{U}\, \overline{\mathsf{p}}_2)$;
  return $\{q \in St \mid \forall a, q' \,.\, a \in A \wedge q \sim_a q' \Rightarrow q' \in Q_3\}$;
end case
```

```
function $trim(M, thischoice)$;
  Returns the CTL model, which includes exactly the transitions that can occur when $A$ execute choices from $thischoice$.

■ $\mathcal{R} := \emptyset$; // the CTL transition relation (contains pairs of states)
■ for each $q \in St$ and tuple $resp$ of choices from $\mathbb{A}\mathrm{gt} \setminus A$, such that $resp[a] \in d(a, q)$:
  – $q' := o(q, thischoice[q], resp)$;
  – $\mathcal{R} := \mathcal{R} \cup \{\langle q, q'\rangle\}$;
■ return $\langle St, \mathcal{R}, \Pi, \pi\rangle$;
```

Fig. 12 Nondeterministic algorithm for model checking formulae of ATL$_{ir}$.

6.2 **NP***-completeness for "Positive ATL"*

We already investigated the complexity of ATL$_{ir}$ model checking in [21], concluding that the problem was **NP**-complete. Unfortunately, our claim was incorrect: the error occurred in the way we handled negation in our model checking algorithm (cf. [29]). Still, the algorithm from [21] *was* correct for "positive" formulae of ATL$_{ir}$: in this case, we can do the same trick as in Section 3.3, and guess all the relevant strategies beforehand. The size of the witness is still polynomial in this case: more precisely, it is $\mathbf{O}(ml)$, where m is the number of transitions, and l is the length of the formula. Thus, the following holds.

Theorem 8 *Model checking of "Positive* ATL$_{ir}$*" is* **NP***-complete with respect to the number of transitions in the model and the length of the formula.*

Proof A nondeterministic algorithm that checks formula φ in model M is presented in Figure 12. Calls to $mcheck_{CTL}$ refer to any established CTL model-checker (e.g. [9]). As for the time necessary to carry out the procedure: guessing the strategies can be done in time $\mathbf{O}(ml)$, while "trimming" the model, checking CTL formulae, and getting rid of the states in which agents may not know that the strategy is successful, can all be done in time $\mathbf{O}(m)$ (recursively for subformulae). Thus, the algorithm terminates in time $\mathbf{O}(ml)$. Combining it with the **NP**-hardness result from [37], we obtain the theorem.

Note that the exhaustive deterministic algorithm that checks all possible strategies runs in time $\mathbf{O}(nd^{kn}l) = \mathbf{O}(n(m/n)^n l)$, even for "Positive ATL$_{ir}$".

$\mathbf{\Delta_2^P}$-hardness for full ATL$_{ir}$ is proved in the next section.

6.3 Model Checking ATL$_{ir}$ Is Indeed $\mathbf{\Delta_2^P}$*-complete*

Let us first recall (after [29]) the definition of **SNSAT**, a typical $\mathbf{\Delta_2^P}$-hard problem.

Definition 5 (SNSAT)
Input: *p sets of propositional variables $X_r = \{x_{1,r}, ..., x_{k,r}\}$, p propositional variables z_r, and p Boolean formulae φ_r in CNF, with each φ_r involving only variables in $X_r \cup \{z_1, ..., z_{r-1}\}$, with the following requirement:*

$$z_r \equiv \textit{ there exists an assignment of variables in } X_r \textit{ such that } \varphi_r \textit{ is true.}$$

We will also write, by abuse of notation, $z_r \equiv\ \exists X_r\ \varphi_r(z_1, ..., z_{r-1}, X_r)$.
Output: *The truth-value of z_p (i.e., $\top$ or $\bot$).*

Let n be the maximal number of clauses in any $\varphi_1, ..., \varphi_p$ from the given input. Now, each φ_r can be written as:

$$\varphi_r \equiv C_1^r \wedge \ldots \wedge C_n^r, \quad \textit{and } C_i^r \equiv x_{1,r}^{s_{i,1}^r} \vee \ldots \vee x_{k,r}^{s_{i,k}^r} \vee z_1^{s_{i,k+1}^r} \vee \ldots z_{r-1}^{s_{i,k+r-1}^r}.$$

Again, $s^r_{i,j} \in \{+,-,0\}$; x^+ denotes a positive occurrence of x, x^- denotes an occurrence of $\neg x$, and x^0 indicates that x does not occur in the clause. Similarly, $s^r_{i,k+j}$ defines the "sign" of z_j in clause C^r_i. Given such an instance of **SNSAT**, we construct a sequence of concurrent game structures M_r for $r = 1,...,p$ in a similar way to the construction in Section 6.1. That is, clauses and variables $x_{i,r}$ are handled in exactly the same way as before. Moreover, if z_i occurs as a positive literal in φ_r, we embed M_i in M_r, and add a transition to the initial state q^i_0 of M_i. If $\neg z_i$ occurs in φ_r, we do almost the same: the only difference is that we split the transition into two steps, with a state neg^r_i (labeled with an ATL$_{ir}$ proposition neg) added in between.

More formally, $M_r = \langle \mathbb{A}\mathrm{gt}, St^r, \Pi, \pi^r, Act^r, d^r, o^r, \sim^r_1, ..., \sim^r_k\rangle$, where:

- $\mathbb{A}\mathrm{gt} = \{\mathbf{v}, \mathbf{r}\}$,
- $St^r = \{q^r_0, q^r_1, \dots, q^r_n, q^r_{1,1}, \dots, q^r_{n,k}, neg^r_1, \dots, neg^r_{r-1}, q_\top\} \cup St^{r-1}$,
- $\Pi = \{\mathsf{yes}, \mathsf{neg}\}, \qquad \pi^r(\mathsf{yes}) = \{q_\top\}, \qquad \pi^r(\mathsf{neg}) = \{neg^j_i \mid i,j = 1,...,r\}$,
- $Act^r = \{1,..., \max(k+r-1, n), \top, \bot\}$,
- $d^r(\mathbf{v}, q^r_0) = d^r(\mathbf{v}, neg^r_i) = d^r(\mathbf{v}, q_\top) = \{1\}, \qquad d^r(\mathbf{v}, q^r_i) = \{1,...,k+r-1\}$,
  $d^r(\mathbf{v}, q^r_{i,j}) = \{\top, \bot\}$,
  $d^r(\mathbf{r}, q) = \{1,...,n\}$ for $q = q^r_0$ and $\{1\}$ for the other $q \in St^r$.
  For $q \in St^{r-1}$, we simply include the function from M_{r-1}: $d^r(a,q) = d^{r-1}(a,q)$;
- $o^r(q^r_0, 1, i) = q^r_i, \qquad o^r(q^r_i, j, 1) = q^r_{i,j}$ for $j \le k$,
  $o^r(q^r_i, k{+}j, 1) = q^j_0$ if $s^r_{i,k+j} = +$, and $o^r(q^r_i, k{+}j, 1) = neg^r_j$ if $s^r_{i,k+j} = -$,
  $o^r(neg^r_j, 1, 1) = q^j_0$,
  $o^r(q^r_{i,j}, \top, 1) = q_\top$ if $s^r_{i,j} = +$, and $q^r_{i,j}$ otherwise,
  $o^r(q^r_{i,j}, \bot, 1) = q_\top$ if $s^r_{i,j} = -$, and $q^r_{i,j}$ otherwise.
  For $q \in St^{r-1}$, we include the transitions from M_{r-1}: $o^r(q,\alpha) = o^{r-1}(q,\alpha)$;
- $q^r_0 \sim_\mathbf{v} q$ iff $q = q^r_0, \qquad q^r_i \sim_\mathbf{v} q$ iff $q = q^r_i, \qquad q^r_{i,j} \sim_\mathbf{v} q$ iff $q = q^r_{i',j}$.
  For $q, q' \in St^{r-1}$, we include the tuples from M_{r-1}: $q \sim^r_\mathbf{v} q'$ iff $q \sim^{r-1}_\mathbf{v} q'$.

As an example, model M_3 for $\varphi_3 \equiv (x_3 \vee \neg z_2) \wedge (\neg x_3 \vee \neg z_1)$, $\varphi_2 \equiv z_1 \wedge \neg z_1$, $\varphi_1 \equiv (x_1 \vee x_2) \wedge \neg x_1$, is presented in Figure 13.

Theorem 9 *Let*

$$\begin{aligned}\Phi_1 &\equiv \langle\langle \mathbf{v} \rangle\rangle_{ir} (\neg \mathsf{neg})\, \mathcal{U}\, \mathsf{yes},\\ \Phi_i &\equiv \langle\langle \mathbf{v} \rangle\rangle_{ir} (\neg \mathsf{neg})\, \mathcal{U}\, (\mathsf{yes} \vee (\mathsf{neg} \wedge \mathsf{A}\bigcirc \neg \Phi_{i-1})).\end{aligned}$$

Now, for all r: z_r is true iff $M_r, q^r_0 \models \Phi_r$.

Before we prove the theorem, we state an important lemma. Lemma 5 says that "overlong" formulae Φ_i do not introduce new properties of model M_r. More precisely, a formula Φ_i that includes more "nestings" than model M_r can be as well reduced to Φ_{i-1} when model checked in M_r, q^r_0.

Lemma 5 *For $i \ge r$: $M_r, q^r_0 \models \Phi_i$ iff $M_r, q^r_0 \models \Phi_{i+1}$.*

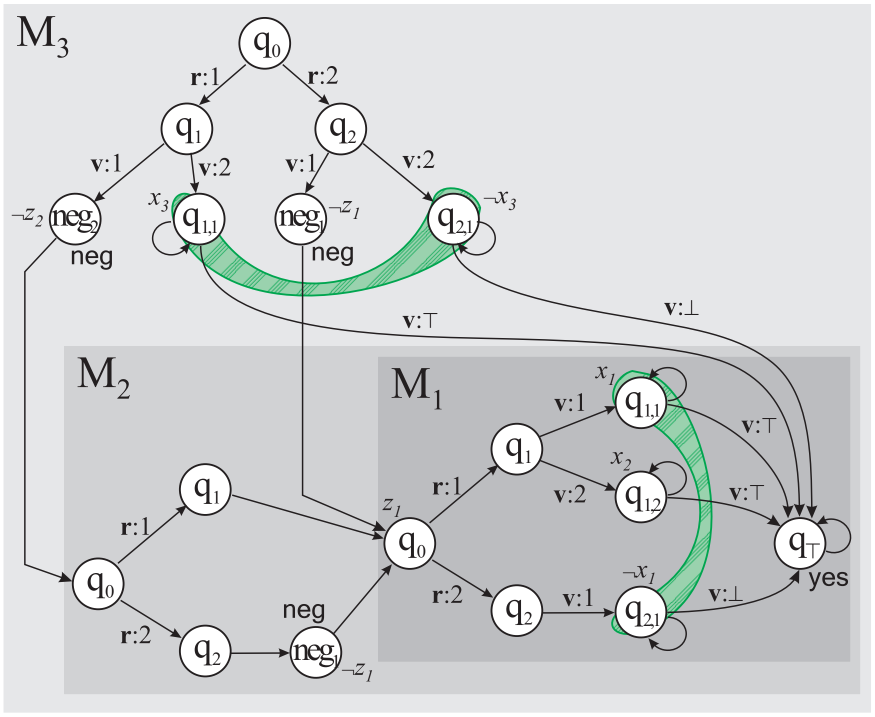


Fig. 13 An *i*-CGS for the reduction of **SNSAT**. The superscripts in state labels are omitted since they can be deduced from the sub-machine in which the state resides.

Proof (induction on r)

1. For $r = 1$: $M_1, q_0^1 \models \Phi_i$ iff $M_1, q_0^1 \models \langle\langle \mathbf{v} \rangle\rangle_{ir} \Diamond \mathsf{yes}$ iff $M_1, q_0^1 \models \Phi_{i+1}$, because M_1 does not include states that satisfy neg.
2. For $r > 1$: $M_r, q_0^r \models \Phi_{i+1} \equiv \langle\langle \mathbf{v} \rangle\rangle_{ir} (\neg \mathsf{neg}) \mathcal{U} (\mathsf{yes} \vee (\mathsf{neg} \wedge \mathsf{A}\bigcirc \neg \Phi_i))$ iff $\exists s_{\mathbf{v}} \forall \lambda \in out(q_0^r, s_{\mathbf{v}}) \exists u \forall w \leq u. \big((M_r, \lambda[u] \models \mathsf{yes}$ *or* $M_r, \lambda[u] \models \mathsf{neg} \wedge \mathsf{A}\bigcirc \neg \Phi_i)$ *and* $(M_r, \lambda[w] \models \neg \mathsf{neg})\big)$. [*]
   However, each state satisfying neg has exactly one outgoing transition, so $M_r, \lambda[u] \models \mathsf{neg} \wedge \mathsf{A}\bigcirc \neg \Phi_i$ is equivalent to $M_r, \lambda[u] \models \mathsf{neg}$ *and* $M_r, \lambda[u + 1] \models \neg \Phi_i$. Thus, [*] iff $\exists s_{\mathbf{v}} \forall \lambda \in out(q_0^r, s_{\mathbf{v}}) \exists u \forall w \leq u. \big((M_r, \lambda[u] \models \mathsf{yes}$ *or* $M_r, \lambda[u] \models \mathsf{neg}$ *and* $M_r, \lambda[u + 1] \models \neg \Phi_i)$ *and* $(M_r, \lambda[w] \models \neg \mathsf{neg})\big)$ [**].
   Note that, by the construction of M_r, $\lambda[u + 1]$ must refer to the initial state q_0^j of some "submodel" M_j, $j < r \leq i$. Thus, $M_r, \lambda[u+1] \models \neg \Phi_i$ iff $M_j, q_0^j \models \neg \Phi_i$ iff (by induction) $M_j, q_0^j \models \neg \Phi_{i-1}$ iff $M_j, \lambda[u+1] \models \neg \Phi_{i-1}$.
   So, [**] iff $\exists s_{\mathbf{v}} \forall \lambda \in out(q_0^r, s_{\mathbf{v}}) \exists u \forall w \leq u. \big((M_r, \lambda[u] \models \mathsf{yes}$ *or* $M_r, \lambda[u] \models \mathsf{neg}$ *and* $M_r, \lambda[u + 1] \models \neg \Phi_{i-1})$ *and* $(M_r, \lambda[w] \models \neg \mathsf{neg})\big)$ iff $M_r, q_0^r \models \langle\langle \mathbf{v} \rangle\rangle_{ir} (\neg \mathsf{neg}) \mathcal{U} (\mathsf{yes} \vee (\mathsf{neg} \wedge \mathsf{A}\bigcirc \neg \Phi_{i-1})) \equiv \Phi_i$.

Proof (of Theorem 9) Induction on r:

1. For $r = 1$: we use the proof of Theorem 7.
2. For $r > 1$:
   For the implication from left to right ($\Rightarrow$): let z_r be true: then, there is a valuation of X_r such that φ_r holds. We construct $s_{\mathbf{v}}$ as in the proof of Theorem 7. In case that some x_i^s has been "chosen" in clause C_i^r, we are done. In case that some z_j^- has been "chosen" in clause C_i^r (note: j must be smaller than i), we have (by induction) that $M_j, q_0^j \models \neg\Phi_j$. By Lemma 5, also $M_j, q_0^j \models \neg\Phi_r$, and hence $M_r, q_0^j \models \neg\Phi_r$. So we can make the same choice (i.e., z_j^-) in $s_{\mathbf{v}}$, and this will lead to state neg_j^r, in which it holds that $\mathsf{neg} \wedge \mathsf{A}\bigcirc\neg\Phi_r$.
   In case that some z_j^+ has been "chosen" in clause C_i^r, we have (by induction) that $M_j, q_0^j \models \Phi_j$, and hence, by Lemma 5, $M_j, q_0^j \models \Phi_r$. That is, there is a strategy $s'_{\mathbf{v}}$ in M_j such that $(\neg\mathsf{neg})\,\mathcal{U}\,(\mathsf{yes} \vee (\mathsf{neg} \wedge \mathsf{A}\bigcirc\neg\Phi_{r-1}))$ holds for all paths from $out(q_0^j, s'_{\mathbf{v}})$. As the states in M_j have no epistemic links to states outside of it, we can merge $s'_{\mathbf{v}}$ into $s_{\mathbf{v}}$.
   For the other direction ($\Leftarrow$): let $M_r, q_0^r \models \Phi_r \equiv \langle\!\langle \mathbf{v} \rangle\!\rangle_{ir}(\neg\mathsf{neg})\,\mathcal{U}\,(\mathsf{yes} \vee (\mathsf{neg} \wedge \mathsf{A}\bigcirc\neg\Phi_{r-1}))$. We take the strategy $s_{\mathbf{v}}$ that enforces $(\neg\mathsf{neg})\,\mathcal{U}\,(\mathsf{yes} \vee (\mathsf{neg} \wedge \mathsf{A}\bigcirc\neg\Phi_{r-1}))$. We first consider the clause C_i^r for which a "propositional" state is chosen by $s_{\mathbf{v}}$. The strategy defines a uniform valuation for X_r that satisfies these clauses. For the other clauses, we have two possibilities:
   - $s_{\mathbf{v}}$ chooses q_0^j in the state corresponding to C_i^r. Neither yes nor neg have been encountered on this path yet, so we can take $s_{\mathbf{v}}$ to demonstrate that $M_r, q_0^j \models \Phi_r$, and hence $M_j, q_0^j \models \Phi_r$. By Lemma 5, also $M_j, q_0^j \models \Phi_j$. By induction, z_j must be true, and hence clause C_i^r is satisfied.
   - $s_{\mathbf{v}}$ chooses neg_j^r in the state corresponding to C_i^r. Then, it must be that $M_r, neg_j^r \models \mathsf{A}\bigcirc\neg\Phi_{r-1}$, and hence $M_j, q_0^j \models \neg\Phi_{r-1}$. By Lemma 5, also $M_j, q_0^j \models \neg\Phi_j$. By induction, z_j must be false, and hence clause C_i^r (containing $\neg z_j$) is also satisfied.

Thus, in order to determine the value of z_p, it is sufficient to model check Φ_p in M_p, q_0^p. Note that model M_p consists of $O(|\varphi|p)$ states and $O(|\varphi|p)$ transitions, where $|\varphi|$ is the maximal length of formulae $\varphi_1, ..., \varphi_p$. Moreover, the length of formula Φ_p is linear in p, and the construction of M_p and Φ_p can be also done in time $O(|\varphi|p)$ and $O(p)$, respectively. In consequence, we obtain a polynomial reduction of **SNSAT** to ATL$_{ir}$ model checking.

Theorem 10 *Model checking* ATL$_{ir}$ *is* $\mathbf{\Delta_2^P}$*-complete with respect to the number of transitions in the model, and the length of the formula. The problem is* $\mathbf{\Delta_2^P}$*-complete even for turn-based models with at most two agents.*

6.4 The Complexity Refined

One of the problems with model checking formulae of ATL is that the number of transitions m in a model is not bounded by n^2, and can be very large: more precisely, $m = \mathbf{O}(nd^k)$ where n is the number of states, k the number of agents, and d the maximal number of decisions per agent per state. Thus, m is exponential in k unless the model is turn-based or the number of agents is fixed. In consequence, ATL model checking becomes $\mathbf{\Delta_3^P}$-complete when n, k, d, l are the parameters of the problem, and model checking "Positive ATL" becomes $\mathbf{\Sigma_2^P}$-complete. In this section, we show that ATL$_{ir}$ model checking is also $\mathbf{\Delta_3^P}$-complete (resp. $\mathbf{\Sigma_2^P}$-complete "Positive ATL$_{ir}$") in the same setting. To prove the lower bounds, it suffices to point out that:

Lemma 6 ATL *is semantically subsumed by* ATL$_{ir}$*. "Positive* ATL*" is semantically subsumed by "Positive* ATL$_{ir}$*".*

Proof In order to transform a concurrent game structure M to a corresponding imperfect information concurrent game structure M', we fix the indistinguishability relations as the minimal total reflexive relations, (i.e. $\sim_a = \{\langle q, q\rangle \mid q \in St\}$ for all $a \in \mathbb{A}\mathrm{gt}$), which means that the agents can distinguish between any two states. Let φ be a formula of ATL, and φ' the result of adding subscript ir in each cooperation modality in φ. Then, $M, q \models \varphi$ iff $M', q \models \varphi'$. Thus, ATL (resp. "Positive ATL") model checking can be seen as a special case of ATL$_{ir}$ (resp. "Positive ATL$_{ir}$") model checking.

To show that the problem is $\mathbf{\Sigma_2^P}$-easy for "Positive ATL$_{ir}$", we present a refinement of the algorithm from Section 6.2 in Figures 14 and 15.

Proposition 11 *Function mcheck$_5$ defines a nondeterministic Turing machine that runs in time $\mathbf{O}(n^2kl)$, making calls to an* **NP** *oracle. The oracle itself is a nondeterministic Turing machine that runs in time $\mathbf{O}(n+k)$. The size of witnesses is never more than $\mathbf{O}(nkl)$.*

Proof The main idea is as follows. Firstly, we guess nondeterministically *all* the strategies for the cooperation modalities that occur in formula φ (we do it beforehand, as in Section 6.2). The strategies must be uniform, so setting $s_a(q)$ fixes automatically $s_a(q')$ for all $q \sim_a q'$. Then we model check φ recursively: for each subformula $\langle\!\langle A\rangle\!\rangle_{ir}\psi$, we assume the respective strategy and check the formula $\langle\!\langle \varnothing\rangle\!\rangle_{ir}\psi$. To do so, we take ATL formula $\langle\!\langle A\rangle\!\rangle\psi$ as input, and employ the standard ATL model checking algorithm from [5] with one important modification: each time function $pre(A, Q_1)$ is called, it assumes the respective A's choices, and checks whether $q \in pre(A, Q_1)$ by calling an **NP** oracle (*"is there a response from the opposition in q that leads to a state outside Q_1?"*) and reversing its answer. Note that the latter amounts to checking $M', q \models \langle\!\langle \varnothing\rangle\!\rangle \bigcirc \mathbf{Q_1}$, where M' is model M with A's actions fixed accordingly, and $\mathbf{Q_1}$ is a new proposition that holds exactly in states Q_1. Finally, we get rid of the states that have indistinguishable counterparts for which the assumed strategy is not successful. Note that, in

```
function mcheck_5(M, φ);
Returns the set of states in M, in which formula φ holds.
■ assign cooperation modalities in $\varphi$ with subsequent numbers $1, ..., c$;
    // note that $c \le l$
    // we will denote the coalition from the $i$th cooperation modality in $\varphi$ as $\varphi[i]$
■ for each $i = 1, ..., c$, assign the agents in $\varphi[i]$ with numbers $1, ..., k_c$;
    // note that $k_c \le k$ and $k_c \le l$
    // we will denote the $j$th agent in $A$ with $A[j]$
■ guess an array choice such that, for each $i = 1, ..., c$, $q \in St$, and $j = 1, ..., k_c$, we have that $choice[i][q][j] \in d_{\varphi[i][j]}(q)$, and for each $q' \in St$ such that $q \sim_{\varphi[i][j]} q'$ we have $choice[i][q][j] = choice[i][q'][j]$;
    // at this point, the optimal choices for all coalitions in $\varphi$ are guessed
    // note that the size of choice is $\mathbf{O}(nkl)$
    // by $choice|_i$, we will denote the array choice with rows $1, ..., i-1$ removed
■ return $eval_5(M, \varphi, choice)$;
```

```
function eval_5(M, φ, choice);
Returns the states in which φ holds, given choices for all the coalitions from φ.
case $\varphi \in \Pi$ : return $\{q \mid \varphi \in \pi(q)\}$;
case $\varphi = \neg\psi$ : return $Q \setminus eval_5(M, \psi, choice)$;
case $\varphi = \psi_1 \vee \psi_2$ : return $eval_5(M, \psi_1, choice) \cup eval_5(M, \psi_2, choice)$;
case $\varphi = \langle\langle A \rangle\rangle \bigcirc \psi$ :
  $Q_1 := pre_5(A, eval_5(M, \psi, choice|_2), M, choice[1])$;
  return $\{q \in St \mid \forall a, q' \,.\, a \in A \wedge q \sim_a q' \Rightarrow q' \in Q_1\}$;
case $\varphi = \langle\langle A \rangle\rangle \Box\psi$ : $Q_1 := St$;   $Q_2 := Q_3 := eval_5(M, \psi, choice|_2)$;
  while $Q_1 \not\subseteq Q_2$ do $Q_1 := Q_1 \cap Q_2$;  $Q_2 := pre_5(A, Q_1, M, choice[1]) \cap Q_3$ od;
  return $\{q \in St \mid \forall a, q' \,.\, a \in A \wedge q \sim_a q' \Rightarrow q' \in Q_1\}$;
case $\varphi = \langle\langle A \rangle\rangle \psi_1 \mathcal{U} \psi_2$ : $c'$ := the number of cooperation modalities in $\psi_1$;
  $Q_1 := \emptyset$;  $Q_2 := eval_5(M, \psi_1, choice|_2)$;  $Q_3 := eval_5(M, \psi_2, choice|_{c'+2})$;
  while $Q_3 \not\subseteq Q_1$ do $Q_1 := Q_1 \cup Q_3$;  $Q_3 := pre_5(A, Q_1, M, choice[1]) \cap Q_2$ od;
  return $\{q \in St \mid \forall a, q' \,.\, a \in A \wedge q \sim_a q' \Rightarrow q' \in Q_1\}$;
end case
```

```
function pre_5(A, Q_1, M, thischoice);
Returns the set of states, for which the A's choices from thischoice enforce that the next state is in Q_1, regardless of what agents from Agt \ A do.
■ $Q_2 := \emptyset$;
■ for each $q \in St$: if $oracle_5(A, Q_1, M, thischoice, q) = yes$ then $Q_2 := Q_2 \cup \{q\}$ fi;
■ return $Q_2$;
```

Fig. 14 The model checking algorithm refined (main part).

```
function $oracle_5(A, Q_1, M, thischoice, q)$;
Returns $yes$ if, and only if, the $A$'s choices from $thischoice$ in $q$ enforce that
the next state is in $Q_1$, regardless of what agents from $\mathbb{A}\mathrm{gt} \setminus A$ do.

■ guess an array $resp$ such that, for each $a \in \mathbb{A}\mathrm{gt} \setminus A$, we have $resp[a] \in d(a, q)$;
  // at this point, the most dangerous response from the opposition is guessed
  // note that the size of $resp$ is $\mathbf{O}(k)$
■ if $o(q, thischoice[q], resp) \in Q_1$ then return $yes$ else return $no$ fi;
```

Fig. 15 The model checking algorithm refined: the oracle.

the middle part of the algorithm, we use an adaptation of the ATL model checking procedure, which *iterates* over states of the system. This kind of iterative solution is possible because $\langle\langle\varnothing\rangle\rangle_{ir}\psi \equiv \langle\langle\varnothing\rangle\rangle\psi$ (although, of course, the analogous property does not hold for $\langle\langle A\rangle\rangle_{ir}$ in general).

The detailed algorithm is shown in Figures 14 and 15. The procedure is very similar to the "Positive ATL" model checking algorithm from Section 3.3. Analogous complexity analysis applies: first, the number of iterations *within* one single call of function *eval*, as well as the number of calls to *pre*, is $\mathbf{O}(n)$; next, function *pre* runs in $\mathbf{O}(n)$ steps, including calls to the oracle; removing the states for which a member of the coalition can have any doubts can be done in time $\mathbf{O}(n^2k)$; finally, *eval* is called at most $\mathbf{O}(l)$ times. In consequence, we get a nondeterministic polynomial algorithm that makes calls to an **NP** oracle.

Like for ATL, the algorithm can be easily adapted to handle arbitrary ATL$_{ir}$ formulae in time $\mathbf{\Delta_3^P}$ (strategies are guessed for each $\langle\langle A\rangle\rangle_{ir}$ separately, and not in advance). Thus, we get the following.

Theorem 11 *Model checking* ATL$_{ir}$ *formulae over i-*CGS *is* $\mathbf{\Delta_3^P}$*-complete with respect to the number of states, decisions and agents, and the length of formulae. Model checking "Positive* ATL$_{ir}$*" is* $\mathbf{\Sigma_2^P}$*-complete.*

6.5 Discussion

The result has been somewhat surprising to us, since it turns out that a *fine grained* analysis puts checking strategic abilities of agents under imperfect information in the same complexity class as for perfect information games. It is surprising because the first case appears to be *strictly* harder than the latter when we approach it from a more "distant" perspective (i.e. when the input parameters are less detailed).

Let us recall from Section 3 that the hardness of model checking ATL is due to simultaneous actions of agents, and can be demonstrated even for scenarios that consist of a single step. It turns out that restricting agents' strategies to uniform strategies only does not increase model checking complexity *enough* to shift it to a higher complexity class. Even the size of witnesses is the same in both cases.

What is different then, that makes model checking of ATL$_{ir}$ *harder than* ATL *in relation to the number of transitions?*

Definitely *not* the number of transitions itself, because CGS can be seen as a special case of *i*-CGS. Comparison of model checking complexity for turn-based structures[16] can give us a hint in this respect. Note that, for such structures, $m = \mathbf{O}(nd)$ and we can use the model checking algorithms from Section 6.2 and from [5] to model-check formulae of ATL$_{ir}$ and ATL, respectively.

Proposition 12 *Model checking* ATL$_{ir}$ *over turn-based i-*CGS *is* $\mathbf{\Delta_2^P}$*-complete (*$\mathbf{NP}$*-complete for "Positive* ATL$_{ir}$*"), while model checking* ATL *over turn-based* CGS *can be done deterministically in time* $\mathbf{O}(ndl)$*. Since* $d \leq n$ *for turn-based structures, the latter bound can be replaced by* $\mathbf{O}(n^2 l)$*.*

The result can be generalised to systems in which only a fixed (or bounded) number of agents is acting in each state; we propose to call such systems *semi-turn-based* concurrent game structures. Note that systems with a fixed (or bounded) number of agents are a special case of semi-turn-based CGS.

Proposition 13 *Model checking* ATL$_{ir}$ *over semi-turn-based i-*CGS *is* $\mathbf{\Delta_2^P}$*-complete (*$\mathbf{NP}$*-complete for "Positive* ATL$_{ir}$*"), while model checking* ATL *over semi-turn-based* CGS *can be done deterministically in time* $\mathbf{O}(n^2 l)$*.*

Moreover, the exhaustive model checking of ATL formulae can be done in time $\mathbf{O}(nd^k l)$, while, for ATL$_{ir}$ formulae, it can be done in $\mathbf{O}(nd^{kn} l)$ steps. This is due to the fact that $\langle\langle A\rangle\rangle \Box\varphi \equiv \varphi \wedge \langle\langle A\rangle\rangle \bigcirc \langle\langle A\rangle\rangle \Box\varphi$ and $\langle\langle A\rangle\rangle \varphi \mathcal{U} \psi \equiv \psi \vee \varphi \wedge \langle\langle A\rangle\rangle \bigcirc \langle\langle A\rangle\rangle \varphi \mathcal{U} \varphi$ in ATL, whereas analogous fixpoint characterisations do not hold for ATL$_{ir}$ modalities. Thus, successful ATL strategies can be computed incrementally, state by state. By contrast, uniform strategies must be considered *as a whole*, which requires much more backtracking if we check the possibilities exhaustively.

Nevertheless, we believe that the results in this section indicate that agent logics with imperfect information might not be unfeasible. If ATL formulae can be feasibly model-checked then agents with imperfect information are not *that* far away. And there already exist running model-checkers for ATL [6,1], based on OBDD (Ordered Binary Decision Diagrams). Also, new model checking techniques, based on the idea of *Unbounded Model Checking*, are under development [26].

[16] I.e., structures in which at each state there is a single agent who decides upon the next transition; this can be modeled by requiring that $d(a, q)$ is a singleton for all but one agent.

7 Conclusions

In this article, we discussed model checking complexity for several variants of alternating-time temporal logic ATL. We analyzed the complexity of model checking for explicit models when the size of models is defined in terms of states rather than transitions, and the number of agents is considered a parameter of the problem. Most importantly, we proved that the problem is intractable for all studied variants of the logic. First of all, we showed that model checking "Positive ATL" (i.e., ATL where the use of negation is restricted to literals) over concurrent game structures is $\mathbf{\Sigma_2^P}$-complete. Moreover, for the previous semantics based on alternating transition systems, the problem is "only" **NP**-complete. Using our results, Laroussinie, Markey and Oreiby proved subsequently in [29] that model checking of full ATL is $\mathbf{\Delta_3^P}$-complete for CGS and $\mathbf{\Delta_2^P}$-complete for ATS. All these results suggest that using ATS may have some advantage over CGS. Secondly, we showed that ATL model checking over the broader class of nondeterministic alternating transition systems is still **NP**-complete for the "positive" formulae (and $\mathbf{\Delta_2^P}$-complete in the general case), and hence the ATS-based semantics might perhaps be used in a more convenient way than until now.

Finally, we proved that:

1. Model checking ATL$_{ir}$ (i.e., ATL with imperfect information) is $\mathbf{\Delta_2^P}$-complete in the number of transitions and the length of the formula (therefore closing a gap in existing research);
2. Model checking "Positive ATL$_{ir}$" is **NP**-complete in the number of transitions and the length of the formula;
3. Model checking ATL$_{ir}$ is $\mathbf{\Delta_3^P}$-complete when the size of models is defined in terms of states rather than transitions;
4. Model checking "Positive ATL$_{ir}$" is $\mathbf{\Sigma_2^P}$-complete in the same setting.

Thus, checking strategic ability under imperfect information falls in the same complexity class as checking strategic ability for perfect information agents, when a more refined analysis is conducted – which we consider somewhat surprising. We summarise the existing results on model checking ATL-related logics in Figure 7.

Additionally, we presented a truth-preserving translation of ATL models and formulae. The resulting models are always *turn-based*, which usually means an exponential decrease in the number of transitions. As turn-based alternating transition systems are very close to CTL models, as well as extensive form games with perfect information, one may hope that some interesting techniques can be transferred from CTL model checking and/or game theory this way. Moreover, our translation of models is independent from the translation of formulae, which allows for "pre-compiling" models when one wants to check various properties of a particular multi-agent system.

We would like to thank: Thomas Eiter for his help in the **NP**-hardness proof of *Sfc-SAT*; Rafał Somla for the discussions about model checking ATL and its various extensions; Nils Bulling for checking our checking of "weak

		m, l	n, k, l	n_{local}, k, l
CTL		P [9]	P [9]	PSPACE [27]
P-ATL	ATS	P [4]	**NP** (Section 4)	EXPTIME [39]
	NATS	P (Section 4.5)	**NP** (Section 4.5)	
	CGS	P [5]	$\mathbf{\Sigma_2^P}$ (Section 3)	
ATL	ATS	P [4]	$\mathbf{\Delta_2^P}$ [29]	
	NATS	P (Section 4.5)	$\mathbf{\Delta_2^P}$ (Section 4.5)	
	CGS	P [5]	$\mathbf{\Delta_3^P}$ [29]	
P-ATL$_{ir}$		NP ([37] & Sec. 6.2)	$\mathbf{\Sigma_2^P}$ (Section 6.4)	?
ATL$_{ir}$		$\mathbf{\Delta_2^P}$ ([37] & Sec. 6.3)	$\mathbf{\Delta_3^P}$ (Section 6.4)	

Fig. 16 Model checking complexity: completeness results for various settings of input parameters. Symbols n, k, m stand for the number od states, agents and transitions in the explicit model, l is the length of the formula, and n_{local} is the number of local states in a concurrent program. P-ATL stands for "Positive ATL", and P-ATL$_{ir}$ for "Positive ATL$_{ir}$".

until", and for checking our proof of Theorem 9; Pierre-Yves Schobbens for comments; last, but not least, the anonymous reviewers of MFCS'05, CEEMAS'05, ICTCS'05, BNAIC'05, and TOCS for their helpful remarks.

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