Extending Fuzzy Logics to Support Decisions

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Abstract. In this paper a concept of evaluation schema for uncertain statements is proposed. The aim is to represent and process knowledge and to support decision making under uncertainty, involving the information with unmeasurable uncertainty in the reasoning process. This 'uncertainty logic' is based on fuzzy logics and Łukasiewicz three-valued logic, but it's underlied by a richer structure that makes more subtle analysis of statements possible. The logic is a kind of L-fuzzy logic formally. However, its ontological motivations are different.

Keywords: knowledge representation, reasoning under uncertainty,
decision making, fuzzy logics, three-valued logic.

1 Motivations

1.1 Philosophical Motivations

According to (Klir & Folger 1988), situations of uncertainty arise either in a world that's vague or in a world of imperfect information (ambiguity; uncertainty in the narrow sense). There are some approaches to the problem of representing and processing uncertain information; fuzzy measures, fuzzy sets (as well as fuzzy logics) and three-valued logic among them. Fuzzy logic describes the logical value of any proposition with a number from $[0, 1]$ set: 0 is assigned to evidently false sentences, while 1 stands for complete truth. The three-valued logic, proposed by Jan Łukasiewicz in the 1920s, has only 3 logical values: 0 for false, 1 for true and *u* for undefined – neither false nor clearly true – statements (originally, Łukasiewicz used $\frac{1}{2}$ instead of *u*, but it's better not to confuse 'half true' or 'half certain' sentences with sentences of *unknown* truth value). However, Łukasiewicz logic isn't merely a simpler version of the fuzzy logic. They represent different ontological categories: the fuzzy model describes only *measurable uncertainty* while the 3-valued logic refers to the *unmeasurable* one. In other words, the fuzzy model offers no opportunity to express our belief that something is possible (true to some degree) without stating *how* possible (true) it is. On the other hand, the 3-valued model doesn't allow us to measure the possibility (veracity) in any way. Thus, putting these two approaches in one box and making them work together might prove useful.

In this paper such a unification is proposed in a form of a logical system **UL** (uncertainty logic). An obvious application of the system is the decision-making process – decisions may be evaluated in a more subtle way than in any of the approaches alone. The framework is being considered on the propositional level; first order uncertainty logic should be investigated in the future.

1.2 Fuzzy Logic

In fuzzy logics, $[0, 1]$ is the set of logical values. The logical value of a compound formula is defined as:

- $\text{val}(\alpha \wedge \beta) = \text{val}(\alpha) \overset{T}{*} \text{val}(\beta)$,
- $\text{val}(\alpha \vee \beta) = \text{val}(\alpha) \overset{T}{+} \text{val}(\beta)$,
- $\text{val}(\neg\alpha) = \overset{T}{-} \text{val}(\alpha)$,

where $\overset{T}{*}$ and $\overset{T}{+}$ are called *T-norm* and *T-conorm* respectively. Many norms are used for different purposes, some of them are presented below:

$a_1 \overset{T}{*} a_2$	$a_1 \overset{T}{+} a_2$
$\min(a_1, a_2)$	$\max(a_1, a_2)$
$a_1 a_2$	$a_1 + a_2 - a_1 a_2$
$\max(0, a_1 + a_2 - 1)$	$\min(a_1 + a_2, 1)$
a_1 *if* $a_2=1$; a_2 *if* $a_1=1$; 0 *otherwise*	a_1 *if* $a_2=0$; a_2 *if* $a_1=0$; 1 *otherwise*

Fuzzy negation is most often defined as $\overset{T}{-}a = 1 - a$.

Multivalued logics provide a language for describing both vague and uncertain situations – in the case when we can measure our uncertainty, of course. Unfortunately, a reliable logic of beliefs cannot be truth-functional as fuzzy logics are. Thus, the fuzzy logics deal with vagueness generally. Within the fuzziness theory *fuzzy measures* are used to describe imperfect knowledge. A particular fuzzy measure is constrained to beliefs centered on one fixed object of the universum only, and we cannot combine measures for different objects if we want to keep our reasoning on a high level of reliability. However, when we have no reliable logic of beliefs at hand, it's often clever to use a 'non-reliable' one instead. Consequently, there is a place for truth-functional fuzzy logics in describing uncertain beliefs as well.

1.3 Three-Valued Logic

Jan Łukasiewicz was one of the first researchers who investigated multivalued logics. He developed a group of logics (L_n); each of n logical values $0, \frac{1}{n-1}, \frac{1}{n-2}, ..., 1$. Among them, L_3 is the most interesting. As the logical value of $\frac{1}{2}$ does not provide any measure of truthfulness/possibility, it is often replaced with symbol *u* (undefined) to avoid confusion. Basic operators are defined as follows:

val(α)	val(β)	val($\alpha \wedge \beta$)
0	0	0
0	u	0
0	1	0
u	0	0
u	u	u
u	1	u
1	0	0
1	u	u
1	1	1

val(α)	val(β)	val($\alpha \vee \beta$)
0	0	0
0	u	u
0	1	1
u	0	u
u	u	u
u	1	1
1	0	1
1	u	1
1	1	1

$val(\alpha)$	val($\neg\alpha$)
0	1
u	u
1	0

Since L_3 is also truth-functional, it may be employed for situations of unmeasurable vagueness/uncertainty under the same assumptions [1].

1.4 A Practical Example

Let's trace the problem on the following example. A Producer (P) has to decide whether he wants to produce pencils or markers. An expert proposed such decision rules:

if P knows everything about pencils **then** P should produce pencils

if P has a very good marketing specialist **then** P should produce both pencils & markers

if P knows everything about markers **or** market coefficients are high
then P should start with markers.

People often propose rules for decision making in a similar way.

According to some experience, P can evaluate his knowing everything of pencils as well as his knowledge about markers to 0.4, and the 'very goodness' of his marketing manager - to 0.3. Whether the coefficients are high, P doesn't know at all. He even doesn't know what coefficients they should be... Which decision should he take?

Of course, P shouldn't produce pencils and markers simultaneously, since the premise for such action is the weakest one (0.3 against at least 0.4). Any fuzzy decision system would conclude so without a problem. But – for any intelligent person – it should also be obvious that markers are better than pencils, because the evaluation of 'P knowing everything about pencils' is exactly 0.4 while the logical value of 'P being an expert in markers or market coefficients being high' may be 0.4, but may also amount to a *somewhat higher* value.

[1] There are several other 3-valued logics, most notably the one developed by Kleene.

2 The Logic

It is strongly intuitive that, if val$(\alpha) \neq 0$ and val$(\beta) < 1$, then $\alpha \vee \beta$ should have a higher value than β. Similarly, if val$(\alpha) \neq 0$, val$(\beta) \neq 0$ and α is more true/possible than β, then $\gamma \wedge \alpha$ ($\gamma \vee \alpha$) should be more true/possible than $\gamma \wedge \beta$ ($\gamma \vee \beta$ respectively), and so on. To satisfy these intuitions we propose a logical system, which is an extension of both fuzzy logic and three-valued logic. The set of logical values LV contains pairs:

$$\mathbf{LV} = \big([0,1) \times [0,\infty]\big) \cup \{(1,0)\}$$

and it allows to evaluate the value of a formula along two orthogonal dimensions - the first corresponds with its measurable (un)certainty while the other refers to the unmeasurable one. Like a compound number, such a logical value (a,b) may be presented as $a + bu$, where u must be defined as $(0,1)$. Of course, there should be no logical value beyond $\mathbf{1} = (1,0)$ (certainly true) and below $\mathbf{0} = (0,0)$ (evidently false) neither. The intuition is to let the fuzzy dimension behave in the usual way, and to make the second dimension accumulate as new unmeasurable alternative appear (if val$(\alpha) = u =$ val(β), then val$(\alpha \vee \beta) = u + u = 2u$; on the other hand, val$(\alpha \wedge \beta) = u * u = u$).

So, operations $\odot$ and $\oplus$ that correspond to logical conjunctions $\wedge$, $\vee$ can be proposed now. Let

$$\text{val}(\alpha) = a_1 + b_1 u, \qquad \text{val}(\beta) = a_2 + b_2 u.$$

Then let

$$\begin{aligned}
\text{val}(\alpha \wedge \beta) &= (a_1 + b_1 u) \odot (a_2 + b_2 u) = \\
&= a_1 \odot a_2 + a_1 \odot b_2 u + \\
&\quad + b_1 u \odot a_2 + b_1 u \odot b_2 u = \\
&= a_1 \overset{T}{*} a_2 + a_1 b_2 u + a_2 b_1 u + b_1 b_2 uu = \\
&= (a_1 \overset{T}{*} a_2) + (a_1 b_2 + a_2 b_1 + b_1 b_2) u,
\end{aligned}$$

$$\begin{aligned}
\text{val}(\alpha \vee \beta) &= (a_1 + b_1 u) \oplus (a_2 + b_2 u) = \\
&= min(1, a_1 \oplus a_2 + b_1 u \oplus b_2 u) = \\
&= \begin{cases} 1 & \text{if } (a_1 \overset{T}{+} a_2) = 1 \\ (a_1 \overset{T}{+} a_2) + (b_1 + b_2) u & \text{else} \end{cases}
\end{aligned}$$

Note that $1 \odot \sigma = \sigma$, $1 \oplus \sigma = 1$, $0 \odot \sigma = 0$, $0 \oplus \sigma = \sigma$ and $u \odot u = u$. $u \oplus u = 2u$ etc.

The measurable uncertainty/vagueness value behaves as if the second dimension was absent. On the other hand, the second coefficient grows as more unmeasurable premises arise. We may ignore this value completely – and then we land in the fuzzy world again.

Of course, it isn't any attempt to *measure* the unmeasureable (un)certainty, this is more about approximation. It's just a way of extracting some more information from knowledge that is incomplete or concepts which are too vague for a given observer in a given context.

2.1 Inference System

For a propositional logic like that you may use the truth-functionality assumption as a basis to evaluate any statement. The reasoning is straightforward then – you should just take the logical values of simple formulae and calculate the values for compound ones using the $\odot$ and $\oplus$ operators defined above. However, we do not necessarily have the values of *all* simple propositions. Instead, people are often able to identify a bunch of relevant dependencies in the form: *if* something is true/certain to some extent *then* something should be true/certain to the same extent at least (implication as inclusion). Thus, an inference system seems to be a more general form of presenting the reasoning algorithm.

In fact, the inference machine should consist of two layers: the reasoning layer, which allows generating statements about the actual situation, and the decision layer, responsible for selecting best decisions from the set of potential ones. Two set of axioms (one - describing knowledge of the actual situation, the other - defining optimal behavior according to various situations) would form the starting point of the inference process.

Reasoning about Situation On that level assertions about the situation are processed. The assertions are statements in a form:

$$\vdash_k \alpha,$$

where k∈ LV, and α is a propositional formula.

The reasoning starts with a set of axioms that represent these pieces of knowledge we have gathered from the outside through some information processing or some kind of 'illumination'. The following inference rules are proposed:

$$[comp_\wedge]\frac{\vdash_k p,\ \vdash_l q}{\vdash_{k\odot l}\ p\wedge q} \qquad [comp_\vee]\frac{\vdash_k p,\ \vdash_l q}{\vdash_{k\oplus l}\ p\vee q}$$

$$[MP]\frac{\vdash_k p,\ \vdash_l p\Rightarrow q}{\vdash_{(k\oplus u)\odot l}\ q}$$

The aim of the reasoning process is to infer the logical value of a given propositional formula.

There are some problems with the consistency of every particular theory (defined with a particular set of axioms). What would it mean if both $\vdash_k\ \alpha$ and $\vdash_{k'}\ \alpha,\ k\neq k'$ could be inferred? Well, we may reject the whole theory as inconsistent but most of descriptions prepared by humans would be inconsistent in that sense. Another solution is to take the maximum ($max(k,k')$) as the derived value for statement α. However, a problem of complexity arises in this case: the amount of all inference paths for $\vdash_?\ \alpha$ should rather be finite (and preferably small, in fact). Thus, rules for composition are the most welcome – since they lead from simpler formulae to more complex ones. On the other hand, the Modus Ponens rule may lead to a loop (as in $\vdash_{0.5} p, \vdash_1 p\Rightarrow q, \vdash_1 q\Rightarrow p$) but such a looping descriptions can be detected and signaled automatically.

In both cases we should decide whether we follow the closed world (val(α) $=0$ by default) or the open world (val(α) $=u$ by default) assumption.

Choosing the Best Decision Let **Act** be the set of potential actions. We define actual decision principles as a set of decision axioms:

$$\mathrel{|\!\sim}\ \alpha\ \rightsquigarrow\ A,$$

where α is a propositional formula, and A∈ Act. The action connected with the most possible premise is considered to be the best[2]. Thus, the algorithm for finding the best decision should consist of the following steps:

- for every decision axiom $\mathrel{|\!\sim}\ \alpha\ \rightsquigarrow\ A$, investigate the goal $\vdash_?\ \alpha$, deriving a logical value for formula α;
- select that action A, for which the logical value (expectancy value, respectively) is maximal according to previously chosen ordering relation on LV.

Which value is maximal? Various ordering relations may be used. For instance, we may consider $k_1 = a_1 + b_1$ superior to $k_2 = a_2 + b_2$ iff $a_1 \geq a_2$ and $b_1 \geq b_2$ – in fact, it is the most intuitive relation. But this order is only partial. If we need a stronger tool for comparing logical values, we may use one of the orders presented below, or construct our own one:

[2] There are some other methods, too: the best decision may be defined as the mean or the gravity center of the fuzzy set of decisions, for instance. If we obtain a payoff function (assigning a payoff to various decisions in various situations), we may also seek for the maximum expectancy decision.

["safe" order] $a_1 + b_1 u \preceq_s a_2 + b_2 u$ iff $a_1 \leq a_2$ or ($a_1 = a_2$ and $b_1 \leq b_2$).
This order defines a strategy that ignores the unmeasurable uncertainty coefficient unless the measurable (fuzzy) one leads to more than one decision;
[k-order] $a_1 + b_1 u \preceq_k a_2 + b_2 u$ iff $a_1 + k * b_1 \leq a_2 + k * b_2$ for an arbitrary small k, for example $k = 0.001$;
["gamble" order] $a_1 + b_1 u \preceq_g a_2 + b_2 u$ iff $a_1 + \frac{2\arctan(b_1)}{\pi} \leq a_2 + \frac{2\arctan(b_2)}{\pi}$.
This strategy combines the measurable aspect and the approximation of the unmeasurable with equal weights. $\frac{2\arctan(b)}{\pi}$ maps $[0, \infty)$ set to $[0, 1)$ interval.

First strategy is the safest; third one - the most risky of them.

2.2 Examples

The Practical Example Revisited First we will try to solve the Producer's problem (presented in section 1.3) more formally. We'll take

$$a_1 \overset{T}{*} a_2 = \min(a_1, a_2); \qquad a_2 \overset{T}{+} a_2 = \max(a_1, a_2)$$

as T-norm and T-conorm respectively, and $\preceq_s$ as the decision ordering relation. We have such axioms:

$\vdash_{0.4}$ 'P knows everything about pencils'
$\vdash_{0.4}$ 'P knows everything about markers'
$\vdash_{0.3}$ 'P has a very good marketing specialist'
$\vdash_u$ 'market coefficients are high'
$\mid\!\sim$ 'P knows everything about pencils' $\rightsquigarrow$
$\rightsquigarrow$ 'produce pencils'
$\mid\!\sim$ 'P has a very good marketing specialist' $\rightsquigarrow$
$\rightsquigarrow$ 'produce pencils & markers'
$\mid\!\sim$ 'P knows everything about markers'$\vee$
$\vee$'market coefficients are high' $\rightsquigarrow$
$\rightsquigarrow$ 'start with markers'

Now we must evaluate the goals:

$\vdash_?$ 'P knows everything about pencils',
$\vdash_?$ 'P has a very good marketing specialist',
$\vdash_?$ 'P knows everything about markers'$\vee$
$\vee$'market coefficients are high'.

We get $\vdash_{0.4}$ 'P knows everything about pencils', $\vdash_{0.3}$ 'P has a very good marketing specialist' directly from axioms, and $\vdash_{0.4+u}$ 'P knows everything about markers' $\vee$ 'market coefficients are high' through the $[comp_\vee]$ inference rule. Both 0.3 and $0.4 \preceq_s 0.4 + u$ so the best decision for Producer is to start with markers.

Example 2 Let's consider the situation of an Operator of some Very Complex Machine. He has to decide whether to take a nap (action *Sleep*), to watch the machine carefully (action *Watch*), or to run away quickly (*Escape!*). He can observe and evaluate the following statements:

- 'the temperature is high' (*highTemp*),
- 'the temperature is not high' (*notHighTemp*),
- 'the reaction time is short' (*fastReact*),

- 'the reaction time is not short' (*notFastReact*),
- 'the Boss will come to check me' (*bossCome*),
- 'the Boss will not come' (*bossNotCome*),
- 'there are some mice in the Machine' (*mice*),
- 'there are no mice in the Machine' (*noMice*).

The Operator can also consider more abstract assertions:

- 'everything is OK' (*certainlyOK*),
- 'the Machine is going to explode' (*goingExplode*).

The actual knowledge of the Operator:

$\vdash_{0.6}$ highTemp
$\vdash_{0.4}$ notHighTemp
$\vdash_{0.7}$ fastReact
$\vdash_{0.3}$ notFastReact
$\vdash_{u}$ bossCome
$\vdash_{u}$ bossNotCome
$\vdash_{u}$ mice
$\vdash_{u}$ noMice

$\vdash_{0.9}$ notHighTemp $\wedge$ fastReact $\wedge$ noMice $\Rightarrow$
$\Rightarrow$ certainlyOK
$\vdash_{0.5}$ highTemp $\wedge$ notFastReact $\Rightarrow$ goingExplode

$\mid\!\sim$ certainlyOK $\vee$ fastReact $\wedge$ bossNotCome $\rightsquigarrow$ Sleep
$\mid\!\sim$ highTemp $\wedge$ fastReact$\vee$
$\vee$notHighTemp $\wedge$ bossCome $\rightsquigarrow$ Watch
$\mid\!\sim$ goingExplode $\rightsquigarrow$ Escape!

Let $a_1 \overset{T}{*} a_2 = a_1 a_2$, $a_1 \overset{T}{+} a_2 = a_1 + a_2 - a_1 a_2$, and let $\preceq_{0.01}$ stand for decision ordering. Then we can infer:

$\vdash_{1.152u}$ certainlyOK
$\vdash_{0.7u}$ fastReact $\wedge$ bossNotCome

$\vdash_{1.852u}$ certainlyOK $\vee$ fastReact $\wedge$ bossNotCome
$\vdash_{0.42+0.4u}$ highTemp $\wedge$ fastReact $\vee$ notHighTemp $\wedge$ bossCome
$\vdash_{0.09+0.5u}$ goingExplode

Since $1.152u \preceq_{0.01} 0.42 + 0.4u$ and $0.09 + 0.5u \preceq_{0.01} 0.42 + 0.4u$ – the best decision for Operator is to watch the Machine carefully.

The Operator's knowledge is partly fuzzy. This fuzziness may be a consequence of using vague notions (*high* temperature). On the other hand, the Operator may precisely know what *high* means (from 1300K up, for instance) but his input data may be ambigous because the control board is a bit out of order... When the language of UL suffices for expressing the knowledge we posses, uncertain statements can be processed together regardless of the source of their uncertainty.

3 Final Remarks

In this paper a framework is presented. UL is an extension of fuzzy logic and 3-valued logic; it provides us with a richer structure for representing the truthfulness level of statements. Differences (previously invisible) between decisions may be detected while using this structure, thus it allows to make better decisions through more subtle analysis.

Another feature of UL is a new, more intuitive Modus Ponens inference rule. The intuition, drawn from the classical implication, is that if the premise is true on some level k then the conclusion must be true on *at least* the same level ($k \oplus u$: a level *somewhat* higher than k) – what's difficult to express in fuzzy logics. On the other hand, there are problems with negation in UL. This can be coped with – as it's been done in the second example – by removing negation out of every formulae except literals, but it isn't necessarily a satisfying solution.

There are several lines of further research in perspective:

- completing the system: adding negation, adding rules for alternative and conjunction reduction;
- investigation of its theoretical properties (completeness, decidability etc.);
- UL development into the First Order level;
- UL application to real-life problems (control systems, games with incomplete information, strategic situations of vague concepts or/and uncertain beliefs).

As stated by Hajek (Hajek 1998), a reliable logic of beliefs should be a kind of modal logic. It cannot be truth-functional, as fuzzy logics are. However, a truth-functional multivalued logic may be a good tool for approximate reasoning under uncertainty. Such a logic of beliefs is suboptimal because it ignores the interconnections between the componets of a logical formula (while the events referred to by the components simply doesn't have to be independent!). Yet – when we want to measure possibilities – the logic may define a reasoning schema with a complexity positively less than a modal reasoning.

Moreover, to use the modal schema we must obtain a precise information on the possible worlds, their 'likelihood', and some satisfaction measure for atom formulae – i.e. which propositions are satisfied in which worlds and to what degree. In some cases, though, such a precise model is impossible to construct, especially when we acquire the description of the actual situation from human operators. The operator is often able to provide an intuitive 'likelihood measure' for various possible events, but he/she can present no deeper analysis. In other words, the operator can propose a bunch of intuitions, but he/she cannot specify on what ground the intuitions are generated. If we are provided in such statements only (and some decision is to be taken), even a suboptimal, approximate reasoning may be valuable. It is supposed to do better than no analysis at all, isn't it?

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